SOME REMARKABLE EXAMPLES IN EIGENVALUE PERTURBATION THEORY ☆

I.W. HERBST
Department of Mathematics, University of Virginia, Charlottesville, VA, USA

and

B. SIMON
Departments of Mathematics and Physics, Princeton University, Princeton, NJ, USA

Received 21 June 1978

We discuss the ground state energy, $E(g)$, of some anharmonic oscillators $p^2 + x^2 - 1 + P(x, g)$, where $P(x, g)$ is a polynomial in x and g. Included is an example with a convergent perturbation series converging to the wrong answer and counterexamples to Dyson's argument on instability implying the divergence of perturbation theory and to the assertion that terms lower order in x to the same order in g do not effect the asymptotics of the perturbation coefficients a_n as $n \to \infty$.

In this note, we want to remark on the behavior of the ground state energy, $E^{(i)}(g)$, of two ‡1 model hamiltonians,

$$H^{(1)}(g) = p^2 - 1 + g^{-2}((gx+1)^2 - 1)^2 - 2gx \ , \qquad (1)$$

$$H^{(2)}(g) = p^2 - 1 + x^2(g^2x^2 + 1)^2 - 3g^2x^2 \ . \qquad (2)$$

Note that as $g \downarrow 0$ these both become harmonic oscillators since expanding the polynomial gives

$$H^{(1)}(g) = p^2 + x^2 - 1 + g^2x^4 + 2gx^3 - 2gx \ , \qquad (1')$$

$$H^{(2)}(g) = p^2 + x^2 - 1 + g^4x^6 + 2g^2x^4 - 3g^2x^2 \ . \qquad (2')$$

We are interested in the behavior of $E(g)$ at $g = 0$ and the associated Rayleigh–Schrödinger perturbation series ‡2 $\Sigma a_n^{(i)} g^{2n}$. The results which we prove below are very simple and unexpected:

(I) $0 < E^{(1)}(g) < C\exp(-Dg^{-2})$;

$C, D > 0, \quad a_n^{(1)} = 0$,

(II) $E^{(2)}(g) \equiv 0, \quad a_n^{(2)} = 0$.

These results are in striking contrast to many of the heuristic arguments and expectations of eigenvalue perturbation theory; in particular:

(1) For $H^{(1)}$, the Rayleigh–Schrödinger series is convergent but to the wrong answer since $E^{(1)}(g) > 0$.

(2) There is a celebrated heuristic argument of Dyson [3] ‡3 on the divergence of the perturbation series in QED: he notes that for $e^2 < 0$, there is no decent physical theory since the vacuum is unstable under breakup into a large number of pairs. Since power series converge in circles and there is no reasonable answer for $e^2 < 0$, one expects divergence

☆ Research partially supported by USNSF Grant MPS-75-11864.

‡1 We express the hamiltonians in such a way that $H = p^2 + g^{-2}f(gx) + h(gx)$. We are indebted to S.B. Treiman for suggesting example (2) to us when we described example (1) to him.

‡2 Explicitly, one obtains the coefficients by using the usual series with $V = H(g) - (p^2 + x^2 - 1)$ and then collecting all terms of a given order in g.

‡3 The argument has also been used to predict divergence of the series for the simplest anharmonic oscillator $p^2 + x^2 + g^2x^2$, see Gottfried [2]. A rigorous proof of divergence in this case is contained in Bender and Wu [2], who count Feynman diagrams. A different proof based on analiticity properties is found in ref. [4].

for all e^2. In the above examples the systems are formally unstable [‡4] for suitable g.

(3) Our analysis below shows that while the series for the ground state energy $E^{(2)}(g)$ is convergent, the series for the corresponding eigenvector, $\Omega^{(2)}(g)$, is divergent at least in norm sense [‡5].

(4) Lipatov and then the Saclay group have recently [5 ‡6,7] made a rather beautiful analysis, based on path space saddle points, of the asymptotics of the Rayleigh–Schrödinger and Feynman series, a_n, for anharmonic oscillators and related quantum field theories. In discussing the effects of renormalization, it is argued that terms which are lower order in the field to the same order in g will not affect the asymptotics so that for the formal study of asymptotics, one can ignore the effects of renormalization. This has been questioned for renormalizable theories but not for super renormalizable theories. The above examples show that lower order terms can indeed affect the asymptotics of the a_n's in super renormalizable theories, for the series obtained by dropping the last terms in hamiltonians (1) and (2) surely [‡7] grow at $n!$. Since the significance of our examples to the Lipatov analysis is the most important consequence of our examples, we discuss it further below.

Next, we turn to the proof of the assertions (I) and (II). One first needs to prove that the series $\Sigma a_n^{(i)} g^{2n}$ is asymptotic to the function $E^{(i)}(g)$. For $H^{(2)}$, which is a single-welled oscillator, this follows from the most standard analysis (see, e.g., the paper of Simon [4]). Since the potential in $H^{(1)}$ has a second minimum near $x = -g^{-1}$, where $V(x) \sim 1$, the analysis is not quite standard; however the proof of the asymptotic nature of the series for the double well in ref. [8] carries over with no change. Thus, the assertions about $E^{(i)}$ imply those about $a^{(i)}$.

The assertions about $E^{(i)}(g)$ depend on the fact that

$$H^{(i)}(g) = [A^{(i)}(g)]^* A^{(i)}(g) ,$$

where

$$A^{(1)}(g) = \mathrm{d}/\mathrm{d}x + x + gx^2 , \tag{3}$$

$$A^{(2)}(g) = \mathrm{d}/\mathrm{d}x + x + gx^3 . \tag{4}$$

This shows that $E^{(i)}(g) \geqslant 0$ and $E^{(i)}(g) = 0$ if and only if the solution $\eta^{(i)}(g, x)$ of $A^{(i)}(g)\, \eta^{(i)}(g, x) = 0$ is square integrable. In case (2), $\eta^{(2)}(g, x) = \exp(-\frac{1}{2}x^2 - \frac{1}{4}g^2x^4)$ is square integrable showing that $E^{(2)}(g) = 0$; $\eta^{(2)}$ is the ground state. The solution $\eta^{(1)}$ which we normalize by:

$$\eta^{(1)}(g, x) = \exp(-\tfrac{1}{2}x^2 - \tfrac{1}{3}gx^3 + \tfrac{1}{6}g^{-2}) , \tag{5}$$

is not square integrable at $-\infty$, so $E^{(1)}(g) > 0$. However, $\eta^{(1)}(g, x)$ obeys

$$\eta^{(1)}(g, x = -\frac{1}{g}) = 1 , \quad \frac{\mathrm{d}}{\mathrm{d}x}\eta^{(1)}(g, x = -\frac{1}{g}) = 0 . \tag{6}$$

Pick a function φ which is C^∞ with compact support so that $\varphi(0) = 1$, $\varphi'(0) = 0$ and let $\psi_g(x)$ be the trial function

$$\psi_g(x) = \eta^{(1)}(g, x) , \qquad x \geqslant -1/g ,$$

$$= \varphi(x + 1/g) , \qquad x \leqslant -1/g .$$

Then $(\psi, H^{(1)}(g)\psi) = \|A^{(1)}(g)\psi\|^2 = O(g^2)$ but $(\psi, \psi) = O(\exp(+\frac{1}{3}g^{-2})$, so the bound $E^{(1)}(g) \leqslant C\exp(-Dg^{-2})$ follows from the variational principle. This concludes the proofs of assertions (I) and (II).

To return to the discussion initiated in point (4) above, we begin by noting that the fact that the potential in (2) is a single well is somewhat deceptive. For under the change $g \to ig$, $H^{(1)}$ becomes:

$$H^{(3)} = p^2 + x^2 - 1 + g^4x^6 - 2g^2x^4 + 3g^2x^2$$

$$= A^{(3)}(g)^* A^{(3)}(g) ,$$

with $A^{(3)}(g) = \mathrm{d}/\mathrm{d}x + x - g^2x^3$. Again, as in example 1, $\eta^{(3)}(x) = \exp(-\frac{1}{2}x^2 + \frac{1}{4}g^2x^4 + \frac{1}{4}g^{-2})$ is not square integrable, but since $\eta^{(3)}(\pm 1/g) = 1$, $(\mathrm{d}/\mathrm{d}x)\eta^{(3)}(\pm 1/g) = 0$, we have that $0 < E^{(3)}(g) < D' \exp(-cg^{-2})$. The potential in $H^{(3)}$ has three wells at $x = 0$, $V(x) = 0$, and at $x \sim \pm 1/g$, $V(x) \sim 3$. If the lower order term (i.e. the last term in each H) is

‡4 When g^2 is negative, the potential is no longer real. The instability is indicated by the fact that the numerical range of $H(g)$ (i.e. the set of expectation values $(\psi, H(g)\psi)$ with $\|\psi\| = 1$) is all complex numbers.

‡5 However, one can normalize $\Omega^{(2)}$ so that the power series for x *fixed*, $\Omega^{(2)}(g, x)$ does converge.

‡6 Earlier, Bender and Wu [6] have computed the asymptotics of the simple x^4 oscillator.

‡7 In ref. [7], the a_n's resulting when the last terms in eqs. (1) and (2) are dropped are computed within the Lipatov framework obtaining $n!$ growth. Moreover, the first 78 terms for eq. (1) with the $-2gx$ dropped are explicitly computed and the numbers confirm the $n!$ growth. However, we do not have a rigorous proof of $n!$ growth.

dropped, we have degenerate wells of the type discussed in ref. [7], i.e. multiple saddle points. Since the Lipatov method considers all complex g, even example 2 has a multiple saddle point. Moreover, in each case, there is a degeneracy of expected states: In case (1), the limiting eigenvalues for the well at $x = 0$ are $2n$ and for the well at $x = -g^{-1}$, they are $2n + 2$. In case (3), the limiting eigenvalues at $x = 0$ are $2n$, at $x = \pm g^{-1}$, they are $4n + 4$. These features suggest that asymptotic degeneracy of excited states might be necessary for the kind of phenomena discussed here but this is not necessary. For example, if $H(g) = A(g)^*A(g)$ with $A(g) = \mathrm{d}/\mathrm{d}x + x(xg - 1) \times (xg + \pi)$, then $E(g) = 0$; the asymptotic eigenvalues are $2\pi n$, $2(\pi + 1)(n + 1)$ and $2\pi(\pi + 1)(n + 1)$ and no state is asymptotically degenerate. However, all the examples that one can construct of the form $H(g) = A(g)^*A(g)$ with $A(g) = \mathrm{d}/\mathrm{d}x + xQ(xg)$, Q a polynomial, have the property that $H(g) = -\mathrm{d}^2/\mathrm{d}x^2 + g^{-2}V(gx) + W(gx)$ with V having a double minimum for suitable arg g, i.e. degenerate saddle points. This suggests it *might* be possible to ignore lower order terms unless these are multiple saddle points. Indeed Brezin and Zinn-Justin [11] [‡8] have analyzed our model (1) and find that in the Lipatov analysis if one properly treats the second zero eigenvalue present in the gaussian approximation, then the leading $3^K K!$ which is present when the $2gx$ term is dropped is cancelled by oscillations along this zero mode direction.

We should mention another "application" of the upper bound $E(g) \leqslant C\exp(-Dg^{-2})$; indeed, it was this application that led us to consider these hamiltonians originally in connection with our work [8,10] on the Stark effect in atoms. In this connection, one of us discovered [9] that the operator $p^2 + \epsilon x$ has no spectrum (1) if ϵ is not real. However, it appeared that a "memory" of the spectrum shows up in that $\|(p^2 + \epsilon x - z)^{-1}\|$ diverges as $\epsilon \to 0$ so long as z is to the right of the line arg z = arg ϵ. We came across example (1) in studying the rate of the divergence of $(p^2 + \epsilon x - z)^{-1}$. Clearly, one need only consider $(\mathrm{i}\epsilon^{-1}p^2 + \mathrm{i}x - w)^{-1}$, and then by utilizing the invariances, $x \to x - a$ (conjugation with $\mathrm{e}^{\mathrm{i}pa}$); $x \to \alpha p^2 + x$, $p \to p$ (conjugation with $\mathrm{e}^{\mathrm{i}\alpha p^3/3}$); and $x \to \lambda x$, $p \to \lambda^{-1}p$ (conjugation with $\mathrm{e}^{\mathrm{i}D}$, $D = (\ln\lambda)\frac{1}{2}[xp + px]$), one need only consider $(gp^2 + \mathrm{i}x - 1/g)^{-1}$ as $g \to 0$. Finally, taking $p \to (p + 1/g)$, $x \to x$, and then $x \to -p, p \to x$, we are left to consideration of $(\mathrm{d}/\mathrm{d}x + x + gx^2)^{-1}$. Thus the divergence of $(p^2 + \epsilon x - z)^{-1}$ as $\epsilon \to 0$ is directly related to the lowest eigenvalue of $H^{(1)}(g)$! The net result [‡9] is that $(p^2 + \epsilon x - z)^{-1}$ diverges at least as $\exp(+a/\epsilon)$ as $\epsilon \to 0$!

Finally, we cannot resist remarking on one feature of the hamiltonian (1) which leads to rather speculative possibilities. Namely, (1) is a combination of a tunnelling hamiltonian together with a small explicit symmetry breaking; such combinations are believed to occur in the real world. Perhaps the rather striking cancellations also occur in a quantity of greater physical interest than the ground state energy of an anharmonic oscillator.

‡8 We are grateful to them for their comments and interest.

‡9 One does not need to make all the above changes to obtain this result. One need only translate x to make z real and then try a trial function $\varphi(p) = \exp(-\mathrm{i}\epsilon^{-1}(\frac{1}{3}p^3 - zp))$ cutoff about one of the two points $p = \pm\sqrt{z}$ in $\|(p^2 + \epsilon x - z)\varphi(p)\|$; see ref. [9].

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