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Convergence of Regularized, Renormalized Perturbation Series for Super-Renormalizable Field Theories (*).

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Summary. — It is shown for the two-dimensional scalar Yukawa interaction, that the renormalized perturbation series has at least a finite radius of convergence *when regularized* (cut-off in space-time and momentum space). This is accomplished by writing down the explicit renormalized series, studying some of its associated combinatorics, and applying Caianiello's standard arguments on the unrenormalized series ([16]). The result extends to any super-renormalizable theory of the $\varphi\bar{\psi}\psi$ form.

1. – Introduction.

One of the most important open questions in a purely perturbative approach to quantum field theory involves the convergence of the usual Gell-Mann–Low series. Even if one regards quantum electrodynamics (Q.E.D.) as a semi-phenomenological set of rules, one cannot regard the theory as giving well-defined results until either convergence of the renormalized series or some substitute ([1]) is proven.

While there are a large number of very nice results for convergence of nonrelativistic (Rayleigh-Schrödinger) perturbation theory ([2,3]) all the mathe-

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([1]) A suitable substitute might be some additional principle which picked out one function from the many with the perturbation series as its asymptotic expansion.
([2]) T. KATO: *Perturbation Theory for Linear Operators* (New York, 1966).
([3]) Unfortunately, a large number of competent theoretical physicists seem unaware of this; see *e.g.* K. GOTTFRIED: *Quantum Mechanics*, I (New York, 1966), p. 361.

matically suitable results in the relativistic (Feynman) case are of a negative nature [4].

The earliest mathematically complete results proved that cubic and quartic scalar *boson* couplings yielded *divergent* series [5]. As FRANK [6] has remarked, this is not surprising: the analogous nonrelativistic problem, namely a harmonic oscillator with x^3 or x^4 perturbation does not have a convergent perturbation theory either. Besides the fact of divergence, nothing is known about the exact nature of the series; for example, it is not known if it is an asymptotic expansion, nor if the exact solution [7] has only an isolated singularity at zero coupling. In fact, these questions are open even in the corresponding nonrelativistic problem [9].

Divergence of the perturbation series for Dirac particles in a static external (nonquantized) electromagnetic field is a result contained implicitly [10] in Schwinger's work [11], where an exact solution nonanalytic at zero coupling is to be found. The exact nature of the divergence has been studied somewhat by OGIEVSKI [12] who showed that the perturbation series is Borel summable to the correct answer. On the other hand, CAPRI [13] has shown that electrodynamics in an external field of compact support in space *and time* has a convergent perturbation series.

For the case of a theory with fermions there are no definitive results, but there is a rash of half-solutions which generally can be regarded as evidence for convergence or divergence at one's whim (this paper falls solidly into the half-solution class). The earliest statements about these theories remarked

(4) In fact the best text in the field—J. BJORKEN and S. DRELL: ***Relativistic Quantum Mechanics*** (New York, 1964) and ***Relativistic Quantum Fields*** (New York, 1965)—settles for ignoring the problem rather than trotting out many conflicting partial answers.

(5) C. A. HURST: *Proc. Camb. Phil. Soc.*, **48**, 625 (1952); W. THIRRING: *Helv. Phys. Acta*, **26**, 33 (1953); A. PETERMAN: *Helv. Phys. Acta*, **26**, 291 (1953).

(6) W. FRANK: *Ann. of Phys.*, **29**, 175 (1964).

(7) Assuming there is an exact solution—which seems likely, at least for φ^4 in two dimensions, given the recent work of GLIMM and JAFFE (8).

(8) J. GLIMM: *Commun. Math. Phys.*, **5**, 343 (1967); **6**, 120 (1967); **8**, 12 (1968); M.I.T. preprints; A. JAFFE: Thesis, Princeton University (1965); A. JAFFE and R. POWERS: *Commun. Math. Phys.*, **7**, 218 (1968); A. JAFFE and J. GLIMM: M.I.T. preprints.

(9) There are however recent results of C. BENDER and T. T. WU: *Phys. Rev., Lett.*, **21**, 406 (1968), which are of a highly questionable mathematical nature. These results indicate severe nonanalyticity at zero coupling.

(10) This was first explicitly remarked in B. L. IOFFE: *Doklady Akad. Nauk. SSR (N.S.)*, **94**, 437 (1954).

(11) J. SCHWINGER: *Phys. Rev.*, **82**, 664 (1951).

(12) V. I. OGIEVSKI: *Doklady Akad. Nauk. SSR (N.S.)*, **109**, 919 (1956).

(13) A. CAPRI: Univ. of Alberta preprint.

that barring cancellations due to alternating signs, one would get divergence as in a φ^3 theory. Of course, the Pauli principle predicts large cancellations and for this reason, this problem is qualitatively different from the scalar case. The first person to control these cancellations was CAIANIELLO. As a by-product of his elegant formulation of the Gell-Mann–Low series in terms of Haffnians and determinants [14], he proved that a $\bar{\psi}\psi\varphi$ theory [15] cut-off in momentum space and ordinary space-time [16] converges in at least a finite circle [17]. Two years, later, YENNIE and GARTENHAUS [18], using nothing but the boundedness of fermion creation operators, proved the convergence of the cut-off series in the entire plane [19]. Shortly thereafter, CAIANIELLO and BUCCAFURRI [20] were able to obtain the Yennie-Gartenhaus result through the use of rather elaborate estimates within the Caianiello formalism [21]. There are more recent results on the problem [6,22-24, others] but none go significantly further than the classics.

Our brief summary of the classic arguments on convergence *vs.* divergence would be incomplete without a mention of the often-quoted article of DYSON [25]. He argues that the series for Q.E.D. cannot converge because for imaginary values of the charge, the Coulomb force is attractive between like particles and so the vacuum is unstable under decay into a large number of pairs. It cannot be emphasized too greatly that this argument must be considered unacceptable for the problem at hand. The perturbation series at unphysical values of the coupling [26] is unphysical and as a result cannot be treated by physical arguments. A simple parable to show that the uncritical use of physical arguments will lead one astray can be obtained from a λ^2x^4 perturbation to a harmonic oscillator p^2+x^2 in nonrelativistic quantum mechanics. One might conclude from physical grounds that since the potential misbehaves for all negative λ,

(14) E. R. CAIANIELLO: *Nuovo Cimento*, **10**, 1634 (1953); **11**, 493 (1954).

(15) This would include both scalar and pseudoscalar Yukawa interactions as well as Q.E.D.

(16) Such a theory is called regularized.

(17) E. R. CAIANIELLO: *Nuovo Cimento*, **3**, 223 (1956).

(18) D. R. YENNIE and S. GARTENHAUS: *Nuovo Cimento*, **9**, 59 (1958).

(19) Actually, they proved that the ordinary connected graph sum is the ratio of the two entire functions and hence meromorphic.

(20) A. BUCCAFURRI and E. R. CAIANIELLO: *Nuovo Cimento*, **8**, 170 (1958).

(21) The cut-off methods used by Caianiello-Buccafurri are very different from those of Yennie-Gartenhaus; it is difficult to compare them, but those of the latter appear to be less stringent.

(22) W. FRANK: *Journ. Math. Phys.*, **8**, 194 (1967).

(23) W. FRANK: *Ann. of Phys.*, **29**, 217 (1964).

(24) F. GUERRA and M. MARINARO: *Nuovo Cimento*, **42** A, 285 (1966).

(25) F. DYSON: *Phys. Rev.*, **85**, 631 (1952).

(26) A conclusive way of seeing that an imaginary charge is unphysical is to remark that it leads to a non-Hermitian Hamiltonian.

that the eigenvalues cannot be analytic at $\lambda = \infty$. Nevertheless they can be shown ([27]) to be analytic there.

Another physical argument favoring divergence of the perturbation series for Q.E.D. is due to FRAUTSCHI ([28]) and makes very clever use of renormalization group techniques. FRAUTSCHI only proves that the infinite momentum value of the « scalar photon propagator, » d ([29]), cannot be continuous at zero physical charge, *i.e.*

$$\lim_{e\to 0}\,[\lim_{q^2\to\infty} d(e^2, q^2)] \neq \lim_{q^2\to\infty} d(0, q^2)\,.$$

However, it requires a rather loose sense of mathematical propriety to use this to make any predictions about the finite q^2 behavior.

In order to explain the motivation behind studying the model we will treat, we explain the CAIANIELLO proof of convergence ([17]) in a little more detail. Rather than deal with the expansion of the Green's function ([30]) as the sum of all connected diagrams, he used the expansion as the ratio of all diagrams (connected or disconnected) to the vacuum diagrams (see, *e.g.* ([4]), Chap. 17). In this form, even the intermediate states obey the Pauli principle and so « maximal » cancellations are to be expected. But one pays a price for using this form. Connected vacuum diagrams diverge linearly in the space-time volume and disconnected vacuum diagrams diverge with a higher order. For this reason a cut-off of the space-time integrals is needed, at least initially ([31]). In addition, one must introduce the standard momentum-space cut-off to control ultra-violet divergence. Once this has been done one proves convergence easily; one takes the Caianiello determinant-Haffnian formulae (discussed in Sect. 3) and applies Hadamard's inequality ([32]) for an $n\times n$ matrix:

$$|\det(a_{ij})| \leq n^{n/2}(\max_{i,j}|a_{ij}|)^n\,. \tag{1}$$

If one now wishes to remove the momentum-space cut-off, one must first renormalize the series. But renormalization is typically accomplished by a

([27]) Private communication from K. SYMANZIK via A. DICKE and A. S. WIGTHMAN.

([28]) S. C. I. FRAUTSCHI: *Progr. Theor. Phys.*, **22**, 882 (1959).

([29]) d is defined so $D_{\mu\nu} = (g_{\mu\nu}/q^2)\,d +$ « gauge terms ».

([30]) By Green's function, we mean a vacuum expectation value of a time-ordered product of fields.

([31]) In this paper, we will not deal at all with removing this cut-off. Hopefully, by using the fact that the sum of the vacuum diagrams is the exponential of the sum of the connected vacuum diagrams, one could isolate the volume dependence in a convenient form and eventually eliminate the volume cut-off from the Green's function formulae. Admittedly, this is a rather optomistic hope.

([32]) J. HADAMARD: *Compt. Rend.*, **66**, 1500 (1893).

partial suppression of certain diagrams and these modifications will destroy some of the cancellations ([33]) that led to Hadmard's inequality. The first step in any attempt at a proof of convergence based on successive removal of Caianiello's cut-offs must be to examine whether renormalization is « sufficiently gentle » to avoid destroying the cancellations needed for convergence. In this paper we show that for a Yukawa interaction in two-dimensional space-time renormalization does not destroy the convergence of the *regularized* series in some neighborhood of zero coupling constant. I initially hoped that this would just be a first step in the complete removal of the ultra-violet cut-off but this hope has not been realized.

It is my belief that this model is the natural place to try to settle the question of convergence. This belief is predicated on the fact that the theory is super-renormalizable (S.R.), that is, there are only a finite number of primitive divergences and these can be explicitly eliminated to yield closed formulae. The use of S.R. theories is not really new to the problem; in his proof of divergence of φ^3, HURST ([5]) used the S.R. property of that theory. However, because there are no four-dimensional S.R. theories with fermions, there has been no previous work on convergence of specific renormalized theories arising from local fermion Lagrangians.

It is an idea presented forcefully by WIGHTMAN that the natural place to try to understand many of the questions of quantum field theory is in two dimensions, where the standard Lagrangians with the exception of the Fermi type ([34]) are S.R. In fact, the S.R. property of the φ^4 and Yukawa interactions is used decisively in the important recent work of JAFFE and GLIMM ([8]) which seems to be leading towards the first examples of nontrivial local fields obeying the Wightman axioms! These results of JAFFE and GLIMM, if they ever reach complete fruition, may solve the question of convergence of the perturbation series by obtaining the actual answer in some form—the answer may then be examined directly for singularities at zero coupling.

We remark that while a proof of divergence for our theory would be extremely indicative for the four-dimensional theories, a proof of convergence would not necessarily indicate convergence in the four-dimensional case ([35]). However, a convergence proof would have « fringe benefits » far outweighing this disadvantage. For, if one could prove convergence, one would have candidates for the Green's functions of an interacting field. It is a well-

([33]) To see that cancellations are responsible in (1), one need only remark that the formula is false for permanents.

([34]) That is, a $\bar\psi_1\gamma_\mu\psi_1\bar\psi_2\gamma^\mu\psi_2$ interaction; however, this theory is at least renormalizable in two dimensions, while it is nonrenormalizable in three or more dimensions.

([35]) Since the four-dimensional non-S.R. theories are apparantly not as well-behaved as the two-dimensional S.R. theories.

known fact [36] that the axioms for a field are expressible in terms of the vacuum expectation values and the folklore of axiomatic field theory assures us that this should carry over to the time-ordered vacuum expectation values as well [37]. Thus, to prove that one actually had a field, one would only have to verify certain properties, most of which are immediate [38].

Finally, we briefly summarize the content of the remaining Sections.

In Sect. 2, the renormalization is carried out explicitly for the theory under consideration—the two-dimensional scalar Yukawa interaction.

In Sect. 3, we present the combinatorics necessary to replace Hadamard's inequality.

In Sect. 4, we prove the convergence of the *regularized* renormalized series.

And in Sect. 5, we discuss the significance of the result for the unregularized problem.

2. – Explicit renormalization of the theory.

We treat a model with Lagrangian

$$L=\frac{1}{2}:\left(\frac{\partial\varphi}{\partial x^\nu}\frac{\partial\varphi}{\partial x_\nu}-\mu_0^2\varphi^2\right):+:\bar\psi(i\nabla-m)\psi:+g_0:\bar\psi\psi\varphi:\,,$$

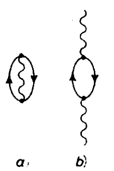


Fig. 1.

with φ a scalar field and ψ a spinor Dirac field in two dimensions. Our metric is chosen so that $a^2=a_0^2-a_1^2$ and our γ-matrices follow Wightman's conventions [39]. Our Lagrangian will eventually have $\mu_0^2\to\infty$ with $\mu^2=\mu_0^2+\delta\mu^2$ [40].

Simple power counting shows only two primitive divergent diagrams (Fig. 1). We can elimitate 1 *a*) from the ratio of all diagrams divided by the vacuum diagrams by simply suppressing it, since all disconnected diagrams cancel out anyway [41]. However 1 *b*) requires a mass renormalization. Let us define $\pi(p)$ by requiring 1 *b*) to have the value

$$\frac{i}{p^2-\mu_0^2+i\varepsilon}\left(-i\pi(p)\right)\frac{i}{p^2-\mu_0^2+i\varepsilon}\equiv \tag{2}$$

[36] A. S. WIGHTMAN: *Phys. Rev.*, **101**, 860 (1956); R. F. STREATER and A. S. WIGHTMAN: *PCT, Spin and Statistics and All That* (New York, 1964), p. 117.

[37] D. RUELLE: *Nuovo Cimento*, **19**, 356 (1961).

[38] The only difficult property to verify would presumably be positive definiteness.

[39] *Cargèse Lectures in Theoretical Physics*, LEVY Ed. (New York, 1967), p. 219, eq. (4.53).

[40] μ will still differ from the physical mass by a finite renormalization.

[41] Dropping this can be viewed as an infinite renormalization of the vacuum energy density.

when the «external» lines have momentum p and we include their propagators; thus

$$-i\pi(p)=-(-ig_0)^2\int\frac{d^2k}{(2\pi)^2}\,\mathrm{Tr}\left(\frac{i}{\gamma\cdot k-m+i\varepsilon}\,\frac{i}{\gamma\cdot k+\gamma\cdot p-m+i\varepsilon}\right). \tag{3}$$

This may be computed with a Pauli-Villars regularization in a series of steps almost identical to [4], Vol. I, p. 155. The end result is

$$\pi(p)=\frac{-g_0^2}{2\pi}\log\left(\frac{M^2}{m^2}\right)+\pi_f(p)\,,$$

where M is the regularization mass and

$$\pi_f(p)=\frac{g_0^2}{2\pi}\int_0^1 dz\log\left[1-\frac{p^2}{m^2}z(1-z)\right]. \tag{4}$$

Explicitly [42]

$$\pi_f(p)=\frac{g_0^2}{\pi}\left[S(p)\arcsin\left(\frac{p}{2m}\right)-1\right],\qquad \text{if } 0<p^2<4m^2\,, \tag{5a}$$

$$=\frac{g_0^2}{\pi}\left\{S(p)\log\left[\frac{p}{2m}\left(1+S(p)\right)\right]-1\right\}-i\frac{g_0^2}{2}S(p)\,,\qquad \text{if } p^2>4m^2\,, \tag{5b}$$

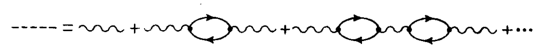
$$=\frac{g_0^2}{\pi}\left\{S(p)\log\left[\frac{\sqrt{-p^2}}{2m}\left(1+S(p)\right)\right]-1\right\}\qquad \text{if } p^2<0\,, \tag{5c}$$

where

$$S(p)=\left|1-\frac{4m^2}{p^2}\right|^{\frac{1}{2}}. \tag{5d}$$

The renormalization is now easy; we first symbolically sum all the divergent bubbles, *i.e.*

(6)



[42] There is really an $i\varepsilon$ in (4) which determines the sign of the imaginary part of (5*b*).

Using (2), we find

$$----=\frac{i}{p^2-\mu_0^2-\pi(p)+i\varepsilon}\,.$$

If we write $\mu^2=\mu_0^2+\delta\mu^2$ with $\delta\mu^2=(-g_0^2/2\pi)\log(M^2/m^2)<0$ [43] and fix μ^2 as finite [40], we see

$$(7a)\qquad ----=\frac{i}{p^2-\mu^2-\pi_f(p)}\,.$$

We will call ---- the « aërated propagator ». It is clear that a simple reduced graph formulation will now describe the complete renormalization. We throw away all diagrams containing either 1 *a*) or 1 *b*) as a subgraph and in the remaining graphs we replace the ordinary propagator by the aërated propagator. That the result is a formal summation followed by renormalization of the original series is evident [44].

We remark that since the aërated propagator is dependent on g_0, the renormalized series is no longer strictly a power series and its region of convergence (if any) is no longer necessarily a circle.

The x-space form of the aërated propagator is, of course

$$(7b)\qquad \tilde{\Delta}_F^{(g_0)}(x'-x)=\int\frac{\mathrm{d}^2k}{(2\pi)^2}\exp[-ik\cdot(x'-x)]\frac{1}{k^2-\mu^2-\pi_f(k)+i\varepsilon}\,.$$

If we put the system in a box of volume V with periodic boundary conditions and restricted our system to a finite number of boson modes, we would have

$$(7c)\qquad \tilde{\Delta}_F^{(V,B,g_0)}(x)=\sum_{|n|\leqslant B}\frac{1}{V}\exp[ik_nx_1]\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{(2\pi)}\frac{\exp[-ipx_0]}{p^2-k_n^2-\mu^2-\pi_f(p,k_n)+i\varepsilon}\,,$$

where $k_n=2\pi n/V$.

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(43) $\mu^2<\mu_0^2$ in accordance with general considerations of H. LEHMANN: *Nuovo Cimento*, **11**, 342 (1954), eq. (32).

(44) Actually all that is clear is that the series of linked diagrams, *i.e.* the ordinary Feynman series, is properly described. In fact, it can be argued that by dropping

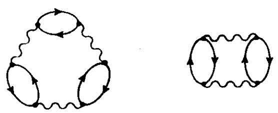

Fig. 2.

diagrams like those in Fig. 2 we have not treated the vacuum graphs properly. Since all the physics is in the linked diagrams and the vacuum amplitude drops out anyway, the argument over treatment of the vacuum amplitude is fruitless.

We will later need a lemma.

Lemma 1. Let V and B be fixed, and let us fix a compact subset D of the complex plane. Then, there is constant C such that for all x and for all g_0 in D, we have

$$|\tilde{\Delta}_F^{(V,B,g_0)}(x)|<C.$$

We do not include a proof of Lemma 1, as it is tedious but reasonably straightforward. Moreover, Lemma 1 will only be needed for a side remark.

3. – Algebraic considerations.

We now embark on a proof that the regularized form of the renormalized series above converges in a neighborhood of $g_0=0$. The basic formula we use is that of Gell-Mann–Low (see [4], eq. (17.22)) as written by CAIANIELLO [14], namely

$$(8)\qquad \langle 0|T[\bar{\psi}_{\beta_1}(x_1)\psi_{\alpha_1}(y_1)\dots\bar{\psi}_{\beta_N}(x_N)\psi_{\alpha_N}(y_N)\varphi(t_1)\dots\varphi(t_\nu)]|0\rangle=$$
$$=\frac{K_{\beta_1,\dots\beta_N;\alpha_1,\dots,\alpha_N}\begin{pmatrix}x_1\dots x_N\\ y_1\dots y_N\end{pmatrix}\!\left|\,t_1\dots t_\nu\right)}{K_{00}}\,.$$

We will show that K_{00} has a finite radius of convergence and will sketch in Sect. 4 how the argument would have to be modified to cover the kernels with external lines. Before renormalization, K_{00} is just the sum of all vacuum graphs; it can be written

$$(9)\qquad K_{00}=\sum_{m=0}^{\infty}\frac{(-ig_0)^{2m}}{(2m)!}\int\dots\int\mathrm{d}y_1\dots\mathrm{d}y_{2m}[1\dots 2m]\begin{pmatrix}1\dots 2m\\ 1\dots 2m\end{pmatrix}=$$
$$=\sum_{m=0}^{\infty}\left(-\frac{g_0^2}{2}\right)^m\frac{1}{m!}\int\dots\int\mathrm{d}y_1\dots\mathrm{d}y_{2m}[12]\dots[2m-1\;\,2m]\begin{pmatrix}1\dots 2m\\ 1\dots 2m\end{pmatrix}.$$

$[1\dots 2m]$ is the Haffnian $\sum[i_1j_1]\dots[i_mj_m]$ with the sum over all possible sets $i_1,\dots,i_m;\,j_1,\dots j_m$ with $i_1<\dots<i_m;\;i_1<j_1;\dots;i_m<j_m$ and all i's and j's distinct and where

$$(10)\qquad [ij]=i\Delta_F(y_i-y_j)\,.$$

(11) $\begin{pmatrix}1\dots 2m\\ 1\dots 2m\end{pmatrix}$ is the determinant

$$\begin{pmatrix}1\dots 2m\\ 1\dots 2m\end{pmatrix}=\sum_{\pi\in\mathcal{S}_{2m}}(\operatorname{sgn}\pi)(1\pi(1))(2\pi(2))\dots(2m\pi(2m))\,,$$

where σ_{2m} is the set of all permutations on $2m$ letters. We have

$$(ij)=\begin{cases} iS_{F\alpha_i\alpha_j}(y_j-y_i) & i\neq j \\ 0 & i=j \end{cases}. \tag{12}$$

(The α's are spinor indices and $\int \mathrm{d}y_i$ is meant to include a sum over $\alpha_i=0,1$.)

The effects of the renormalization are twofold. The change to the aërated propagator just changes (10). The dropping of all diagrams with bubbles (*i.e.* 1 *a*) and 1 *b*)) is equivalent to dropping certain terms in the determinant, namely those terms corresponding to permutations with 2-cycles in their decomposition into disjoint cycles. This follows from the direct analysis of the total $2m$-th order term of (9) as the sum of all Feynman diagrams of order $2m$ [45]. To summarize, we replace (9), (10) and (11) by

$$K_{00}=\sum_{m=0}^{\infty}\left(\frac{-g_0^2}{2}\right)^m\frac{1}{m!}\int\dots\int \mathrm{d}y_1\dots\mathrm{d}y_{2m}[12]'\dots[2m-1\;\;2m]'\begin{pmatrix}1\dots 2m\\1\dots 2m\end{pmatrix}', \tag{9$'$}$$

where

$$[ij]'=i\tilde{\Delta}_F(y_i-y_j) \tag{10$'$}$$

is the aërated propagator (7*b*) and

$$\begin{pmatrix}1\dots 2m\\1\dots 2m\end{pmatrix}'=\sum_{\pi\in Q}(\operatorname{sgn}\pi)(1\pi(1))(2\pi(2))\dots(2m\pi(2m)) \tag{11$'$}$$

with Q the set all permutations on $2m$ letters whose decomposition into disjoint cycles contains *no* 2-cycles.

We call the object defined by (11′) a «bubblessian». The prime consideration of this Section will be to find an analogue of Hadamard's inequality for the bubblessian. We first find an expression for a bubblesian in terms of determinants:

Theorem 1. (the bubblessian formula)

$$\begin{pmatrix}1\dots n\\1\dots n\end{pmatrix}'=\begin{pmatrix}1\dots n\\1\dots n\end{pmatrix}+\sum_{i<j}(ij)(ji)\begin{pmatrix}1\dots\hat{i}\dots\hat{j}\dots n\\1\dots\hat{i}\dots\hat{j}\dots n\end{pmatrix}+ \\ +\dots+\sum_k(i_1j_1)(j_1i_1)\dots(i_kj_k)(j_ki_k)\begin{pmatrix}1\dots\hat{i}_\alpha\dots\hat{j}_\alpha\dots n\\1\dots\hat{i}_\alpha\dots\hat{j}_\alpha\dots n\end{pmatrix}+\dots, \tag{13}$$

(45) A. Buccafurri and G. Fano: *Nuovo Cimento*, 13, 628 (1959).

with $k=0,1,\dots,[n/2]$; here $\sum_k$ is the sum over all sets of distinct numbers $i_1,\dots,i_k,j_1,\dots,j_k$ with $i_1<\dots<i_k$; $i_1<j_1;\dots;i_k<j_k$. The last symbol in (13) means to delete all the i's and j's from the determinant.

Proof. We first consider the term with $k=1$ above. If we expand the determinant, we see that every term is of the form

$$(ij)\;(ji)\;(1\pi(1))\;(2(\pi 2))\dots(n\pi(n))\;(\operatorname{sgn}\pi)$$

where π is a permutation of $1,\dots,\hat{i},\dots,\hat{j},\dots,n$. This is the same as

$$(1\pi'(1))\;(2\pi'(2))\dots(2\pi'(2))\dots(n\pi'(n))(-1)\;(\operatorname{sgn}\pi'),$$

where $\pi'(k)=\pi(k)$, if $k\neq i,j$ and $\pi'(i)=j$, $\pi'(j)=i$. Thus, if a π' on n letters has exactly one 2-cycle its contribution to the determinant above is subtracted off by the $k=1$ term. However, a permutation with two 2-cycles is subtracted twice, etc. A similar analysis of the k-th term of (13) shows it is $(-1)^k$ times those terms in the determinant coming from a permutation with at least k 2-cycles. If π' has exactly m 2-cycles and $m\geq k$, it will appear $\binom{m}{k}$ distinct times [46] in the k-th term of (13). Thus π' will be weighted by a factor of

$$\sum_{k=0}^{m}\binom{m}{k}(-1)^k=(1-1)^m=0$$

if $m>0$. As a result, the right-hand side of (13) will be precisely the sum of all contributions to the determinant coming from permutations with no 2-cycles, *i.e.*, just the bubblessian.

Theorem 2. (Hadamard's inequality for bubblessians)

$$\left|\begin{pmatrix}1\dots n\\1\dots n\end{pmatrix}'\right|\leq\left(\frac{9n}{4}\right)^{n/2}\left(\max_{i,j}|(ij)|\right)^n \tag{14}$$

(compare with formula (1)).

Proof. Without loss of generality, we may assume that $\max|(ij)|=1$. Then, by theorem 1,

$$\left|\begin{pmatrix}1\dots n\\1\dots n\end{pmatrix}'\right|\leq\sum_{k=0}^{[n/2]}\sum_k|(i_1j_1)\dots(j_ki_k)|\left|\begin{pmatrix}1\dots\hat{i}_\alpha\dots\hat{j}_\alpha\dots n\\1\dots\hat{i}_\alpha\dots\hat{j}_\alpha\dots n\end{pmatrix}\right|\leq\sum_{k=0}^{[n/2]}\#(k)(n-2k)^{n/2-k}, \tag{15}$$

(46) We can pick the k 2-cycles $(i_1j_1),\dots,(i_kj_k)$ from the m 2-cycles in this many ways.

where $\#(k)$ is the number of terms in $\sum_k$. We have used the ordinary Hadamard's inequality, (1), to obtain (15).

But $\#(k) = \binom{n}{2k}(2k)!!$, for $\binom{n}{2k}$ is the number of ways of choosing the $2k$ numbers $i_1, \ldots, i_k; j_1, \ldots, j_k$ and $(2k)!! = (2k-1)(2k-3)\ldots(1)$ is the number of ways of pairing the $2n$ numbers off. Thus

$$\left|\begin{pmatrix}1 \ldots n\\ 1 \ldots n\end{pmatrix}'\right| \leqslant \sum_{k=0}^{[n/2]} \frac{n!}{(2k)!\,(n-2k)!}\,\frac{(2k)!}{k!\,2^k}\,(n-2k)^{n/2-k} =$$

$$= \sum_{k=0}^{[n/2]} \frac{n!}{k!\,(n-k)!}\,\frac{1}{2^k}\left[\frac{(n-k)!}{(n-2k)!}\,(n-2k)^{n/2-k}\right] \leqslant$$

$$\leqslant (n)^{n/2}\sum_{k=0}^{[n/2]}\frac{1}{2^k}\binom{n}{k} \leqslant (n)^{n/2}\sum_{k=0}^{n}\frac{1}{2^k}\binom{n}{k} = (n)^{n/2}\left(\frac{3}{2}\right)^n = \left(\frac{9n}{4}\right)^{n/2}.$$

In the above we have used

$$\frac{(n-k)!}{(n-2k)!}\,(n-2k)^{n/2-k} \leqslant \frac{(n-k)!}{(n-2k)!}\,(n)^{n/2-k} =$$

$$= (n)^{n/2}\left(\frac{n-k}{n}\right)\left(\frac{n-k-1}{n}\right)\ldots\left(\frac{n-2k+1}{n}\right) \leqslant n^{n/2}.$$

We remark that the methods used to probe Theorems 1 and 2 are not dependent on the fact that 2-cycles are involved. If one were analyzing a 3-dimensional Yukawa interaction (which is also S.R.), one would have to delete bubbles with two or three Fermi lines; however, an analogous analysis for a modified determinant missing two- and three-cycles would be possible.

The method above also enables us to understand some of Frank's remarks on « point loop renormaliztion » [6,22,23]. PLR is related to the fact that in (11) we set $(jj) = 0$. This is important since $S(0)$ is not defined. However if we regularize the theory $S(0)$ becomes finite and nonzero. Frank's « augmented S-matrix » includes these diagonal terms [47] and PLR is the removal of them. In [6] and [22] FRANK has two distinct arguments to show that if one can majorize the augumented S-matrix, one can majorize the ordinary S-matrix. We can produce a third argument by using the analogue of theorem 1 with 1-cycles. Eliminating the 1-cycles is equivalent to setting the diagonal terms equal to 0. Theorem 1 with 1-cycles then lets us express the ordinary S-matrix element

[47] Including these corresponds to failing to Wick order the interaction Lagrangian.

in terms of the augmented, in fact if

$$\begin{vmatrix} S(0) & S(x_1-x_2) & \ldots & S(x_1-x_n)\\ S(x_2-x_1) & S(0) & \ldots & S(x_2-x_n)\\ \vdots & \vdots & & \vdots\\ S(x_n-x_1) & S(x_n-x_2) & \ldots & S(0)\end{vmatrix} \leqslant (S(0))^n,$$

then the analogue to Theorem 1 shows

$$\begin{vmatrix} 0 & S(x_1-x_2) & \ldots & S(x_1-x_n)\\ S(x_2-x_1) & 0 & \ldots & S(x_2-x_n)\\ \vdots & \vdots & & \vdots\\ S(x_n-x_1) & S(x_n-x_2) & \ldots & 0\end{vmatrix} \leqslant (2S(0))^n.$$

This is just eq. (1) of [23].

4. – The proof of convergence for the regularized series.

The next step is to regularize the theory. This is done by introducing two cut-offs

A) A space-time cut-off.

B) An ultra-violet cut-off.

As we have remarked before, cut-off A) is essential if eq. (8) is to make sense. On the other hand, the series in (8) after renormalization makes sense without cut-off B), but we will need B) to prove convergence.

The effects of A) are twofold. First, the integrals in (9′) now only extend over a finite region. Secondly, we replace the x-space form of the propagator with a Fourier sum rather than transform (see eq. (7*c*)).

Cut-off B) can be realized in two « natural » ways: $B1$) we can cut off the propagators to be uniformly bounded in x-space. $B2$) We can restrict all Fourier sums to a finite number of modes. Lemma 1 shows that condition $B2$) will imply $B1$). In the classic arguments of CAIANIELLO [17,20] for the *unrenormalized* series $B1$) implies convergence in a neighborhood of zero coupling and $B2$) implies entirety of the kernels K_{00} and $K_{\beta;\alpha}\left(\begin{matrix}x\\ y\end{matrix}\middle|t\right)$. For the case of the renormalized series, we will show that $B1$) still implies convergence in a neighborhood. However, the standard arguments for entirety from $B2$) do not *appear* to carry over to this case.

The proof of convergence is easy. If $B1$) is true so that

$$|[ij]'| < B; \ |(ij)| < F$$

then the m-th term of (9') is bounded by

$$\left(\frac{|g_0|^2}{2}\right)^m \frac{1}{m!} 2^{2m} (VT)^{2m} B^m \left(\frac{9m}{2}\right)^m F^{2m} .$$

The factor of $(9m/2)^m$ comes from using Theorem 2 on the bubblessian; VT is the volume of the space-time box and the factor of 2^{2m} comes from the sum over spinor indices. If we let $x = 9(VT)^2 BF^2$, then (9') is majorized by

$$\sum_{m=0}^{\infty} \frac{m^m}{m!} x^m |g_0|^{2m} .$$

This series converges for $|g_0| < (ex)^{-2}$ and thus (9') converges in a finite circle.

Let us sketch briefly the modification necessary to prove convergence for the $K_{\beta;\alpha}\begin{pmatrix} x \\ y \end{pmatrix} t \Big)$. The renormalization does not eliminate all permutations with 2-cycles but only those with 2-cycles between « internal indices ». Theorem 1 will hold for this new object if the sums over i and j are only over those « internal indices ». The remaining part of the proof now holds with obvious modifications.

As we have already remarked, the proof of entirety does not go through. This is due to the last term in the bubblessian formula (13). This last term contains $(2m)!/m!2^m$ factors (for $n = 2m$) each of order F^{2m}. For large m, $2m!/m!m!2^m \sim 2^m$ and thus this term alone is enough to prevent entirety unless we can better control it. However, since there are no determinants we cannot use Hadamard's inequality or the improved estimates of Buccafurri-Caianiello [20]. One might think of trying to prove these improved estimates directly for bubblessians; but their proof depends heavily on the symmetry of determinants [48], a symmetry not shared by bubblessians. The last term of (13) also prevents one from using the argument of YENNIE-GARTENHAUS [18] to obtain entirety [49]. It should be remarked however that there are cancellations between different terms in (13) and these cancellations have been ignored. If one could control them, then entirety might be provable.

[48] Through their use of Arnaldi's theorem.

[49] Although the Yennie-Gartenhaus argument and Theorem 1 will provide an alternate proof of convergence in a neighborhood of zero coupling.

5. – Discussion of results.

Let us briefly recapitulate. We showed for a two-dimensional scalar Yukawa interaction that renormalization could be accomplished by replacing the boson propagator with a modified form (7) and by replacing the determinant in Caianello's kernels by a modified object called a bubblessian. We then showed that a modification of Hadamard's inequality holds for bubblessians (Theorem 2) and used this to prove convergence of the *regularized*, renormalized series by a method analogous to Caianiello's 1956 approach [17].

The arguments used will hold just as well for a two-dimensional pseudoscalar Yukawa interaction or for two-dimensional Q.E.D. [50]—the only difference is that π_f will not have as simple a closed form as (5). The discussion following Theorem 2, makes it fairly evident that the convergence argument will go through for any S.R. theory with only $\bar{\psi}\psi\varphi$ interactions.

The significance of this result for the question of convergence of the unregularized series is unclear. On the one hand, the rearrangements of renormalization do not destroy convergence, on the other hand, the rearrangements *may* destroy entirety [51]. It can be argued [52] and with much merit that this convergence is to be expected even while the actual series diverges: there is an unphysical model of GUERRA and MARINARO [24] which requires renormalization and which is exactly solvable. For this model the actual answer is nonanalytic at zero coupling while the regularized renormalized perturbation series converges. However, this model involves a nonlocal field spread over all space-time. Thus in a *very rough* way, this model is analogous to a static external field problem, while local quantum theories are analogous to the compact support problem. If there is any validly in this analogy, it is dangerous to drawn conclusions from the Guerra-Marinaro model for the general case [53].

Then, the question of convergence of nonregularized fermion theories is about as unsettled as it was before this discussion. But, to end on an optomistic note, let me reiterate my belief that the problem can be settled by study of one of the S.R. theories.

* * *

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[50] Because we are in a box, infra-red divergences will not appear.

[51] Since our radius of convergence $(xe)^{-2}$ goes to zero as the ultra-violet cut-off goes away, if this is an estimate of the real radius of convergence, convergence is doomed.

[52] E. R. CAIANIELLO: private communication.

[53] However, the Guerra-Marinaro model does indicate that any proof of convergence will have to use the rapid fall-off of the boson Feynman propagators in spacelike directions.