Commun. Math. Phys. 80, 181–216 (1981)

Communications in Mathematical Physics

Dilation Analyticity in Constant Electric Field

II. N-Body Problem, Borel Summability

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Abstract. We extend the analysis of Paper I from two body dilation analytic systems in constant electric field to N-body systems in constant electric field. Particular attention is paid to what happens to isolated eigenvalues of an atomic or molecular system in zero field when the field is turned on. We prove that the corresponding eigenvalue of the complex scaled Hamiltonian is stable and becomes a resonance. We study analyticity properties of the levels as a function of the field and also Borel summability.

1. Introduction

Our goals in this paper are to extend the formalism developed by Herbst [16] (which describes complex scaling in the presence of constant electric field) from two body systems to N-body systems, to recover the beautiful results of Graffi and Grecchi [13] on Borel Summability of the hydrogen Stark problem within this framework and to extend these summability results to multielectron atoms. Some of our results were announced in [17]. Subsequently, Graffi and Grecchi [37] developed a different formalism which allows a discussion of certain N-body systems in electric field. Their analysis appears to require that all particles have charges with the same sign. Moreover, their method does not reduce to ordinary complex scaling when the electric field is absent. However, since they need only treat *strictly* sectorial operators, their method has some technical advantage over ours.

While it is probable that with some extra effort, we could handle quadratic form perturbations and non-local potentials, we will use the operator class of Aguilar–Balslev-Combes [2, 8] and restrict to local potentials. As usual, let $\mathfrak{H}=L^2(\mathbb{R}^\nu)$, $t=-\Delta$, $(u(\theta)f)(r)=e^{\nu\theta/2}f(e^\theta r)$ for θ real and $\mathfrak{H}_{+1}=D(t)$ with the graph norm. Notice that $u(\theta)$ is bounded from $\mathfrak{H}_{+1}$ to itself.

* Research partially supported by USNSF grant MCS 78-00101
** Research partially supported by USNSF grant MCS 78-01885

0010-3616/81/0080/0181/$07.20

Definition. Fix $\phi > 0$. A self-adjoint multiplication operator, V, on $L^2(\mathbb{R}^\nu)$ is said to lie in C_ϕ^M if and only if:

(i) $D(V) \supset D(t)$, so V can be viewed as an operator in $\mathcal{L}(\mathfrak{H}_{+1}, \mathfrak{H})$.

(ii) As an operator in $\mathcal{L}(\mathfrak{H}_{+1}, \mathfrak{H})$, V is compact, i.e., $V(t+1)^{-1}$ is compact from $\mathfrak{H}$ to $\mathfrak{H}$.

(iii) $V(\theta) \equiv u(\theta) V u(\theta)^{-1}$ as an $\mathcal{L}(\mathfrak{H}_{+1}, \mathfrak{H})$-valued function has an analytic continuation from $\theta \in (-\infty, \infty)$ to $\{\theta \| \operatorname{Im} \theta| < \phi\}$.

Aguilar and Combes [2] developed the complex scaling theory for $-\Delta + V$ with $V \in C_\phi^M$ and Balslev and Combes [8] the corresponding multibody theory (neither were restricted to multiplication operators). Let $\hat{e}$ be a fixed unit vector in $\mathbb{R}^\nu$. The Aguilar–Balslev–Combes theory does not extend to $-\Delta + V(\mathbf{x}) + f\hat{e}\cdot\mathbf{x}$ and, at first sight, it appears hopeless to try to extend complex scaling to this setting because of the singular nature of $\hat{e}\cdot\mathbf{x}$. However, Reinhardt [24] boldly tried calculations for this problem and this motivated [16]. While we will freely use technical lemmas from [16], let us summarize its results to put those in this paper into perspective. Fix $V \in C_\phi^M$; $f \neq 0$. For θ with $\operatorname{Im}\theta \in [0, \pi/3)$ and $\operatorname{Im}\theta < \phi$, let

$$h_0(\theta) = -e^{-2\theta}\Delta + f e^{\theta} \hat{e}\cdot\mathbf{x}$$
$$h(\theta) = h_0(\theta) + V(\theta).$$

Then:

(1) For $\operatorname{Im}\theta > 0$, $h_0(\theta)$ and $h(\theta)$ are closed operators on $D(t) \cap D(\hat{e}\cdot\mathbf{x})$ and $ie^{-\theta} h_0(\theta)$, $ie^{-\theta} h(\theta)$ are generators respectively of contraction and exponentially bounded semigroups. For $\theta = 0$, the operators are essentially self-adjoint on $D(t) \cap D(\hat{e}\cdot\mathbf{x})$.

(2) For $\operatorname{Im}\theta > 0$, $\sigma(h_0(\theta))$ is empty and $\sigma(h(\theta))$ is purely discrete, i.e., isolated eigenvalues of finite algebraic and geometric multiplicity. In this region, $\sigma(h(\theta))$ is θ independent and in the lower half-plane.

(3) For fixed z with $\operatorname{Im} z > 0$,

$$\operatorname*{s\text{-}lim}_{\operatorname{Im}\theta \downarrow 0;\, \theta \to 0} (h(\theta) - z)^{-1} = (h(\theta = 0) - z)^{-1}.$$

(4) If $z < 0$ is an eigenvalue of $t + V$ of multiplicity d, then for all f sufficiently small and θ with $\operatorname{Im}\theta \neq 0$, $h(\theta)$ has at most d eigenvalues near z and their combined multiplicities is exactly d.

All these results are from [16] except for two. First, it is not noted there that $ie^{-\theta} h(\theta)$ generates an exponentially bounded semigroup. This follows from the analysis of $ie^{-\theta} h_0(\theta)$ in [16], the quadratic estimates there [16] and the theory of perturbation of semigroups [22]. The self-adjointness of $h(\theta = 0)$ follows from ideas of Faris and Lavine [12]. Secondly, (3) was only proved in [16] under an extra hypothesis. Here is a general proof of (3): By the result of Faris and Lavine [12], $(h(\theta = 0) - z)\,[\mathcal{S}]$ is dense. Since for $\eta \in \mathcal{S}$, $[h(\theta) - h(\theta = 0)]\eta \to 0$ as $\theta \to 0$ we need only show that $\|(h(\theta) - z)^{-1}\|$ is uniformly bounded if $\operatorname{Im} z > 0$, $\operatorname{Im}\theta > 0$, and $|\theta|$ is sufficiently small. We will show that there is an $E_0 > 0$ so that for $\operatorname{Im}\theta > 0$ and sufficiently small, the numerical range of $h(\theta)$ lies below the line $\{-E_0 + e^{\theta} s : s \in \mathbb{R}\}$ and by sectoriality considerations this will complete the proof. By the unitarity of $u(\theta)$ for θ real it thus suffices to show

$$\operatorname{Im}(\eta, (h(i\alpha) + E_0) e^{-i\alpha} \eta) \leqq 0 \tag{1.1}$$

for $\alpha>0$ sufficiently small. Choose $\alpha_0>0$ with $\alpha_0<\mathrm{Min}\{\phi,\pi/3\}$ so that $\sin\alpha \geqq \alpha/2$ and $\sin 3\alpha \geqq \alpha/2$ for $\alpha\in[0,\alpha_0]$. Since $e^{-\theta}V(\theta)\,(t+1)^{-1}$ is compact and analytic in $|\mathrm{Im}\,\theta|<\phi$ we can choose $\|V(\theta)e^{-\theta}(t+E)^{-1}\|$ to be as small as we like for $\mathrm{Im}\,\theta\in[-\alpha_0,\alpha_0]$, $|\mathrm{Re}\,\theta|\leqq 1$ by taking E large enough. For E_0 large enough we can thus obtain the bound

$$\left\|\frac{d}{d\alpha}V(i\alpha)e^{-i\alpha}(t+E_0)^{-1}\right\|\leqq 1/2\,;\qquad \alpha\in[-\alpha_0,\alpha_0] \tag{1.2}$$

by a Cauchy estimate. (1.2) implies (by integration)

$$\|(V(i\alpha)e^{-i\alpha}-V(0))\,(t+E_0)^{-1}\|\leqq|\alpha|/2\,;\qquad \alpha\in[-\alpha_0,\alpha_0] \tag{1.3}$$

and by interpolation

$$\|(t+E_0)^{-1/2}(V(i\alpha)e^{-i\alpha}-V(0))\,(t+E_0)^{-1/2}\|\leqq\alpha/2\,;\qquad \alpha\in[0,\alpha_0]\,. \tag{1.4}$$

Thus

$$\mathrm{Im}(\eta,V(i\alpha)e^{-i\alpha}\eta)=\mathrm{Im}(\eta,(V(i\alpha)e^{-i\alpha}-V(0))\eta)\leqq\frac{\alpha}{2}(\eta,(t+E_0)\eta)\,. \tag{1.5}$$

Hence for $\alpha\in[0,\alpha_0]$, (1.5) gives

$$\mathrm{Im}(\eta,(h(i\alpha)e^{-i\alpha}+E_0e^{-i\alpha})\eta)\leqq\left(\eta,\left\{-t\sin 3\alpha+\frac{\alpha}{2}(t+E_0)-E_0\sin\alpha\right\}\eta\right)$$

which is $\leqq 0$ by our choice of α_0.

We emphasize the condition above that $\mathrm{Im}\,\theta<\pi/3$, since at $\mathrm{Im}\,\theta=\pi/3$ the spectrum is again continuous on account of $h_0(i\pi/3)$ being unitarily equivalent (under $x\to -x$) to $-e^{i\pi/3}h_0(\theta=0)$. Also the convergence in (3) is definitely not norm convergence, nor does it hold for z with $0>\mathrm{Arg}\,z>-\pi$.

In this paper, we wish to consider operators on $L^2(\mathbb{R}^{N\nu})$ of the form:

$$-\sum_{i=1}^{N}(2m_i)^{-1}\Delta_i+\sum_{i<j}V_{ij}(\mathbf{r}_i-\mathbf{r}_j)+\sum_{i=1}^{N}V_i(\mathbf{r}_i)+f\sum_{i=1}^{N}q_i\hat{e}\cdot\mathbf{r}_i$$

or the slightly more general operators that arise from removing the center of mass motion from general N-body systems. The extension of properties (1)–(3) to this setting will be fairly easy: the only real restriction other than the obvious one that all potentials lie in some C_ϕ^M will be that certain requirements on the q_i's and m_i's are necessary for (2) to extend – for example, in the above case where there is an infinite mass particle we need to know that there exists no non-trivial breakup into two or more clusters each of which is neutral. We will accomplish this in Sect. 2.

The extension of property (4) will not be so easy for the following reason. In [16], the resolvent equation $(h(\theta)-z)^{-1}=(h_0(\theta)-z)^{-1}\cdot(1+V(\theta)\,(h_0(\theta)-z)^{-1})^{-1}$ was used extensively. A key role was played by the fact that the numerical range of $h_0(\theta)$ was precisely $\{(\Sigma+y)e^{-2\theta}+xe^{\theta}:y\geqq 0,\ x\in\mathbb{R}\}\equiv K$ with $\Sigma=\inf\mathrm{spec}(h_0(\theta=0,$ $f=0))$, i.e., zero. This equality for Σ is true because for $f=0$, $h_0(\theta)$ is normal so that its numerical range is the convex hull of its spectrum. For N-body systems, the natural replacement of the resolvent equation is a Weinberg–van Winter or other

N-body equation with compact kernel. This will involve operators $H_D(\theta)$ built out of subsystems and we will need a bound on $(H_D(\theta)-z)^{-1}$ uniform in small f for $z \notin K$ where K is the above set with Σ now the lowest threshold of $H(f=0)$. Because $H_D(\theta, f=0)$ is *not* normal, the numerical range of $H_D(\theta)$ will not lie in K in general, even though K contains the convex hull of the spectrum of $H_D(\theta, f=0)$, so the bound on $(H_D(\theta)-z)^{-1}$ will be more subtle to obtain. In Sect. 3, we will show how to obtain a bound of the form:

$$\|(H_D(\theta, f)-z)^{-1}\| \leqq C_\varepsilon[\operatorname{dist}(z, K)-\varepsilon]^{-1} \tag{1.6}$$

for all z with $\operatorname{dist}(z, K) > \varepsilon$. Here C_ε is an ε dependent constant and (1.6) will only hold for $|f| < F(\theta, \varepsilon)$. (1.6) will follow from an estimate

$$\|\exp[-t(ie^{-\theta} H_D(\theta, f)]\| \leqq \tilde{C}_\varepsilon \exp\left(\left[\operatorname{Re}(ie^{-\theta}\Sigma)+\frac{\varepsilon}{2}\right] t\right). \tag{1.7}$$

(1.7) will be proven by developing an equation for the semigroup analogous to the Weinberg–van Winter equation; indeed it will just be the inverse Laplace transform of that equation.

Once we have the estimate in (1.6) the stability method of Avron et al. [4] exploited in [16] will yield stability in the N-body case for eigenvalues below the lowest threshold. Indeed, we will be able to obtain a resonance eigenvalue in a sector of the complex plane. Specifically, so long as all potentials lie in C_ϕ^M with ϕ sufficiently large (e.g., Coulomb potentials), we will know that for any non-degenerate eigenvalue, E_0, of $H(\theta=0, f=0)$, there will be an eigenvalue, $E(f)$, of $H(\theta, f)$ for f small, real and positive and any 0 with $0 < \operatorname{Im}\theta < \pi/3$. Then $E(f)$ is analytic in regions of the form $\{f | 0 < |f| < R_\delta, -\pi/2+\delta < \arg f < 3\pi/2-\delta\}$. We will also obtain analogous results for some degenerate eigenvalues E_0. In particular, this will allow us to recover the Graffi–Grecchi result [13] on Borel summability of the Stark eigenvalue in Hydrogen and a similar result in atoms and molecules. We also relate the width of the resonance to the growth of the coefficients in the perturbation series. These results on analyticity of eigenvalues may be found in Sect. 4. The necessary estimates to complete the proof of Borel summability can be found in Appendix A. In Sect. 5, we describe falloff properties of eigenfunctions.

It is known that for hydrogen in a non-zero (real) electric field, the (unscaled) Hamiltonian has no eigenvalues, i.e., imaginary parts of resonance energies are non-zero. This is a result of Titchmarsh [32]; see also [1, 3]. So far as we can determine, their proofs do not extend to multielectron atoms. In [17], we announced a result to the effect that discrete eigenvalues of atoms turn into resonances whose imaginary part is necessarily non-zero. Unfortunately our method of proving this last fact ran into certain technical difficulties in the multiparticle case. Since we feel the basic scheme which relies on ideas of Balslev [7] and Simon [25] may be sound and since the difficulties illustrate our ignorance of certain questions in operator theory, we describe the method for two bodies and some of the problems in extending to N-bodies in Sect. 6.

In Appendix C, we describe some basic estimates which we found and which one of us has used elsewhere [28].

We refer the reader to [43] where the subject of Schrödinger operators with electric fields is reviewed. The Weinberg–van Winter semigroup analysis of this paper is replaced there by a simpler techniqe.

It is a pleasure to thank S. Graffi, V. Grecchi, E. Harrell, and W. Reinhardt for valuable discussions.

2. Basic Spectral Analysis

We want to consider $N+1$ particles in ν dimensions with masses $m_1,\dots,m_{N+1}$ and charges $q_1,\dots,q_{N+1}$. The basic Hamiltonian in a constant electric field $-|f|\hat{e}$ ($\hat{e}$ a unit vector in $\mathbb{R}^\nu$) is thus:

$$\tilde{H}_0(|f|)=-\sum_{j=1}^{N+1}(2m_j)^{-1}\Delta_j+|f|\hat{e}\cdot\left(\sum_{j=1}^{N+1}q_j\mathbf{r}_j\right). \tag{2.1}$$

We begin by discussing removal of the center of mass motion, a subject already explained briefly in [5]. As usual we define

$$\mathbf{R}=M^{-1}\sum_{j=1}^{N+1}m_j\mathbf{r}_j$$

$$M=\sum_{j=1}^{N+1}m_j$$

and let ζ be a generic symbol for a linear function of the $\mathbf{r}_i-\mathbf{r}_j$. Also as usual the basic Hilbert space, $\tilde{\mathfrak{H}}=L^2(\mathbb{R}^{\nu(N+1)})$ is factored as

$$\tilde{\mathfrak{H}}=\mathfrak{H}_{CM}\otimes\mathfrak{H}$$

with $\mathfrak{H}_{CM}\cong L^2(\mathbb{R}^\nu)$ functions of $\mathbf{R}$ and $\mathfrak{H}\cong L^2(\mathbb{R}^{\nu N})$ functions of the ζ's. $\tilde{H}_0(|f|)$ factors as:

$$\tilde{H}_0(|f|)=H_0^{CM}\otimes 1+1\otimes H_0(|f|),$$

where

$$H_0^{CM}=-(2M)^{-1}\Delta_{\mathbf{R}}+|f|\hat{e}\cdot(Q\mathbf{R}) \tag{2.2}$$

with

$$Q=\sum_{j=1}^{N+1}q_j.$$

The exact formula for $H_0(|f|)$ will not concern us.

For later purposes, we note the condition that all the electric field terms are absorbed in H_0^{CM}.

Proposition 2.1. *$H_0(|f|)$ is independent of $|f|$ if and only if $q_i/m_i=Q/M$ for all i.*

Proof. One clearly has independence if and only if

$$Q\mathbf{R}=\sum_{j=1}^{N+1}q_j\mathbf{r}_j.$$

The formula for **R** completes the proof. □

The operator on $L^2(\mathbb{R}^{N\nu})$ obtained from (2.1) by suppressing Δ_{N+1} and $q_{N+1}\mathbf{r}_{N+1}$ will also be denoted by $H_0(|f|)$. In that case, we say that "m_{N+1} is infinite".

For θ complex and η real, we also introduce the symbol $H_0(|f|, \theta, \eta)$ for the object obtained by removing the center of mass from

$$\tilde{H}_0(|f|, \theta, \eta) = -e^{-2\theta} \sum_{j=1}^{N+1} (2m_j)^{-1} \Delta_j + |f| e^{i\eta} e^{\theta} \hat{e} \cdot \sum_{j=1}^{N+1} q_j \mathbf{r}_j . \tag{2.3}$$

Given $\frac{1}{2}N(N+1)$ potentials $V_{ij} \in C_\phi^M$, the Combes class of Sect. 1, we identify V_{ij} as a function of $\mathbf{r}_i - \mathbf{r}_j$ (if $m_{N+1} = \infty$, $V_{i,N+1}$ is a function of $\mathbf{r}_i$) and let for $|\operatorname{Im}\theta| < \phi$

$$V(\theta) = \sum_{i<j} V_{ij}(\theta)$$

$$H(|f|) = H_0(|f|) + V$$

$$H(|f|, \theta, \eta) = H_0(|f|, \theta, \eta) + V(\theta) .$$

The reason for including an η in the above is to allow $f = |f| e^{i\eta}$ to be non-real when we consider analyticity properties in f.

Theorem 2.2. (a) *$H(|f|)$ is essentially self-adjoint on $\mathscr{S}(\mathbb{R}^{\nu N})$.*

(b) *The operator*

$$L_0(|f|, \theta, \eta) \equiv i e^{-i\eta - \theta} H_0(|f|, \theta, \eta) \tag{2.4}$$

defined on $D(-\Delta) \cap D(\hat{e} \cdot \sum q_j(\mathbf{r}_j - \mathbf{R})) \equiv \mathscr{D}$ is closed and the generator of a contraction semigroup so long as $|f| > 0$ and $\operatorname{Re}(ie^{-3\theta - i\eta}) > 0$, i.e., $0 < 3\operatorname{Im}\theta + \eta < \pi$.

(c) *The operator*

$$L(|f|, \theta, \eta) \equiv i e^{-i\eta - \theta} H(|f|, \theta, \eta) \tag{2.5}$$

defined on $\mathscr{D}$ is closed and the generator of an exponentially bounded semigroup so long as $|f| > 0$, $0 < 3\operatorname{Im}\theta + \eta < \pi$ and $|\operatorname{Im}\theta| < \phi$, the angle in C_ϕ^M.

(d) *$H_0(|f|, \theta, \eta)$ and $H(|f|, \theta, \eta)$ are holomorphic families of type (A) as functions of θ and $f \equiv |f| e^{i\eta}$ in the region*

$$\{(\theta, f) | 0 < 3\operatorname{Im}\theta + \eta < \pi, |\operatorname{Im}\theta| < \phi, |f| > 0\} \equiv \mathscr{R} . \tag{2.6}$$

Proof. (a) This follows from the Faris-Lavine theorem; see [12] or [22; Sect. X.5].

(b) This is a restatement of results of Herbst [16]; see Theorems II.1 and II.3 of that paper.

(c) By hypothesis, $V(\theta)$ is an $H_0(|f| = 0)$-bounded operator with relative bound zero. By the quadratic estimates, Proposition II.4 of [16], it is automatically also $H_0(|f|, \theta, \eta)$-bounded with relative bound zero. Moreover, it is easy to see that $H(|f|, \theta, \eta)$ has numerical range in a half-plane. Thus the claimed results follow from standard ones in the perturbation theory of semigroups; see e.g. [22].

(d) This follows from the definition of type (A) family [18, 23] and the fact that in the region in question, H and H_0 are closed on the fixed domain, $\mathscr{D}$. □

Definition. We say a system of charges and masses $(q_1, m_1), \ldots, (q_{N+1}, m_{N+1})$ is ineffective if and only if there is a non-trivial decomposition $D = \{C_1, \ldots, C_\ell\}$ of $\{1, \ldots, N+1\}$ so that $Q_\alpha/M_\alpha = Q/M$ for all α where $Q_\alpha = \sum_{j \in C_\alpha} q_j$; $M_\alpha = \sum_{j \in C_\alpha} m_j$. If $m_{N+1} = \infty$, we say the set is *ineffective* if and only if there is a non-trivial decomposition $\{C_1, \ldots, C_l\}$ of $\{1, \ldots, N+1\}$ with $N+1$ in C_1 such that $Q_\alpha = 0$ for $\alpha > 1$. If the system is not ineffective, we call it effective.

Note that a system with one positive and the remaining charges strictly negative with the same charge to mass ratio is always effective even when the positive charge has infinite mass. Notice also that since q_{N+1} drops out if $m_{N+1} = \infty$, we can take a set of fixed charges and $m_1, \ldots, m_N$ and take m_{N+1} to infinity so that the limit is ineffective even though the approximates are effective. Given the theorem below this says there is an instability of essential spectrum as m_{N+1} goes to infinity which is associated to the instability of this spectrum as $|f| \to 0$.

Theorem 2.3. *Let $(\theta, |f|e^{i\eta})$ lie in the region $\mathcal{R}$ and suppose that the masses and charges are effective. Then $H(|f|, \theta, \eta)$ has purely discrete spectrum.*

Proof. We describe the details for $m_{N+1} \neq \infty$. The modifications if $m_{N+1} = \infty$ are easy. For a given decomposition D of $\{1, \ldots, N+1\}$, we define H_D in the usual way by dropping all V_{ij} with i and j in distinct clusters. As usual, there is a decomposition $\mathfrak{H} = \bigotimes_\alpha \mathfrak{H}_{C_\alpha} \otimes \mathfrak{H}_D$

$$H_D = \sum_{C_\alpha \in D} [H(C_\alpha) \otimes I] + [h_D \otimes I], \tag{2.7}$$

where $\mathfrak{H}_{C_\alpha}$ are functions of the internal coordinates of C_α and $\mathfrak{H}_D$ are functions of differences of center of masses of clusters. In (2.7) the symbol $[A \otimes I]$ indicates the tensor product of operators A on one of the factors and I on all other factors. Each factor is different: $\mathfrak{H}_\alpha$ for $H(C_\alpha)$ and $\mathfrak{H}_D$ for h_D.

Since L_D, $L(C_\alpha)$, ℓ_D all generate exponentially bounded semigroups, it is easy to see that

$$e^{-tL_D} = \left[\bigotimes_\alpha e^{-tL(C_\alpha)}\right] \otimes e^{-t\ell_D}.$$

Moreover, h_D is just what results from removal of the center of mass from

$$\tilde{h}_D = -e^{-2\theta} \sum_\alpha (2M_\alpha)^{-1} \Delta_\alpha + |f| e^{+i\eta} e^\theta \hat{e} \cdot \left(\sum_\alpha Q_\alpha \mathbf{R}_\alpha\right).$$

By Proposition 2.1, and the effectiveness hypothesis, h_D still has a non-zero electric field term and so by Proposition II.2 of [16].

$$\|e^{-t\ell_D}\| \leqq \exp(-Ct^3)$$

for $C > 0$. Since $\|e^{-tL(C_\alpha)}\| \leqq e^{B(\alpha)t}$ for some $B(\alpha)$, we see that

$$\int \|e^{zt} e^{-tL_D}\| dt < \infty$$

for all z in $\mathbb{C}$. Thus, $(H_D - z)^{-1}$ is an entire function.

By Theorem 2.2, one can go to a region in the complex plane where the perturbation series in $V(\theta)$ for H converges and make the necessary rearrangements to obtain Weinberg–van Winter equations for H, i.e.,

$$(H-z)^{-1}=D(z)+I(z)(H-z)^{-1}.$$

Since $D(z)$, $I(z)$ are made up of $(H_D-z)^{-1}$'s, they are entire functions. Moreover, by the quadratic estimates (Proposition II.4 of [16]), and the hypothesis on V_{ij}, $I(z)$ is compact in the region of perturbation series convergence and so in the whole complex plane. Thus, in the usual way, $(H-z)^{-1}$ is meromorphic with finite rank residues, i.e., H has purely discrete spectrum. □.

For ineffective charges and masses, the essential spectrum may not be empty but one can still identify it. The following theorem is proved in Appendix B:

Theorem 2.4. *Suppose* $\eta=0$, $0<\operatorname{Im}\theta<\operatorname{Min}(\phi,\pi/3)$, $|f|>0$ *and suppose the charges and masses are ineffective. Then*

$$\sigma_{\mathrm{ess}}(H(|f|,\theta))=\bigcup\{\lambda_1+\ldots+\lambda_k+e^{-2\theta}\mu\},$$

where the λ_α *are discrete eigenvalues of* $H(C_\alpha)$, $\mu\geqq 0$ *and we run only through those decompositions* $D=\{C_\alpha\}$ *with* $Q_\alpha/M_\alpha=Q/M$.

By an argument identical to that given in the Introduction for the one-body case we have

Theorem 2.5. *Fix* $\eta=0$ *and* $|f|$. *Then for* $\operatorname{Im}z>0$

$$\operatorname*{s-lim}_{\operatorname{Im}\theta\downarrow 0,\,\theta\to 0}(H(|f|,\theta)-z)^{-1}=(H(|f|)-z)^{-1}.$$

Remark. One can also obtain a result of this genre on the limit of the resolvent as $|f|\downarrow 0$ but as we will prove a stronger result in the next section we don't write that down here.

In the usual way one obtains:

Corollary 2.6. *For* $\eta=0$, $0<\operatorname{Im}\theta<\operatorname{Min}(\phi,\pi/3)$, *we have* $\sigma(H(|f|,\theta))\subseteqq\{z:\operatorname{Im}z\leqq 0\}$ *and the point spectrum of* $H(|f|,\theta=0)$ *away from* $\sigma_{\mathrm{ess}}(H(|f|,\theta))\cap\mathbb{R}$ *is identical to* $\sigma_{\mathrm{disc}}(H(|f|,\theta))\cap\mathbb{R}$.

3. A Semigroup Weinberg–van Winter Analysis

In this section we will find it convenient to be more systematic in our notation. We denote the operator appearing in (2.3) by $\tilde{H}_0(f,\theta)$ where $f=e^{i\eta}|f|$. An operator without a tilde as usual has its center of mass removed. We write

$$H(f,\theta)=H_0(f,\theta)+V(\theta)$$

$$L(f,\theta)=ie^{-i\eta}e^{-\theta}H(f,\theta)$$

$$H(\theta)=H(0,\theta)$$

$$\Sigma=\inf\sigma(H(0))$$

and introduce the sets

$$K(\theta)=\{\Sigma+\lambda e^{-2\theta}+\mu:\lambda\geqq 0,\mu>0\}$$
$$K(\eta,\theta)=\{z+xe^{\theta}e^{i\eta}:z\in K(\theta),x\in\mathbb{R}\}$$
$$K_L(\eta,\theta)=ie^{-i\eta}e^{-\theta}K(\eta,\theta)$$
$$\mathscr{S}_0=\{(\eta,\theta):|\mathrm{Im}\,\theta|<\phi,0\leqq\eta+\mathrm{Im}\,\theta\leqq\pi,0<\eta+3\,\mathrm{Im}\,\theta<\pi\}.$$

Note that if $|\mathrm{Im}\,\theta|<\pi/2$, $K(\theta)$ contains the convex hull of the spectrum of $H(\theta)$. The conditions $0<\eta+3\,\mathrm{Im}\,\theta<\pi$, $|\mathrm{Im}\,\theta|<\phi$ are natural because if we also demand $|f|>0(f=|f|e^{i\eta})$ then $H_0(f,\theta)$ satisfies the quadratic estimate of [16] and $H(f,\theta)$ is analytic. The additional condition $0\leqq\eta+\mathrm{Im}\,\theta\leqq\pi$ guarantees that $K(\eta,\theta)$ is a half-plane. Thus if $(\eta,\theta)\in\mathscr{S}_0$ with $0<\eta+\mathrm{Im}\,\theta<\pi$

$$K(\eta,\theta)=\{\Sigma+xe^{\theta}e^{i\eta}+\mu:x\in\mathbb{R},\ \mu\geqq 0\} \tag{3.1}$$

while with $\eta+\mathrm{Im}\,\theta=0$ (respectively π), $K(\eta,\theta)$ is the lower (respectively upper) half-plane.

Also note that if $(\eta,\theta)\in\mathscr{S}_0$ then $|\mathrm{Im}\,\theta|<\pi/2$.

Our goal in this section is to prove the following:

Theorem 3.1. *Fix a compact subset $\mathscr{S}_1$ of $\mathscr{S}_0\cap\{(\eta,\theta):0<\eta+\mathrm{Im}\,\theta<\pi\}$. Then for any $\varepsilon_1>0$ there is an $f_{\varepsilon_1}>0$ and a $C_{\varepsilon_1}<\infty$ so that*

$$\|(H(f,\theta)-z)^{-1}\|\leqq C_{\varepsilon_1}$$

so long as $0<|f|<f_{\varepsilon_1}$, $(\eta,\theta)\in\mathscr{S}_1$ and

$$\mathrm{Re}[ie^{-i\eta-\theta}(z-\Sigma+\varepsilon_1)]\leqq 0. \tag{3.2}$$

Theorem 3.1 follows from

Theorem 3.2. *Fix a compact subset $\mathscr{S}$ of $\mathscr{S}_0$. Then for any $\varepsilon>0$ there is an $F_\varepsilon>0$ and a $\tilde{C}_\varepsilon<\infty$ so that if $0<|f|<F_\varepsilon$, $(\eta,\theta)\in\mathscr{S}$*

$$\|e^{-tL(f,\theta)}\|\leqq\tilde{C}_\varepsilon e^{-t(\Sigma(\eta,\theta)-\varepsilon)}, \tag{3.3}$$

where $\Sigma(\eta,\theta)\equiv\mathrm{Re}(ie^{-i\eta}e^{-\theta}\Sigma)=\inf\{\mathrm{Re}\,z:z\in K_L(\eta,\theta)\}$.

Proof of Theorem 3.1. Given $\varepsilon_1>0$ and $\mathscr{S}_1$, as in Theorem 3.1, choose $\varepsilon=\frac{1}{2}\inf\{\sin(\mathrm{Im}\,\theta+\eta)e^{-\mathrm{Re}\,\theta}:(\eta,\theta)\in\mathscr{S}_1\}$. Then if z satisfies (3.2) and $(\eta,\theta)\in\mathscr{S}_1$

$$\begin{aligned}\mathrm{Re}(\Sigma(\eta,\theta)-ie^{-i\eta}e^{-\theta}z)&=\mathrm{Re}(ie^{-i\eta}e^{-\theta}(\Sigma-z))\\&\geqq\varepsilon_1 e^{-\mathrm{Re}\,\theta}\sin(\mathrm{Im}\,\theta+\eta)\geqq 2\varepsilon.\end{aligned}$$

Thus for $(\eta,\theta)\in\mathscr{S}_1$ (we choose $\mathscr{S}=\mathscr{S}_1$) and $0<|f|<F_\varepsilon$

$$\begin{aligned}e^{\mathrm{Re}\,\theta}\|(H(f,\theta)-z)^{-1}\|&=\left\|\int_0^\infty e^{-t(L(f,\theta)-zie^{-i\eta}e^{-\theta})}dt\right\|\\&\leqq\tilde{C}_\varepsilon\int_0^\infty e^{-t\mathrm{Re}(\Sigma(\eta,\theta)-zie^{-i\eta}e^{-\theta}-\varepsilon)}dt\\&\leqq\tilde{C}_\varepsilon\int_0^\infty e^{-t(2\varepsilon-\varepsilon)}dt=\tilde{C}_\varepsilon/\varepsilon.\quad\square\end{aligned}$$

Theorem 3.1 which is needed for subsystem Hamiltonians in the next section, says essentially that as long as we take an extra ε width and $|f|$ small, we can control the norm of the resolvent outside the region of sums of the form $\mu+\lambda$ with μ in the convex hull of $\sigma(H(\theta))$ and $\lambda\in\sigma(e^{i\eta}e^{\theta}x)$. It is thus a replacement for numerical range arguments of [16] which are possible when $N=1$.

Before beginning the proof of Theorem 3.2 we will need some further notation. Since we will be taking $|f|\to 0$ in the operator $L(f,\theta)$ we introduce

$$L'(\eta,\theta)=ie^{-i\eta}e^{-\theta}H(\theta).$$

By convention $L(0,\theta)=L(\eta,\theta)$. In addition we will need

$$H_0=H_0(0,0)$$

$$\mathbf{r}_{ij}=\mathbf{r}_i-\mathbf{r}_j$$

$$W_{ij}(\alpha)=\text{multiplication by }\exp(\alpha\sqrt{r_{ij}^2+1})$$

We begin with two technical lemmas:

Lemma 3.3. *Suppose* $\tilde{\mathscr{S}}$ *is a compact subset of* $\{(|f|,\eta,\theta):0<3\operatorname{Im}\theta+\eta<\pi,\ |\operatorname{Im}\theta|<\phi\}$ *and J is a subset of the pairs* $\{(i,j):1\leqq i<j\leqq N+1\}$. *Define for* $\alpha_{ij}\in[-1,1]$, $W(\boldsymbol{\alpha})=\prod_{(i,j)\in J}W_{ij}(\alpha_{ij})$ *and* $L(f,\theta,\boldsymbol{\alpha})=W(\boldsymbol{\alpha})L(f,\theta)W(-\boldsymbol{\alpha})$ *with domain* $C_0^\infty(\mathbb{R}^{\nu N})$. *Then there is an* $E>0$ *so that for all* $(|f|,\theta,\eta)\in\tilde{\mathscr{S}}$ *and all* $\alpha_{ij}\in(-1,1)$

(i) $L(f,\theta,\boldsymbol{\alpha})$ *is closable. Denote its closure by* $\bar{L}(f,\theta,\boldsymbol{\alpha})$.

(ii) $\bar{L}(f,\theta,\boldsymbol{\alpha})+E$ *is maximal accretive.*

(iii) *If* $\phi\in\mathscr{D}(W(-\boldsymbol{\alpha}))$, $e^{-t\bar{L}(f,\theta,\boldsymbol{\alpha})}\phi\in\mathscr{D}(W(-\boldsymbol{\alpha}))$ *and*

$$e^{-tL(f,\theta)}W(-\boldsymbol{\alpha})\phi=W(-\boldsymbol{\alpha})e^{-t\bar{L}(f,\theta,\boldsymbol{\alpha})}\phi \tag{3.4}$$

(iv) $\|H_0e^{-t\bar{L}(f,\theta,\boldsymbol{\alpha})}\|\leqq Ct^{-1}e^{tE}$.

Lemma 3.4. *Suppose* $B\in C_\phi^M$. *Then as* $\delta\downarrow 0$

$$\exp(-\delta\sqrt{r^2+1})B(\theta)(-\Delta+1)^{-1}\xrightarrow{\|\cdot\|}B(\theta)(-\Delta+1)^{-1}$$

uniformly on compacts of $\{\theta:|\operatorname{Im}\theta|<\phi\}$.

We leave the proof of Lemma 3.4 to the reader.

Proof of Lemma 3.3. We first show that under the stated conditions, E can be chosen so that $L(f,\theta,\boldsymbol{\alpha})+E$ has numerical range in the right half-plane: First note that ($\mathbf{p}_k=-i\nabla_k$)

$$W(\boldsymbol{\alpha})\mathbf{p}_kW(-\boldsymbol{\alpha})=\mathbf{p}_k+i\mathbf{g}_k(\mathbf{r}),$$

where

$$\mathbf{g}_k(\mathbf{r})=\sum_{(k,j)\in J}\alpha_{kj}\mathbf{r}_{kj}/\sqrt{r_{kj}^2+1}-\sum_{(j,k)\in J}\alpha_{jk}\mathbf{r}_{jk}/\sqrt{r_{jk}^2+1}.$$

Thus

$$W(\boldsymbol{\alpha})H_0W(-\boldsymbol{\alpha})=H_0-\sum_k|\mathbf{g}_k|^2/2m_k+i\sum_k(\mathbf{p}_k\cdot\mathbf{g}_k+\mathbf{g}_k\cdot\mathbf{p}_k)/2m_k=H_0+G(\boldsymbol{\alpha}),$$

where

$$G(\boldsymbol{\alpha}) = -\sum_k |\mathbf{g}_k|^2/2m_k + \frac{i}{2}\sum_{(j,k)\in J} \alpha_{jk}((\mathbf{p}_j/m_j) - (\mathbf{p}_k/m_k))\cdot \mathbf{r}_{jk}/\sqrt{r_{jk}^2+1}\,.$$

We have

$$W(\boldsymbol{\alpha})(H_0 + V(\theta)e^{2\theta})W(-\boldsymbol{\alpha}) = H_0 + V(\theta)e^{2\theta} + G(\boldsymbol{\alpha})\,.$$

Because of the compactness and analyticity of $V_{ij}(\theta)(-\Delta+1)^{-1}$, we have $\lim_{\lambda\to\infty} \|V(\theta)\ (H_0+\lambda)^{-1}\| = 0$ uniformly for θ in compacts of $\{\theta : |\mathrm{Im}\,\theta| < \phi\}$. In addition, $\lim_{\lambda\to\infty} \|G(\boldsymbol{\alpha})\ (H_0+\lambda)^{-1}\| = 0$ uniformly for $\boldsymbol{\alpha}$ in compact sets. Thus there is a $\lambda_0 > 0$ such that for $\theta\in D \equiv \{\theta : (|f|,\ \eta,\ \theta)\in\tilde{\mathscr{S}}\}$ and $\alpha_{ij}\in[-1,1]$ all $(i,j)\in J$

$$\begin{aligned}&|\mathrm{Im}(\phi, W(\boldsymbol{\alpha})(H_0 + V(\theta)e^{2\theta} + \lambda_0)W(-\boldsymbol{\alpha})\phi)|\\ &\quad\leqq \tan\gamma\ \mathrm{Re}(\phi, W(\boldsymbol{\alpha})(H_0 + V(\theta)e^{2\theta} + \lambda_0)\ W(-\boldsymbol{\alpha})\phi)\,,\end{aligned}$$

where $\gamma = \mathrm{Inf}\{\eta + 3\,\mathrm{Im}\,\theta, \pi - (\eta + 3\,\mathrm{Im}\,\theta) : (|f|, \eta, \theta)\in\mathscr{S}\}$. In fact, if

$$c(\lambda) = \sup_{\theta\in D,\,\alpha_{ij}\in[-1,1]} \|(G(\boldsymbol{\alpha}) + e^{2\theta}V(\theta))(H_0+\lambda)^{-1}\|$$

we need only take λ_0 so that $c(\lambda_0)/1 - c(\lambda_0) \leqq \tan\gamma$. Thus $W(\boldsymbol{\alpha})\ (H_0 + V(\theta)e^{2\theta} + \lambda_0)W(-\boldsymbol{\alpha})$ has numerical range in a sector between the angles $\pm\gamma$. Now

$$L(f,\theta,\boldsymbol{\alpha}) = ie^{-3\theta}e^{-i\eta}W(\boldsymbol{\alpha})(H_0 + V(\theta)e^{2\theta})W(-\boldsymbol{\alpha}) + i|f|X$$

with $X = \hat{e}\cdot\left(\sum_{i=1}^{N+1} q_i\mathbf{r}_i - Q\mathbf{R}\right)$ so that

$$L(f,\theta,\boldsymbol{\alpha}) + \lambda_0 ie^{-3\theta}e^{-i\eta}$$

has numerical range in the right half-plane if $\gamma \leqq 3\,\mathrm{Im}\,\theta + \eta \leqq \pi - \gamma$. If we take $E = \sup\{\mathrm{Re}(\lambda_0 ie^{-3\theta}e^{-i\eta}) : (|f|, \theta, \eta)\in\tilde{\mathscr{S}}\}$ then $L(f,\theta,\boldsymbol{\alpha}) + E$ has numerical range in the right half-plane for all $(|f|, \eta, \theta)\in\tilde{\mathscr{S}}$. Let

$$L_0(f,\theta) = ie^{-\theta}e^{-i\eta}H_0(f,\theta)\,.$$

From [16], $C_0^\infty(\mathbb{R}^{\nu N})$ is a core for $L_0(f,\theta)$. Consider the operator

$$M(\boldsymbol{\alpha},\lambda) = (L(f,\theta,\boldsymbol{\alpha}) - L_0(f,\theta))(L_0(f,\theta) + \lambda)^{-1}$$

with domain $(L_0(f,\theta) + \lambda)C_0^\infty(\mathbb{R}^{\nu N})$. By the quadratic estimate of [16], M extends to a bounded analytic operator valued function of $\boldsymbol{\alpha}$ which approaches zero in norm as $\lambda\to\infty$. This has the consequence that $\bar{L}(f,\theta,\boldsymbol{\alpha}) + E$ is maximal accretive with domain equal to $\mathscr{D}(L_0(f,\theta))$ and that

$$(\lambda + \bar{L}(f,\theta,\boldsymbol{\alpha}))^{-1}$$

is analytic in $\boldsymbol{\alpha}$ for $\mathrm{Re}\,\alpha_{ij}\in(-1,1)$ and $\lambda > E$. [Here we use the fact that $W(\boldsymbol{\alpha})$ is unitary for $\boldsymbol{\alpha}$ imaginary.]

We have proved (i) and (ii). To show (iii), first note that for fixed t, $e^{-tL(f,\theta,\boldsymbol{\alpha})}$ is analytic in $\boldsymbol{\alpha}$ for $\mathrm{Re}\,\alpha_{ij}\in(-1,1)$ so that for ϕ, $\psi\in C_0^\infty(\mathbb{R}^{\nu N})$ both sides of

$$(W^{-1}(\boldsymbol{\alpha})\psi, e^{-t\bar{L}(f,\theta,\boldsymbol{\alpha})}\phi)=(\psi, e^{-tL(f,\theta)}W^{-1}(\boldsymbol{\alpha})\phi) \tag{3.5}$$

are analytic in $\boldsymbol{\alpha}$, equal for $\boldsymbol{\alpha}$ imaginary and thus equal for $\alpha_{ij}\in(-1,1)$. By a limiting argument (3.5) holds for $\phi\in\mathscr{D}(W(-\boldsymbol{\alpha}))$ and thus $e^{-t\bar{L}(f,\theta,\boldsymbol{\alpha})}\phi\in\mathscr{D}(W(-\boldsymbol{\alpha}))$ and (3.4) holds.

To prove (iv) we consider the operator

$$\Gamma(t)=\mathscr{U}(t)e^{-t(\bar{L}(f,\theta,\boldsymbol{\alpha})+\lambda)}, \tag{3.6}$$

where $\mathscr{U}(t)=\exp(i|f|Xt)$. If $\phi\in C_0^\infty(\mathbb{R}^{\nu N})$ we can differentiate $x(t)\equiv\Gamma(t)\phi$ to find the evolution equation

$$\frac{dx(t)}{dt}=-A(t)x(t) \tag{3.7}$$

with

$$A(t)=\mathscr{U}(t)\left[ie^{-3\theta}e^{-i\eta}W(\boldsymbol{\alpha})(H_0+e^{2\theta}V(\theta))W(-\boldsymbol{\alpha})+\lambda\right]\mathscr{U}(-t). \tag{3.8}$$

Now $A(t)-1$ is sectorial for large enough $\lambda>0$, with numerical range between $\pi/2-\delta$ and $-\pi/2+\delta$ uniformly for $(|f|,\eta,\theta)\in\tilde{\mathscr{S}}$ and $\alpha_{ij}\in(-1,1)$ if λ is chosen large enough for some $\delta>0$. In addition estimates of the form

$$\|A(t)A(s)^{-1}\|\leqq c_1; \qquad t,s\in[0,1]$$

$$\left\|\frac{d}{dt}\{A(t)(z+A(s))^{-1}\}\right\|\leqq c_1; \qquad t,s\in[0,1], \mathrm{Re}\,z\geqq 0$$

are easy to prove if λ is large enough with c_1 independent of $(|f|,\eta,\theta)\in\tilde{\mathscr{S}}$ $t,s\in[0,1]$, z in the right half-plane.

Under these conditions Tanabe [31] and Sobolevski [29] have investigated the solutions to (3.7). Tanabe's method is outlined in Yosida [36] where it is shown that

$$\|A(t)\Gamma(t)\|\leqq c_2t^{-1}; \qquad t\in[0,1], \tag{3.9}$$

where the constant c_2 depends only on c_1 and the angle δ. Since we have

$$\|H_0\mathscr{U}(-t)A(t)^{-1}\|\leqq c_3$$

uniformly in the relevant region we find

$$\begin{aligned}\|H_0e^{-t(\bar{L}(f,\theta,\boldsymbol{\alpha})+\lambda)}\| &= \|H_0\mathscr{U}(-t)A(t)^{-1}A(t)\Gamma(t)\|\\ &\leqq c_3c_2t^{-1}; \qquad t\in[0,1].\end{aligned}$$

This demonstrates (iv). □

We now have the bounds necessary for a discussion of the terms which will appear in our Weinberg–van Winter semigroup expansion.

Lemma 3.5. *Suppose $\tilde{\mathscr{S}}$ is as in Lemma 3.3. Let $S=(D_{N+1},\dots,D_k)$ be a string of cluster decompositions (where D_ℓ has ℓ clusters and $D_{\ell+1}$ refines D_ℓ). Denote by L_{D_j}*

the operator $L(f,\theta,\boldsymbol{\alpha})$ *with all interactions* $V_{\ell m}(\theta)$ *between different clusters of* D_j *removed. Let* $P_j(t)=\exp(-tL_{D_j})$ *and suppose* $\|P_j(t)\|\leqq c_j e^{\gamma t}$, *for all* $(|f|,\eta,\theta)\in\tilde{\mathscr{S}}$.

Suppose β_j, $j=k,\dots,N$ *are multiplication operators with* $\|\beta_j(H_0+1)^{-1}\|=\gamma_j<\infty$. *Then for each* $\varepsilon>0$ *there is a* $C_\varepsilon<\infty$ *independent of* β_j *so that for all* $t_i>0$

$$\|P_{N+1}(t_{N+1})\beta_N P_N(t_N)\beta_{N-1}\dots\beta_k P_k(t_k)\|$$
$$\leqq C_\varepsilon\left(\prod_{j=k}^{N}\gamma_j\right)\left(\prod_{j=k}^{N+1}t_j\right)^{-(1-1/n_k)}\exp\left[\left(\sum_{j=k}^{N+1}t_j\right)(\gamma+\varepsilon)\right]$$

for all $(|f|,\eta,\theta)\in\tilde{\mathscr{S}}$. *Here* $n_k=N+2-k$.

Proof. We write the operator under consideration as

$$\left(\prod_{j=0}^{N-k}\right)[(1+H_0)^{j/n_k}P_{N+1-j}(t_{N+1-j})(1+H_0)^{1-(j+1)/n_k}]$$
$$\cdot\{(1+H_0)^{-(1-(j+1)/n_k)}\beta_{N-j}(1+H_0)^{-(j+1)/n_k}\}(1+H_0)^{(N-k+1)/n_k}P_k(t)$$

and estimate the result as a product of norms.

By interpolation $\|(1+H_0)^{-x}\beta_j(1+H_0)^{1-x}\|\leqq\gamma_j$ for $x\in[0,1]$. In addition since Lemma 3.3 holds for the adjoint of $\bar{L}_D(f,\theta,\boldsymbol{\alpha})$ as well as for $\bar{L}_D(f,\theta,\boldsymbol{\alpha})$ we have $\|(1+H_0)P_j(t)\|+\|P_j(t)(1+H_0)\|\leqq ct^{-1}$ for $t\in[0,1]$, and thus again by interpolation

$$\|(1+H_0)^xP_j(t)\|+\|P_j(t)(1+H_0)^x\|\leqq ct^{-x}$$

if $x\in[0,1]$ and $0\leqq t\leqq 1$. Thus

$$\|(1+H_0)^xP_j(t)(1+H_0)^{1-x-1/n_k}\|\leqq\|(1+H_0)^xP_j(t/2)\|\,\|P_j(t/2)(1+H_0)^{1-x-1/n_k}\|$$
$$\leqq ct^{-(1-1/n_k)};\qquad t\leqq 1, x\in[0,1-1/n_k].$$

If $t\geqq 1$ we can estimate for the same x

$$\|(1+H_0)^xP_j(t)(1+H_0)^{1-x-1/n_k}\|\leqq c\exp(\gamma t)$$

and thus for any $\varepsilon>0$ the latter norm is bounded by

$$d_\varepsilon t^{-(1-1/n_k)}\exp[(\gamma+\varepsilon)t]$$

for some d_ε. Combining the above estimates results in (3.10). □

In our semigroup expansion we will have to deal with integrals of operators like the one we estimated in Lemma 3.5. The next lemma controls these integrals.

Lemma 3.6. *Under the hypotheses of Lemma 3.5. let*

$$F(s)=\int dt_{N+1}\dots dt_k\delta(t_k+\dots+t_{N+1}-s)P_{N+1}(t_{N+1})\beta_N\dots\beta_kP_k(t_k),$$

where $s>0$ *and the integral is a strong integral. Then for each* $\varepsilon>0$ *there is a* C'_ε *independent of* β_j *so that*

$$\|F(s)\|\leqq C'_\varepsilon\left(\prod_{j=k}^{N}\gamma_j\right)\exp[s(\gamma+\varepsilon)] \tag{3.11}$$

and if $\phi\in C_0^\infty(\mathbb{R}^{\nu N})$, $F(s)\phi$ *is Hölder continuous uniformly on compacts of* $(0,T]$ *for any* $T>0$.

Proof. From Lemma 3.5

$$\|F(s)\| \leqq C_\varepsilon \left(\prod_{j=k}^{N} \gamma_j\right) \int dt_{N+1} \ldots dt_k \delta(t_k + \ldots + t_{N+1} - s) \left(\prod_{j=k}^{N+1} t_j\right)^{-(1-1/n_k)} \exp(s(\gamma+\varepsilon)).$$

The change of variable $t_j = \lambda_j s$ gives

$$\int dt_{N+1} \ldots dt_k \delta(t_k + \ldots + t_{N+1} - s) \left(\prod_{j=k}^{N+1} t_j\right)^{-(1-1/n_k)}$$

$$= \int d\lambda_{N+1} \ldots d\lambda_k \delta(\lambda_k + \ldots + \lambda_{N+1} - 1) \left(\prod_{j=k}^{N+1} \lambda_j\right)^{-(1-1/n_k)} = c$$

with $c < \infty$. Thus (3.11) follows with $C'_\varepsilon = cC_\varepsilon$.

To prove the Hölder continuity we write

$$\begin{aligned} F(s+\lambda)\phi = & \int_{s+\lambda \geqq t_{k+1}+\ldots+t_{N+1} \geqq 0} dt_{N+1} \ldots dt_{k+1} P_{N+1}(t_{N+1}) \beta_N \ldots \\ & P_k(s+\lambda - t_{N+1} \ldots - t_{k+1})\phi \\ = & \int_{s+\lambda \geqq t_{k+1}+\ldots+t_{N+1} \geqq s} dt_{N+1} \ldots dt_{k+1} P_{N+1}(t_{N+1}) \ldots \\ & P_k(s+\lambda - t_{N+1} \ldots - t_{k+1})\phi \\ & + F(s)(P_k(\lambda) - 1)\phi + F(s)\phi . \end{aligned}$$

Thus using (3.11) and Lemma 3.5

$$\begin{aligned} \|(F(s+\lambda) - F(s))\phi\| \leqq c & \int_{s+\lambda \geqq t_{k+1}+\ldots+t_{N+1} \geqq s} dt_{N+1} \ldots dt_{k+1} \\ & \cdot \left(\prod_{j=k+1}^{N+1} t_j\right)^{-(1-1/n_k)} (s+\lambda - t_{k+1} \ldots - t_{N+1})^{-(1-1/n_k)} + c'\lambda \|L_{D_k}\phi\| \end{aligned}$$

for $s \in (0, T]$.

The first term can be estimated using Hölder's inequality by the expression

$$\begin{aligned} & c\left(\int_{s+\lambda \geqq t_{k+1}+\ldots+t_{N+1} \geqq s} dt_{k+1} \ldots dt_{N+1} 1\right)^{1/q} \\ & \cdot \left(\int dt_k \ldots dt_{N+1} \delta(s+\lambda - t_k \ldots - t_{N+1}) \left(\prod_{j=k}^{N+1} t_j\right)^{-\alpha}\right)^{1/p}, \end{aligned}$$

where $\alpha \equiv p(1-1/n_k) < 1$, and $p^{-1} + q^{-1} = 1$. The above expression is bounded by

$$\text{const}\,\lambda^{1/q}(s+\lambda)^{-m}$$

for some m. This gives the required Hölder continuity. □

We can now prove the semigroup analog of the Weinberg–van Winter expansion. This expansion was discussed by Weinberg in [35] where the equation for resolvents is derived.

Proposition 3.7. *Let $P_D(t)=e^{-tL_D(f,\theta)}$ where D is a cluster decomposition and suppose $0<3\operatorname{Im}\theta+\eta<\pi$ and $|\operatorname{Im}\theta|<\phi$. Then for $t>0$*

$$e^{-tL(f,\theta)}=\sum_{S=(D_{N+1},\dots,D_k)}(-1)^{N-k+1}\int dt_{N+1}\dots dt_k\delta(t_k+\dots+t_{N+1}-t)P_{D_{N+1}}(t_{N+1})$$

$$\cdot\beta_N^S P_{D_N}(t_N)\beta_{N-1}^S\dots\beta_k^S P_{D_k}(t_k), \qquad (3.12)$$

where $\beta_\ell^S=ie^{-\theta}e^{-i\eta}\Sigma' V_{ij}(\theta)$ and Σ' is the sum over all pairs (i,j) with $i<j$ which connect the two clusters in $D_{\ell+1}$ which are joined together in D_ℓ.

Remark. If we write $P(t)=e^{-tL(f,\theta)}$ and $D(t)=[$sum of terms on RHS of (3.12) with S disconnected$]$, we can rewrite (3.12)

$$P(t)=D(t)+\int_0^t I(t-s)P(s)ds \qquad (3.12a)$$

or

$$P=D+I*P,$$

where

$$I(s)=\sum_{S=(D_{N+1},\dots,D_1)}(-1)^{N-k+1}\int dt_{N+1}\dots dt_2$$

$$\delta(t_{N+1}+\dots+t_2-s)\,P_{D_{N+1}}(t_{N+1})\beta_N^S\dots\beta_1^S.$$

In some applications it may be helpful to invert (3.12a) and use

$$P=(1-I*)^{-1}D$$

but we will deal directly with the expansion (3.12).

Proof. For $-\operatorname{Re}z$ sufficiently large we have the usual Weinberg–van Winter expansion

$$(z-L(f,\theta))^{-1}=\sum_{S=(D_{N+1},\dots,D_k)}(z-L_{D_{N+1}})^{-1}\beta_N^S(z-L_{D_N})^{-1}\beta_{N-1}^S\dots\beta_k^S(z-L_{D_k})^{-1}.$$

Denoting by $G(t)$ the difference between the left and right sides of (3.12) we have by (3.13)

$$\int_0^\infty(\psi,G(t)\phi)e^{-tz}dt=0$$

for $-\operatorname{Re}z$ sufficiently large and hence $(\psi,G(t)\phi)=0$ for almost all $t>0$. By the continuity result of Lemma 3.6, $G(t)=0$ for all $t>0$. □

We now begin to consider $f\to0$. We denote $L_D(0,\theta)$ by $L'_D(\eta,\theta)$.

Lemma 3.8. *Suppose $\mathcal{S}_2$ is a compact subset of $\{(\eta,\theta):0<3\operatorname{Im}\theta+\eta<\pi,\ |\operatorname{Im}\theta|<\phi\}$. Suppose $S=(D_{N+1},\dots,D_1)$, i.e., S is connected. Then as $|f|\to0$*

$$\int\left(\prod_{j=1}^{N+1}dt_j\right)\delta(t_1+\dots+t_{N+1}-t)e^{-t_{N+1}L_{D_{N+1}}(f,\theta)}\beta_N^S\dots\beta_1^S e^{-t_1L_{D_1}(f,\theta)}$$

$$\xrightarrow{\|\cdot\|}\int\left(\prod_{j=1}^{N+1}dt_j\right)\delta(t_1+\dots+t_{N+1}-t)e^{-t_{N+1}L'_{D_{N+1}}(\eta,\theta)}\beta_N^S\dots\beta_1^S e^{-t_1L'_{D_1}(\eta,\theta)} \qquad (3.14)$$

uniformly for $t\in(0,T]$ and $(\eta,\theta)\in\mathcal{S}_2$ for any $T>0$.

Proof. Let $F(t,f,\theta)$ be the operator on the left hand side of (3.14) and $F'(t,\eta,\theta)$ the operator on the right. Lemmas 3.4 and 3.6 show that if all potentials $V_{\ell m}(\theta)$ appearing in each β_j^S are multiplied by $W_{\ell m}(-\delta)$ to give $\beta_{j,\delta}^S$, the resulting operators $F_\delta(t,f,\theta)$ and $F'_\delta(t,\eta,\theta)$ converge in norm to $F(t,f,\theta)$ and $F'(t,\eta,\theta)$ as $\delta\downarrow 0$ uniformly in the region $0<|f|\leqq 1$, $0<t\leqq T$, $(\eta,\theta)\in\mathcal{S}_2$. [Note that by Lemma 3.3, $L_{D_j}(f,\theta)+E$ is accretive for some E independent of $(\eta,\theta)\in\mathcal{S}_2$.] Hence we need only show that for a fixed $\delta>0$

$$\lim_{|f|\to 0}\|F_\delta(t,f,\theta)-F'_\delta(t,\eta,\theta)\|=0$$

uniformly in (t,η,θ) for $0<t\leqq T$, $(\eta,\theta)\in\mathcal{S}_2$. We first write each $\beta_{j,\delta}^S$ as a sum of potentials (which include decreasing exponentials factors) and consider operators $G_\delta(t,f,\theta)$ and $G'_\delta(t,\eta,\theta)$ resulting from keeping one potential from each $\beta_{j,\delta}^S$. Since S is connected, the product of potentials in $G_\delta(t,f,\theta)$ contains a factor $\prod_{(i,j)\in J} W_{ij}(-\delta)$ where J is such that $c\sum_{(i,j)\in J}|\mathbf{r}_{ij}|\geqq\sum_{i>j}|\mathbf{r}_{ij}|$. We change $e^{-t_jL_{D_j}(f,\theta)}$ to $e^{-t_jL'_{D_j}(\eta,\theta)}$ one at a time using the du Hamel formula

$$e^{-t_jL_{D_j}(f,\theta)}=e^{-t_jL'_{D_j}(\eta,\theta)}-i|f|\int ds_1ds_2\delta(t_j-s_1-s_2)e^{-s_1L_{D_j}(f,\theta)}Xe^{-s_2L'_{D_j}(\eta,\theta)}$$

where $X=\hat{e}\cdot(\Sigma q_i\mathbf{r}_i-Q\mathbf{R})$. We thus write $G_\delta(t,f,\theta)-G'_\delta(t,\eta,\theta)$ as $-i|f|$ times a sum of terms of the form

$$\int\left(\prod_{j=1}^{N+2}dt_j\right)\delta\left(t-\sum_{j=1}^{N+2}t_j\right)P_{N+2}(t_{N+2})\alpha_{N+1}P_{N+1}(t_{N+1})\dots\alpha_1P_1(t_1),\tag{3.15}$$

where one of the α_j's is X and the rest are of the form $ie^{-i\eta}e^{-\theta}W_{ij}(-\delta)V_{ij}(\theta)$ for some (i,j), while $P_j(t)=e^{-tL_D(f,\theta)}$ or $e^{-tL'_D(\eta,\theta)}$ for some D.

We need only show that a term like (3.15) is uniformly bounded as $|f|\to 0$ and (η,θ) and t vary over $\mathcal{S}_2$ and $(0,T]$ respectively. We use Lemma 3.3 to move all W_{ij}'s past operators $P_k(t_k)$ to the point where X occurs. Since $\left\|X\prod_{(i,j)\in J}W_{ij}(-\delta)\right\|<\infty$ the expression (3.15) is uniformly bounded using Lemma 3.6. □

Lemma 3.9. *Let $\mathcal{S}$ and $\Sigma(\eta,\theta)$ be as in Theorem 3.2. Then given any $\varepsilon>0$ there is a $C_\varepsilon<\infty$ so that if $(\eta,\theta)\in\mathcal{S}$*

$$\|e^{-tL'_D(\eta,\theta)}\|\leqq C_\varepsilon e^{-t(\Sigma(\eta,\theta)-\varepsilon)}.\tag{3.16}$$

Proof. As in the proof of Lemma 3.3 we can choose $E>0$ so that if $(\eta,\theta)\in\mathcal{S}$ the numerical range of $L'_D(\eta,\theta)+E$ lies in the sector $\{z:|\arg z|<\pi/2-\gamma/2\}$ where $(\eta,\theta)\in\mathcal{S}$ implies $\gamma\leqq\eta+3\,\mathrm{Im}\,\theta\leqq\pi-\gamma$ for some $\gamma>0$. Thus for $0\leqq t\leqq 1$, $\|e^{-tL'_D(\eta,\theta)}\|\leqq e^E$, and we need only prove (3.16) for $t\geqq 1$. In this case we write

$$e^{-tL'_D(\eta,\theta)}=-\frac{1}{2\pi i}\int_\Gamma e^{-zt}(z-L'_D(\eta,\theta))^{-1}dz,$$

where Γ is the contour shown in Fig. 1. Since $\sigma(H_D(\theta))\subseteq K(\theta)$ for any cluster decomposition D and $\inf\{\mathrm{Re}\,z:z\in ie^{-\theta}e^{-i\eta}K(\theta)\}=\Sigma(\eta,\theta)$, Γ surrounds the spectrum of $L'_D(\eta,\theta)$.

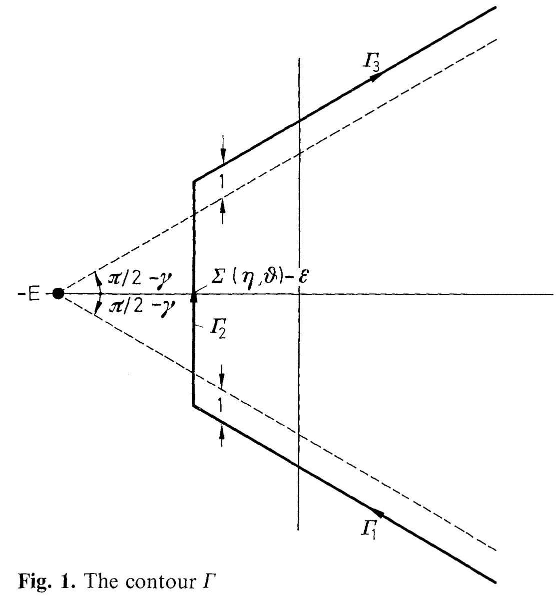

Fig. 1. The contour Γ

The resolvent $(z-L'_D(\eta,\theta))^{-1}$ is uniformly bounded for $z\in\Gamma_1\cup\Gamma_3$ and $(\eta,\theta)\in\mathcal{S}$ by sectoriality and this gives the bound

$$\left\|\int_{\Gamma_1\cup\Gamma_3} e^{-zt}(z-L'_D(\eta,\theta))^{-1}dz\right\|\leqq \text{const} \int_{\Gamma_1\cup\Gamma_3} e^{-\mathrm{Re}\,zt}|dz|\leqq \text{const}\,. \tag{3.17}$$

It is easy to show that if C is a compact subset of $\{(z,\theta):z\in\mathbb{C}, |\mathrm{Im}\,\theta|<\phi\}$ then for any $\delta>0$

$$\sup\{\|(z-H_D(\theta))^{-1}\|:d(z,\sigma(H_D(\theta)))\geqq\delta,(z,\theta)\in C\}<\infty\,. \tag{3.18}$$

Here $d(z,A)$ is the distance from z to the set A. Since for $z\in\Gamma_2$ $d(z,\sigma(L'_D(\eta,\theta)))\geqq\varepsilon$ for all $(\eta,\theta)\in\mathcal{S}$, we have

$$\sup\{\|(z-L'_D(\eta,\theta))^{-1}\|:z\in\Gamma_2,(\eta,\theta)\in\mathcal{S}\}<\infty\,.$$

Thus

$$\left\|\int_{\Gamma_2} e^{-zt}(z-L'_D(\eta,\theta))^{-1}dz\right\|\leqq\text{const}\,e^{-(\Sigma(\eta,\theta)-\varepsilon)t} \tag{3.19}$$

and (3.16) follows from (3.17) and (3.19) since $\Sigma(\eta,\theta)\leqq 0$. □

Proof of Theorem 3.2. Our proof proceeds by induction on $|C|$, the number of particles in a cluster C. Let $\tilde{L}^C(f,\theta)$ denote the operator

$$ie^{-i\eta}e^{-\theta}\left(\sum_{i\in C}(-\Delta_i/2m_i)e^{-2\theta}+\sum_{\substack{i<j\\ i,j\in C}}V_{ij}(\theta)+fe^{\theta}\sum_{i\in C}q_i\mathbf{r}_i\cdot\hat{e}\right)$$

and $L^C(f,\theta)$ the same operator with center of mass removed. $e^{-tL^C(f,\theta)}$ is an operator on $L^2(\mathbb{R}^{\nu(|C|-1)})$. Similarly $H^C(f,\theta)=-ie^{i\eta}e^{\theta}L^C(f,\theta)$. Let

$$\Sigma^C=\inf\sigma(H^C(0,0))\,.$$

If $|C|=1$, the result is trivial. Now assume the result for $|C|\leqq N$. We will prove it for $|C|=N+1$. Thus suppose that if $0<|f|<\tilde{F}_\varepsilon$, $(\eta,\theta)\in\mathscr{S}$

$$\|e^{tL^C(f,\theta)}\|\leqq c_1(\varepsilon)\exp(t\varepsilon/4(N+1))\exp(-t\Sigma^C(\eta,\theta)).$$

Then if $D=\{C_1,\ldots,C_k\}$ with $k>1$ we have

$$\|e^{-tL_D(f,\theta)}\|\leqq c_2(\varepsilon)\exp(t\varepsilon/4)\exp(-t\Sigma(\eta,\theta))$$

because the sum of the Σ^{C_j}'s is at least Σ. From Lemma 3.6 we thus have for each disconnected string $S=(D_{N+1},\ldots,D_k)$

$$\left\|\int e^{-t_{N+1}L_{D_{N+1}}(f,\theta)}\beta^S_N\ldots\beta^S_k e^{-t_kL_{D_k}(f,\theta)}\delta\left(t-\sum_{j=k}^{N+1}t_j\right)\prod_{j=k}^{N+1}dt_j\right\|$$
$$\leqq c_3(\varepsilon)\exp(-t(\Sigma(\eta,\theta)-\varepsilon/2)). \tag{3.20}$$

We now write the Weinberg–van Winter expansion (3.12) as the sum of two terms

$$e^{-tL(f,\theta)}=F_1(t,f,\theta)+F_2(t,f,\theta),$$

where F_1 is the sum over all disconnected strings and F_2 over connected strings. Similarly

$$e^{-tL'(\eta,\theta)}=F_1(t,\eta,\theta)+F_2(t,\eta,\theta).$$

Writing $e^{-tL(f,\theta)}=(F_1(t,f,\theta)+F_2(t,\eta,\theta))+(F_2(t,f,\theta)-F_2(t,\eta,\theta))$ we bound $F_1(t,f,\theta)$ using (3.20) and $F_2(t,\eta,\theta)$ using Lemmas 3.9 and 3.6 to get

$$\|e^{-tL(f,\theta)}\|\leqq c_4(\varepsilon)\exp(-t[\Sigma(\eta,\theta)-\varepsilon/2])+\|F_2(t,f,\theta)-F_2(t,\eta,\theta)\| \tag{3.21}$$

for all $t\geqq 0$ and $(\eta,\theta)\in\mathscr{S}$. Now let

$$\gamma=\mathrm{Max}\{1,c_4(\varepsilon)\}$$
$$T=(4\log(2\gamma))/\varepsilon.$$

Lemma 3.8 implies that we can choose $F_\varepsilon>0$ with $F_\varepsilon\leqq\tilde{F}_\varepsilon$ so that if $0<|f|<F_\varepsilon$, $(\eta,\theta)\in\mathscr{S}$ and $t\in(0,T]$ then

$$\|F_2(t,f,\theta)-F_2(t,\eta,\theta)\|\leqq\gamma e^{-t[\Sigma(\eta,\theta)-\varepsilon/2]}.$$

Under these conditions (3.21) then gives

$$\|e^{-tL(f,\theta)}\|\leqq 2\gamma e^{-t[\Sigma(\eta,\theta)-\varepsilon/2]}. \tag{3.22}$$

Setting $\|e^{-tL(f,\theta)}\|=\exp\varrho(t)$, (3.22) is equivalent to

$$\varrho(t)\leqq\log(2\gamma)-t(\Sigma(\eta,\theta)-\varepsilon/2)$$

so that for $t\in[T/2,T]$

$$\varrho(t)/t\leqq\frac{2}{T}\log(2\gamma)+\frac{\varepsilon}{2}-\Sigma(\eta,\theta)$$

and thus by the choice of T

$$\varrho(t)/t\leqq\varepsilon-\Sigma(\eta,\theta);\qquad t\in[T/2,T]. \tag{3.23}$$

Using the inequality $\|e^{-(t_1+\ldots+t_n)L(f,\theta)}\| \leqq \prod_{j=1}^{n} \|e^{-t_jL(f,\theta)}\|$ we find

$$\varrho(t_1+\ldots+t_n)/(t_1+\ldots+t_n) \leqq \sum_{j=1}^{n} (\varrho(t_j)/t_j)(t_j/t_1+\ldots+t_n).$$

Applying (3.23) for $t_j \in [T/2, T]$ gives

$$\varrho(t)/t \leqq \varepsilon - \Sigma(\eta,\theta) \tag{3.24}$$

for $t \in \left[\frac{nT}{2}, nT\right]$ for all n and thus we get (3.24) for all $t \geqq T/2$. Using the uniform bound $\|e^{-tL(f,\theta)}\| \leqq e^{tE}$ from Lemma 3.3 for $t \leqq T/2$ gives

$$\|e^{-tL(f,\theta)}\| \leqq \tilde{C}_\varepsilon e^{-t[\Sigma(\eta,\theta)-\varepsilon]}$$

for all $t>0$, all $|f|$ with $0<|f|<F_\varepsilon$, and all $(\eta,\theta) \in \mathscr{S}$. □

4. Stability, Analyticity, and Borel Summability of Resonances

In this section, we begin by combining the estimates of Theorem 3.1 with the stability method of Avron et al. [4] to obtain stability of eigenvalues of $H(|f|=0)$ turning into resonances of $H(|f|,\theta,\eta)$. By using the fact that we can allow $\eta \neq 0$ and the freedom to vary θ, we will obtain analyticity of the resonance energies for f in a fairly large region near $f=0$. Even in the two body case, these results are new, except that for the special case $V=r^{-1}$, Graffi and Grecchi [13] obtained the same result that we do with rather different methods. Indeed, their work motivated our discussion in this section. We finally turn to the consequences of our analyticity results for Borel summability recovering the result of Graffi–Grecchi for hydrogen [13] and proving new results for multielectron atoms. In the non-degenerate case or the case where degenercy is broken to first order, we obtain formulas linking the asymptotics of the Rayleigh-Schrödinger coefficients as $n\to\infty$ to the asymptotics of the widths as $f\downarrow 0$.

Definition. $\mathscr{R}_0=\{(\eta,\theta)|\theta=i\gamma,\ \gamma\in\mathbb{R},\ |\gamma|<\phi;\ \eta,\gamma$ obey (4.1–2)$\}$ where

$$0<\eta+3\gamma<\pi, \tag{4.1}$$

$$0<\eta+\gamma<\pi. \tag{4.2}$$

The conditions (4.1) and (4.2) are natural for the following reasons: (4.1) implies that we are in a region where for $|f|>0$ $H(|f|,\theta,\eta)$ is analytic. (4.2) implies that a neighborhood of any real $E_0<\Sigma_1=\inf\sigma_{ess}(H(|f|=\theta=0)$ will lie in the region of z obeying (3.2) for δ sufficiently small and H an H_D. Given these facts and Theorem 3.1 the following result is very easy; its proof is just the same as that of [16] for the case $\eta=0$, $N=1$ and is based on [4]. For this reason, we only sketch the details.

Theorem 4.1. *Let E_0 be a discrete eigenvalue of $H(|f|=0,\theta=0)$ of multiplicity n. Fix $\mathscr{R}_1$ compact in $\mathscr{R}_0$. Then, there exists ε and $F_0>0$ so that for $(\eta,\theta)\in\mathscr{R}_1$ and $|f|<F_0$, $H(|f|,\theta,\eta)$ has at most n spectral points within the disk $\{E|\,|E-E_0|\leqq\varepsilon\}=C$, each is*

an eigenvalue of finite multiplicity and the sum of their multiplicities is exactly n. As $|f|\downarrow 0$ *each of these eigenvalues converges to* E_0 *uniformly for* $(\eta,\theta)\in\mathscr{R}_1$.

Proof. Let $R(|f|,\theta,\eta;z)=(H(|f|,\theta,\eta)-z)^{-1}$. We rearrange the Weinberg–van Winter equation used in Sect. 2, to read

$$R=D+RI,$$

where D, I depend on $|f|,\theta,\eta$ and z. This version of the Weinberg-van Winter equation follows from the one in Sect. 2 for $R(|f|,\bar{\theta},-\eta;\bar{z})$ by taking adjoints. D and I above then have a slightly different meaning. Since $E_0<\Sigma_1$, we can find ε and F_1 so that (i) E_0 is the only spectral point of $H(|f|=\theta=0)$ within the disk C. (ii) For any non-trivial D, $(R_D(|f|,\theta,\eta)-z)^{-1}$ is uniformly bounded for $z\in C$, $(\eta,\theta)\in\mathscr{R}_1$ and $|f|<F_1$. To obtain (ii), we use Theorem 3.1 and $\Sigma_1\leqq\inf\operatorname{spec}(H_D(|f|=\theta=0))$. (iii) E_0 is the only point $z\in C$ where $1-I(|f|=0,\theta)$ is not invertible, for all θ with $(\eta,\theta)\in\mathscr{R}_1$ for some η.

By (ii), we find that D and I are uniformly bounded in the above region. Thus, since I converges *in norm* to $I(|f|=0)$ as $|f|\downarrow 0$ in the region where the perturbation series converges uniformly for $(\eta,\theta)\in\mathscr{R}_1$ (each term converges), I converges in norm uniformly for $z\in C$, $(\eta,\theta)\in\mathscr{R}_1$ by the Vitali theorem. It follows that for some F_0, $(1-I)$ is invertible for $|f|<F_0$, $z\in\partial C$ and $(\eta,\theta)\in\mathscr{R}_1$. Moreover,

$$R-D=DI(1-I)^{-1}$$

converges in norm since I is compact and norm convergent and D is strongly convergent. Since $\oint_{\partial C} R\,dE/2\pi i$ is a spectral projection and $\oint_{\partial C} D\,dE/2\pi i=0$ by analyticity, we get norm convergence of spectral projections. Since the disk C can be shrunk this yields stability. □

Definition. A discrete eigenvalue E of $H(|f|=\theta=0)$ with spectral projection P will be called *normal* if there exist mutually orthogonal projections $Q_1,\ldots,Q_\ell$ each commuting with $H(|f|=\theta=0)$, $X=\hat{e}\cdot\sum_{j=1}^{N+1}q_j(\mathbf{r}_j-\mathbf{R})$ $\left(X=\hat{e}\cdot\sum_{j=1}^{N}q_j\mathbf{r}_j \text{ if } m_{N+1}=\infty\right)$ and $\{U(\theta)|\theta \text{ real}\}$ so that (i) $\operatorname{Ran}P\subset\operatorname{Ran}\left(\sum_1^\ell Q_j\right)$. (ii) For each j, PQ_jXPQ_j is an operator on the finite dimensional subspace $\operatorname{Ran}(PQ_j)$ with distinct eigenvalues.

In practice, the Q_j's are projections onto vectors with certain quantum numbers under a symmetry commuting with dilations and leaving H and X fixed, usually rotations of some kind. Condition (ii) says that on each symmetry subspace, any degeneracy is removed in first order. This removal of degeneracy will prevent non-analyticities possible when one has a degenerate eigenvalue of a non-self adjoint operator like $H(\theta\neq 0)$.

Examples. 1. If E is non-degenerate, i.e., $\dim P=1$, it is automatically normal (take $\ell=1$ and $Q=1$).

2. Suppose that $\nu=3$ and all potentials are rotation invariant and that the rotation group acts irreducibly on $\operatorname{Ran}P$. Let Q_j be the projections on the eigenvectors of $\hat{e}\cdot L$. Then each PQ_j is rank 1 so (ii) is automatic. E is thus normal.

3. *Let* $H(|f|=\theta=0)=-\Delta-1/r$. Let $E=-1/4N^2(N=1,\ 2,\ldots)$. Then E has degeneracy N^2. By using "parabolic quantum numbers", see [15], one can label these N^2 states with quantum numbers $m=0,\ \pm 1,\ \pm 2, \ldots$ and $k, q=1,2,\ldots$ so that $m+k+q-1=N$ (all values allowed). m is just the eigenvalue of L_z so after restricting to suitable symmetry subspaces we have labels $k=1,2,\ldots,N-m$, and $q=N-m-k-1$. The eigenvalues of $PQ_m \times PQ_m$ are exactly $3(k-q)N$ (Lemma 6.4 of [15]) which are distinct for the different values of k on a fixed m subspace. Thus all these eigenvalues are normal.

Theorem 4.2. *Let* $H_0 \equiv H(|f|=\theta=0)$ *be an n-body Hamiltonian with pair potentials in* C_ϕ^M *and let* E_0 *be a normal discrete eigenvalue of* H_0 *with multiplicity* ℓ. *Then there exist for any* $\delta>0$, *an* $R_\delta>0$ *and* ℓ *functions* $E_1(f), \ldots, E_\ell(f)$ *analytic in* $\{f \mid |f|<R_\delta;\ -\min(\phi,\frac{\pi}{2})+\delta<\arg f<\pi+\min(\phi,\frac{\pi}{2})-\delta)\} \equiv K_\delta$ *so that*

(i) *For* θ *small with* $\operatorname{Im}\theta>0$ *and* $|f|$ *small, the* $E_i(|f|)$ *are precisely the eigenvalues of* $H(|f|,\theta,\eta=0)$ *near* E_0.

(ii) *If each two body potential is invariant under* $\mathbf{r}_i \to -\mathbf{r}_i$, *then for each j, there is a k (not necessarily distinct from j) so that for* $|f|$ *real and positive*

$$E_j(-|f|)=\overline{E_k(|f|)}\,. \tag{4.3}$$

(iii) *In the region* K_δ, *each* $E_k(f)$ *has a (Rayleigh-Schrödinger) perturbation series asymptotic to all orders.*

Remarks. 1. Even if E_0 is not normal, it should be possible to prove that (i), (ii) hold.

2. In the case where all E_0 eigenvectors have the same parity which will hold in most atoms, $j=k$ in (4.3). Of course, in hydrogen, where the E's are labelled by quantum numbers m, k, q as above, $E_{m,k,q}(-|f|)=\overline{E_{m,q,k}(+|f|)}$.

3. In an appendix, we prove a detailed bound on the errors in the Rayleigh-Schrödinger series, viz:

$$\left|E_k(f)-\sum_{n=0}^{N} a_n^k f^n\right| \leqq C^{N+1}|f|^{N+1}(N+1)! \tag{4.4}$$

Proof. Let $H(|f|,\theta,\eta)\restriction \operatorname{Ran}Q_j=H_j(|f|,\theta,\eta)$ and suppose the eigenvalue E_0 of $H_j(|f| =\theta=0)$ has multiplicity m_j. Then by the usual dilation analytic machinery for $f=0$, $H_j(|f|=0,\theta)$ has E_0 as a *semisimple* eigenvalue of multiplicity m_j. By our stability result, Theorem 4.1, $H_j(|f|,\theta,\eta)$ has exactly m_j eigenvalues (total algebraic multiplicity) near E_0 for small $|f|$ and they all converge to E_0 as $|f|\downarrow 0$. By a theorem of Kato [18, p. 443] these eigenvalues $E_{jk}(f,\theta)$ $(f=|f|e^{i\eta})$ can be written

$$E_{jk}(f,\theta)=E_0+f\mu_{jk}+o(|f|)\,; \quad k=1,\ldots,m_j\,, \tag{4.5}$$

where the μ_{jk} are the eigenvalues of $Q_jP(\theta)Xe^\theta P(\theta)Q_j\restriction \operatorname{Ran}Q_j$ where

$$P(\theta)=(2\pi i)^{-1} \oint_{|z-E_0|=\varepsilon} dz(z-H(|f|=0,\theta))^{-1}$$

and ε is sufficiently small. By the usual dilation analyticity arguments the μ_{jk} are the eigenvalues of $Q_jPXPQ_j\restriction \operatorname{Ran}Q_j(P=P(0))$ and thus by assumption, they are distinct. An analysis of Kato's argument [18, p. 443] combined with our stability

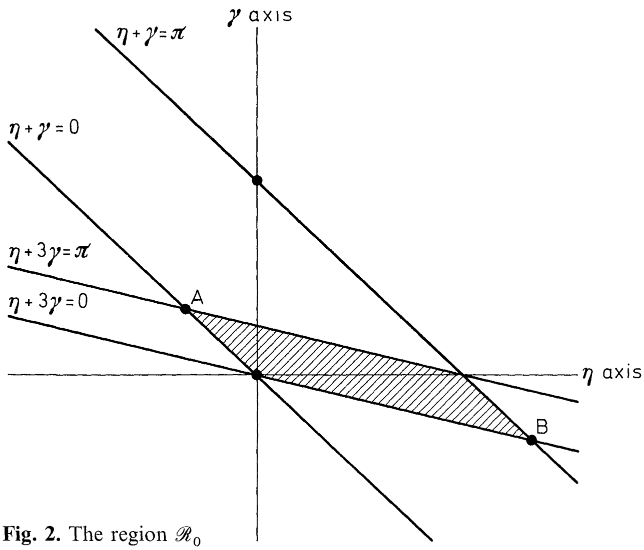

Fig. 2. The region $\mathscr{R}_0$

argument shows that (4.5) holds uniformly in any region of the form

$$\tilde{K}_{\tilde{\delta}} = \{(f,\theta) \mid |f| < F_{\tilde{\delta}}, \tilde{\delta} < \operatorname{Arg} f + \operatorname{Im}\theta < \pi - \tilde{\delta}, \tilde{\delta} < \operatorname{Arg} f + 3\operatorname{Im}\theta < \pi - \tilde{\delta}, |\operatorname{Im}\theta| < \phi < \pi - \tilde{\delta}, |\operatorname{Im}\theta| < \phi - \tilde{\delta}\},$$

where $\tilde{\delta} > 0$ is arbitrary and $F_{\tilde{\delta}}$ is suitably chosen. Because of (4.5) and the fact that the μ_{jk}, $k = 1, \ldots, m_j$ are distinct we can write the spectral projection for eigenvalue $E_{jk}(f,\theta)$ as

$$P_{jk}(f,\theta) = (2\pi i)^{-1} \int_{\Gamma_{jk}} (z - H_j(|f|, \theta, \arg f))^{-1} dz,$$

where Γ_{jk} is a circle surrounding $E_{jk}(f,\theta)$ and no other [of radius $= O(|f|)$]. $P_{jk}(f,\theta)$ is thus analytic in regions of the form $\tilde{K}_{\tilde{\delta}}$ as is

$$E_{jk}(f,\theta) P_{jk}(f,\theta) = (2\pi i)^{-1} \int_{\Gamma_{jk}} z(z - H_j(|f|, \theta, \arg f))^{-1} dz$$

and thus $E_{jk}(f,\theta)$ shares this analyticity property. By the usual arguments $E_{jk}(f,\theta)$ is independent of θ and thus relabelling we have $\ell = \sum_j m_j$ functions, $E_i(f)$, analytic in $\bigcup_{\theta, |\operatorname{Im}\theta| < \phi} \{f \mid (f,\theta) \in \tilde{K}_{\tilde{\delta}}\}$. In case $\phi > \frac{\pi}{2}$, the allowed region $\mathscr{R}_0$ is shown in Fig. 2. The extreme values of η occur at the points $A = (\gamma = \frac{\pi}{2}, \eta = -\frac{\pi}{2})$ and $B = (\gamma = -\frac{\pi}{2}, \eta = \frac{3\pi}{2})$.

If $\phi < \frac{\pi}{2}$, the given region must be sliced with the strip $|\gamma| < \phi$. The extremes lie on the lines $\eta + \gamma = 0$ and $\eta + \gamma = \pi$ and on the lines $\gamma = +\phi$, respectively $-\phi$. In either event, we obtain analyticity in region K_δ by choosing $\tilde{\delta}$ suitably. (iii) follows by the use of the standard theory of stable perturbations once we know that degeneracy is removed at first order.

To prove (ii), let V be the map $(Vf)(\mathbf{r}_i) = f(-\mathbf{r}_i)$ and note that

$$VH(|f|, \theta = 0, \eta = \tfrac{\pi}{2}) V^{-1} = \overline{H(|f|, \theta = 0, \eta = \tfrac{\pi}{2})},$$

where $\bar{A}f \equiv \overline{(A\bar{f})}$ with $\bar{f}$ the complex conjugate of f. Thus for f >0 and small, the set $\{E_1(if), \ldots, E_\ell(if)\}$ is invariant under complex conjugation. Hence for each j there is a k so that $\overline{E_j(-\bar{f})} - E_k(f)$ holds on a determining set (contained in the positive imaginary axis) and thus for all f in $\bigcup_{\delta>0} K_\delta$. In particular we obtain (4.3). □

Given the above analyticity result, the bound (4.4) and Watson's theorem [34, 14] in its sharpest form [20, 30] we obtain:

Theorem 4.3. *Let $E_k(f)$ be one of the eigenvalues given by Theorem 4.2. and let $\sum_{n=0}^{\infty} a_n f^n$ be its formal asymptotic series. Then for suitable $\tilde{R}>0$, the Borel transform*

$$B(z) = \sum_{n=0}^{\infty} a_n z^n/n!$$

defines an analytic function in the region

$$\{z \mid |z| < \tilde{R}\} \cup \{z \mid \tfrac{\pi}{2} - \min(\phi, \tfrac{\pi}{2}) < \arg z < \tfrac{\pi}{2} + \min(\phi, \tfrac{\pi}{2})\}\,.$$

In particular, if $\phi \geqq \frac{\pi}{2}$, $B(z)$ is analytic in the union of a half-plane and a semicircle. Moreover, given $\omega = e^{i\lambda}$ with $|\lambda| < \pi/2$ and f with $-\mathrm{Min}(\phi, \pi/2) < \arg f < \pi + \mathrm{Min}(\phi, \pi/2)$ then if $|\arg f + \lambda - \pi/2| < \min(\phi, \pi/2)$ and $|f|$ is sufficiently small

$$E_k(f) = \omega \int_0^{\infty} e^{-t\omega} B(tf\omega)\,dt\,. \tag{4.6}$$

In particular if $\phi > \pi/2$, and $0 < \lambda < \pi/2$ then (4.6) holds for all $f > 0$ which are sufficiently small.

Remark. Since $B(z)$ is real for z real, it is also automatically analytic in

$$\{z \mid -\pi/2 - \min(\phi, \pi/2) < \arg z < -\pi/2 + \min(\phi, \pi/2)\}\,.$$

In particular, if $\phi \geqq \pi/2$, B will be analytic in the plane with two "cuts" $(\tilde{R}, \infty)$ and $(-\infty, -\tilde{R})$ removed.

Finally, we want to note a relation between the a_n and the width $\Gamma_k(f)$ defined by

$$\Gamma_k(f) = -2\,\mathrm{Im}\,E_k(f)$$

for $f>0$:

Theorem 4.4. *Suppose all two body potentials satisfy $V_{ij}(\mathbf{r}) = V_{ij}(-\mathbf{r})$. Let $E_k(f)$ be one of the eigenvalues given by Theorem 4.2 and let $\sum_{n=0}^{\infty} a_n f^n$ be its formal asymptotic series. Let E_j be the level given by (4.3). Then for all sufficiently small R:*

$$a_n = -(2\pi)^{-1}\left[\int_0^R \lambda^{-n-1}[\Gamma_k(\lambda) + (-1)^n \Gamma_j(\lambda)]\,d\lambda\right] + O(R^{-n})\,. \tag{4.7}$$

Remarks. 1. If $j=k$, then $a_n = 0$ for n odd.

2. For the case of hydrogen, one can rigorously compute Γ asymptotically for small f and so, using (4.6), the asymptotics of a_n for all levels; see [15].

Proof. For n even, we write a Cauchy integral:

$$E_k(i\mu)=(2\pi i)^{-1}\int_C(\mu^2+\lambda^2)^{-1}(2\lambda)E_k(\lambda)d\lambda,$$

where C is the contour shown in Fig. 3. Taking the real part of both sides and

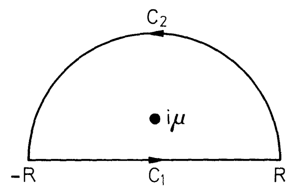

Fig. 3

expanding $(\mu^2+\lambda^2)^{-1}=\lambda^{-2}+(\mathrm{i}\mu)^2\lambda^{-4}+(\mathrm{i}\mu)^4\lambda^{-6}+\ldots$ we see that

$$a_{2n}=(2\pi)^{-1}\int_{-R}^{R}\lambda^{-2n-1}[2\,\mathrm{Im}\,E_k(\lambda)]d\lambda+O(R^{-2n}).$$

Since $2\,\mathrm{Im}\,E_k(\lambda)=-\Gamma_k(\lambda)$ for $\lambda>0$ and $2\,\mathrm{Im}\,E_k(-\lambda)=+\Gamma_j(\lambda)$ for $\lambda>0$, we obtain (4.6) for n even.

For odd n, we proceed in an identical way beginning with

$$E_k(i\mu)=(2\pi i)^{-1}\int_C(\mu^2+\lambda^2)^{-1}(2i\mu)E_k(\lambda)d\lambda$$

and this time take imaginary parts of both sides. □

5. Exponential Falloff of Resonances Wave Functions

In this section, we want to prove the following result:

Theorem 5.1. *Let ψ be an eigenfunction of an $H(|f|,\theta,\eta)$ in a region where this H has only discrete spectrum. Write $\mathbf{r}\equiv(\mathbf{r}_1,\ldots,\mathbf{r}_N)=(a,\mathbf{b})$ where a is the component of $\mathbf{r}$ coupled to the electric field and $\mathbf{b}$ is the orthogonal component. Then for all ε sufficiently small:*

$$\exp(\varepsilon|a|^{3/2}+\varepsilon|\mathbf{b}|)\psi(a,\mathbf{b})\in L^2.$$

Theorem 5.1 is of obvious interest. The proof follows ideas of Combes and Thomas [9] as extended by Simon [26].

Proof of Theorem 5.1. By Proposition II.4 of [16], we have the quadratic estimate

$$\|a\phi\|^2+\|\Delta\phi\|^2\leqq D[\|H_0(|f|,\theta,\eta)\phi\|^2+\|\phi\|^2] \tag{5.1}$$

for suitable D where a is the component of $\mathbf{r}$ coupled to the field. Since V is $-\Delta$-bounded with relative bound zero, (5.1) implies that $\|H_0(\theta,\eta)\phi\|^2$ is dominated by a multiple of $\|H(|f|,\theta,\eta)\phi\|^2+\|\phi\|^2$ so (5.1) holds with $H_0(|f|,\theta,\eta)$ replaced by $H(|f|,\theta,\eta)$.

Now let $F(\mathbf{r})=(a^2+1)^{3/4}+(b^2+1)^{1/2}$. Let $U(\alpha)$ be multiplication by $\exp(i\alpha F(\mathbf{r}))$ and let $H^{(\alpha)}=U(\alpha)HU(\alpha)^{-1}$. Since V and $|f|a$ are multiplication operators

$$H^{(\alpha)}=H^{(0)}+[\alpha\sum B_{ij}\partial_i\partial_jF+\alpha^2\sum C_{ij}(\partial_iF)(\partial_jF)]+\alpha\sum D_{ij}(\partial_iF)\partial_j. \tag{5.2}$$

Now $\partial_i\partial_j F$ is bounded and $\partial_i F$ is bounded by a multiple of $(1+|a|)^{1/2}$. It follows that the term in square brackets in (5.2) is $H^{(0)}$ bounded on account of the extended version of (5.1). Since

$$\begin{aligned}\|f_i\partial_i\phi\|^2 &= -\langle f_i^2\phi, \partial_i^2\phi\rangle - 2\langle f_i\partial_i f_i\phi, \partial_i\phi\rangle \\ &\leqq \tfrac{1}{2}\|f_i^2\phi\|^2 + \tfrac{1}{2}\|\partial_i^2\phi\|^2 + \|(f_i\partial_i f_i)\phi\|^2 + \|\partial_i\phi\|^2\end{aligned}$$

we see that the last term in (5.2) is also $H^{(0)}$ bounded.

Thus $H^{(\alpha)}$ extends from α real to an analytic family in $|\mathrm{Im}\,\alpha|<\delta$ for some $\delta>0$. It follows by the Combes–Thomas–O'Connor method [9, 21] that any discrete eigenvector of $H^{(0)}$ is analytic for $U(\alpha)$, i.e., $e^{\delta F(r)}\psi\in L^2$ for δ small. □

In the next section we will need the following theorem but only in the two-body case. We let $t=\sum_{i,j} a_{ij}\partial_i\partial_j$ with $A=\{a_{ij}\}$ a constant positive definite matrix and X a real linear function of $\mathbf{r}\in\mathbb{R}^n$.

Theorem 5.2. *Suppose W is a complex valued function on $\mathbb{R}^n$ with $t+\mathrm{Re}\,W\geqq$-const and ψ is an eigenfunction of $t+X+W$ in the sense that*

(i) $\psi\in\mathscr{D}(t+X)$.

(ii) *For each* $\phi\in C_0^\infty(\mathbb{R}^n)$, $\phi(t+X)\psi=\phi(E-W)\psi$.

Let $X_+=\mathrm{Max}[0,X]$ *and* $\sum_{i,j} a_{ij}(\partial_i X)(\partial_j X)=\gamma^2>0$. *Then if* $\alpha<\gamma^{-1}$

$$\exp(\tfrac{2}{3}\alpha X_+^{3/2})\psi\in L^2.$$

Proof. We use the identity ($\phi\in C_0^\infty(\mathbb{R}^n)$)

$$\phi t\phi=\tfrac{1}{2}t\phi^2+\tfrac{1}{2}\phi^2 t+\sum_{i,j} a_{ij}\partial_i\phi\partial_j\phi$$

to write (for ϕ real)

$$(\psi,\phi(t+X)\phi\psi)=\tfrac{1}{2}(\phi^2(t+X)\psi,\psi)+\tfrac{1}{2}(\psi,\phi^2(t+X)\psi)+\sum_{i,j} a_{ij}(\psi,\partial_i\phi\partial_j\phi\psi).$$

Using (ii) we have

$$(\psi,\phi(t+X+\mathrm{Re}(W-E))\phi\psi)=\sum_{i,j}(\psi,\partial_i\phi\partial_j\phi\psi)a_{ij}$$

and thus $t+\mathrm{Re}(W-E)\geqq -C$ gives

$$(\psi,\phi(X-C)\phi\psi)\leqq\sum_{i,j}(\psi,\partial_i\phi\partial_j\phi\psi)a_{ij}.$$

Without loss of generality we can take $C=0$ because we can translate X by C. We write $\phi(\mathbf{r})=\phi_0(\lambda\mathbf{r})g(X)$ with $g\in C^\infty(\mathbb{R})$, Xg^2, g' bounded and $\phi_0\in C_0^\infty(\mathbb{R}^n)$ with $\phi_0(0)=1$ and then take $\lambda\downarrow 0$. We find

$$(\psi,(g(X)^2X-\gamma^2 g'(X)^2)\psi)\leqq 0. \tag{5.3}$$

By a limiting argument, (5.3) is also valid if g' is piecewise continuous and bounded and $g^2(X)X$ is bounded. We choose

$$g(x)=\exp(\tfrac{2}{3}\alpha x^{3/2})g_R(x)$$

with

$$\begin{aligned} g_R(x) &= 0 && x \leqq 1/2 \\ &= 1 && 1 \leqq x \leqq R \\ &= \exp(-\tfrac{2}{3}(\alpha+\varepsilon)(x-R)^{3/2}) && R < x, \end{aligned}$$

where $0<\varepsilon<\gamma^{-1}$. Then if $\gamma\alpha<1$

$$\begin{aligned} g(x)^2 x - \gamma^2 g'(x)^2 &= (1-\gamma^2\alpha^2) x \exp(\tfrac{4}{3}\alpha x^{3/2}) && 1 \leqq x < R \\ &> 0 && R \leqq x. \end{aligned}$$

Since the contribution to (5.3) in the region $x>R$ is non-negative we have

$$(1-\gamma^2\alpha^2) \int_{1 \leqq X \leqq R} |\psi|^2 X \exp(\tfrac{4}{3}\alpha X^{3/2}) d\mathbf{r} \leqq \gamma^2 \int_{1/2 \leqq X < 1} |\psi|^2 g'(X)^2 d\mathbf{r} \leqq \tilde{C}, \tag{5.4}$$

where $\tilde{C}$ can be taken independent of R and α if $\alpha \leqq \gamma^{-1}$. Taking $R \uparrow \infty$ in (5.4) gives the desired result.

Remark. A similar technique for proving exponential L^2 bounds was used in [41] and for $X=0$ by Lavine [42].

6. Non-Vanishing of the Width

In this section we will give a proof of the following result:

Theorem 6.1. *Let W be real-valued and in C_ϕ^M for some $\phi>\pi/3$. Suppose in addition that for each $\varepsilon>0$ and each bounded open subset B of $\{\theta : |\mathrm{Im}\,\theta| < \phi - \varepsilon\}$ there is a splitting $W(\theta) = W_1(\theta) + W_2(\theta)$ where*

(i) $W_1(\theta)\,(|\mathbf{r}|+1)\,(-\Delta+1)^{-1}$ is compact and analytic in B.

(ii) $\sup_{\theta \in B} \|W_2(\theta)\| < \varepsilon$.

Let $f>0$ and $h_0 = -\Delta + fa(\mathbf{r})$ where $a(\mathbf{r})$ is a component of $\mathbf{r}$. Suppose that h_0+W obeys a unique continuation principle. Then h_0+W has no negative eigenvalues.

A result of this genre has already been obtained by Avron and Herbst [3] using different methods. (In [3] the eigenvalue is not restricted to be negative.) Here we will mimic an approach of Balslev [7] and Simon [25] and then explain the difficulties in extending the proof to N-body systems.

Proof of Theorem 6.1. We denote $h_0(\theta) = -\Delta e^{-2\theta} + fa(\mathbf{r})e^{\theta}$, $h(\theta) = h_0(\theta) + W(\theta)$. The proof will be broken into several steps:

1. *Resonance in* $|\mathrm{Im}\,\theta| < \pi/3$. Let E be a negative eigenvalue of $h = h(\theta = 0)$. For θ real, let $P(\theta)$ be the spectral projection onto the eigenvectors of $h(\theta)$ with eigenvalue E. By Corollary 2.5 $h(\theta)$ has E as an eigenvalue for $0 < |\mathrm{Im}\,\theta| < \pi/3$. For θ in the latter region define $P(\theta)$ (by a contour integral) as the associated spectral projection. Clearly, for f and g dilation entire and $0 < \mathrm{Im}\,\theta < \pi/3$:

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} i\varepsilon (U(\bar{\theta})f, (E+i\varepsilon-h)^{-1} U(\theta) g) &= \lim_{\varepsilon \downarrow 0} i\varepsilon (f, (E+i\varepsilon - h(\theta))^{-1} g) \\ &= (U(\bar{\theta})f, P(0)\, U(\theta) g) = (f, P(\theta) g) \end{aligned}$$

and similarly for $-\pi/3<\operatorname{Im}\theta\leqq 0$ so that $(f, P(\theta)g)$ is analytic in $|\operatorname{Im}\theta|<\pi/3$. Thus by a Phragmen–Lindelöf argument [2, 8]. $\|P(\theta)\|\leqq\|P(i\pi/6)\|$ for $|\operatorname{Im}\theta|<\pi/6$ and so by a density argument $P(\theta)$ is analytic in $|\operatorname{Im}\theta|<\pi/3$.

2. *Stability at* $\theta=\pm i\pi/3$. By sectoriality considerations, $(h_0(\theta)-z)^{-1}$ is uniformly bounded for $|z-E|<\frac{1}{2}|E|$ and $\frac{\pi}{6}\leqq\operatorname{Im}\theta\leqq\pi/3$ and is strongly continuous in the whole strip including up to $\operatorname{Im}\theta=\pm\pi/3$. We will show that

$$W(\theta)(h_0(\theta)-z)^{-1}\xrightarrow{\|\cdot\|}W(i\pi/3)(h_0(i\pi/3)-z)^{-1} \tag{6.1}$$

as $\theta\to i\pi/3$ with $\operatorname{Im}\theta<\pi/3$, uniformly in z for $|z-E|\leqq|E|/4$. Clearly we can replace W by a W_1 with $W_1(\theta)\,(1+|\mathbf{r}|)\,(-\Delta+1)^{-1}$ analytic and compact in $\{\theta:\frac{\pi}{6}<|\operatorname{Im}\theta|<\pi/3+\varepsilon,\ |\operatorname{Re}\theta|<1\}\equiv B$. The proof of (6.1) will be complete if we can show that for $\theta\in B$ and $|\operatorname{Im}\theta|\leqq\pi/3$

$$W_1(\theta)(h_0(\theta)-z)^{-1}=K_1(\theta,z)+K_2(\theta,z)(z-h_0(\theta))^{-1} \tag{6.2}$$

where K_j is compact and analytic for $\theta\in B$ and $|z-E|<|E|/2$. For then certainly K_j is norm convergent and by the compactness and analyticity of K_2, $K_2(\theta,z)\cdot(z-h_0(\theta))^{-1}$ is norm convergent with the right uniformity. To show (6.2) we use the resolvent equation

$$\begin{aligned}(z-h_0(\theta))^{-1}&=(z+\Delta e^{-2\theta})^{-1}+(z+\Delta e^{-2\theta})^{-1}fa(\mathbf{r})(z-h_0(\theta))^{-1}\\&=(z+\Delta e^{-2\theta})^{-1}+\{fa(\mathbf{r})(z+\Delta e^{-2\theta})^{-1}+f[(z+\Delta e^{-2\theta})^{-1},a(\mathbf{r})]\}\\&\quad\cdot(z-h_0(\theta))^{-1}\end{aligned}$$

and set

$$\begin{aligned}K_1&=W_1(\theta)(z+\Delta e^{-2\theta})^{-1},\\K_2&=W_1(\theta)fa(\mathbf{r})(z+\Delta e^{-2\theta})^{-1}+W_1(\theta)f[(z+\Delta e^{-2\theta})^{-1},a(\mathbf{r})].\end{aligned}$$

By arguments in [3], $W(i\pi/3)(h_0(i\pi/3)-z)^{-1}$ is compact so that $\sigma_{ess}(h(i\pi/3))=\{xe^{i\pi}/3:x\in\mathbb{R}\}$. Choose a circle C of the form $\{z:|z-E|=\delta\}$ with $\delta<|E|/4$, so that $1-W(i\pi/3)\,(z-h_0(i\pi/3))^{-1}$ is invertible on C. Then as in [4]

$$\begin{aligned}P(\theta)&=(2\pi i)^{-1}\int_C(z-h(\theta))^{-1}dz\\&=(2\pi i)^{-1}\int_C[(z-h(\theta))^{-1}-(z-h_0(\theta))^{-1}]dz\end{aligned}$$

is norm continuous as $\theta\to i\pi/3$ and converges to

$$P(i\pi/3)\equiv(2\pi i)^{-1}\int_C(z-h(i\pi/3))^{-1}dz.$$

Similarly

$$0=(h(\theta)-E)P(\theta)=(2\pi i)^{-1}\int_C(z-E)(z-h(\theta))^{-1}dz$$

converges to

$$(2\pi i)^{-1}\int_C(z-E)(z-h(i\pi/3))^{-1}dz=(h(i\pi/3)-E)P(i\pi/3)$$

so that $(h(i\pi/3)-E)P(i\pi/3)=0$.

Similarly we conclude

$$P(\theta) \xrightarrow[\theta \to -i\pi/3]{\|\cdot\|} P(-i\pi/3)$$

with $(h(-i\pi/3)-E)P(-i\pi/3)=0$.

3. Definition of $\eta(\theta)$. By O'Connor's lemma [23], since $P(\theta)$ is finite dimensional, any $\eta \in \operatorname{Ran} P(0)$ is such that $U(\theta)\eta \equiv \eta(\theta)$ has an analytic continuation to $|\operatorname{Im}\theta| < \pi/3$ continuous up to the boundary and $(h(\theta)-E)\eta(\theta)=0$ for $|\operatorname{Im}\theta| \leqq \pi/3$.

4. Falloff of $\eta(\pm i\pi/3)$. By Theorem 5.2, $\exp(\varepsilon a_+^{3/2})\eta(\pm i\pi/3) \in L^2$.

5. Completion of the Argument. We mimic [7] or [25] and conclude by using Carlson's theorem that for any $g \in C_0^\infty$ with $g=0$ for $a(\mathbf{r}) < A$ for some $A>0$ that

$$z(g, U(\tfrac{2}{3}\ln z)\eta)\,; \qquad \operatorname{Re} z \geqq 0$$

is identically zero. In particular $\eta(\mathbf{r})=0$ for $a(\mathbf{r})>0$ and thus by a unique continuation argument, $\eta \equiv 0$.

Since the Balslev–Simon work extends to N-bodies, one might expect the proof of Theorem 6.1 to extend to N-body systems but we have run into a number of technical difficulties, some of which we cannot solve:

A. Definition of $H(i\pi/3)$. Even for Coulomb potentials, it is not clear how to define $H(i\pi/3)$ as an operator sum. The problem is that the usual facts that C A-bounded implies $C \otimes I$ is $A \otimes I + I \otimes B$-bounded only holds in general if B and A are bounded below. However, using form methods, we can make a reasonable definition of $H(i\pi/3)$.

B. HVZ for $H(i\pi/3)$. Even for bounded potentials [when problem (A) is non-existent], we have been unable to prove that $\sigma_{ess}(H(i\pi/3))$ is where it should be. The problem is that Ichinose's lemma $\sigma(A \otimes I + I \otimes B) = \sigma(A) + \sigma(\mathrm{B})$ can fail for operators which only generate contraction semigroups (and not holomorphic semigroups). We note that even geometric methods of analyzing σ_{ess} [11, 27, 10] rely on an Ichinose lemma.

C. Stability at $i\pi/3$. Even if we knew about $\sigma_{ess}(H(i\pi/3))$, we do not see how to gain sufficient control on $(H(\theta)-z)^{-1}$ as $\theta \to i\pi/3$ to conclude continuity of $P(\theta)$.

In spite of these problems, it seems to us that it could be possible to extend the proof. In attempting this, we were shocked at our ignorance of spectral properties of generators of semigroups which fail to be holomorphic. Better understanding of these could be the key to extending Theorem 6.1.

Appendix A. Borel Summability for Degenerate Eigenvalues

In this appendix, we want to explain how to prove the bound (4.4) on normal eigenvalues. Our proof is patterned after the proof in [6] of the analogous result for the Zeeman problem (this same proof also works for bounds on the coefficients in the $1/R$ expansion of molecular physics [19]). We begin with the case where the unperturbed eigenvalue is simple (or simple after restriction to a symmetry subspace) and then discuss the more complicated case where the unperturbed eigenvalue is not simple but the degeneracy is removed to first order.

In either case one writes $E(f)=F(f)/G(f)$. In the non-degenerate case

$$F(f)=E_0G(f)+\oint_{|\lambda-E_0|=\varepsilon}(\phi,fW(H_0+fW-\lambda)^{-1}\phi)d\lambda, \tag{A.1}$$

$$G(f)=\oint_{|\lambda-E_0|=\varepsilon}(\phi,(H_0+fW-\lambda)^{-1}\phi)d\lambda \tag{A.2}$$

with $H_0=H(f=0,\theta,\eta)$ and $W=Xe^{\theta}$ the electric field term. ϕ is the unperturbed eigenvector. The formula $E(f)=F(f)/G(f)$ is a standard result in perturbation theory [23]. The perturbation series is obtained by expanding $(H_0-fW-\lambda)^{-1}$ in a series in f. To prove (4.4) we must control the error separately in F and G. We will prove the bound on the F; the same method controls G. Thus, we will prove that

$$\sup_{|\lambda-E_0|=\varepsilon}\|(H_0+fW-\lambda)^{-1}[W(H_0-\lambda)^{-1}]^n\phi\|\leqq C^{n+1}n!. \tag{A.3}$$

We prove (A.3) for fixed (η,θ) in $\mathscr{R}_0$ but it is easy to see that the estimates are uniform over compacts $\mathscr{R}_1$ in (η,θ) and hence (4.4) follows for f in sectors,

$$\{|f|<R_\delta;\,-\min(\tfrac{\pi}{2},\phi)+\delta<\arg f<\pi+\min(\tfrac{\pi}{2},\phi)-\delta\}.$$

(A.3) depends on the use of the scale of spaces $\mathfrak{H}_a=\{f|e^{a|x|}f\in L^2\}$ with the obvious norm. Since, for f small $\sup_{|\lambda-E_0|=\varepsilon}\|(H_0+fW-\lambda)^{-1}\|<\infty$ by stability, (A.3) requires that we show that

$$\|[W(H_0-\lambda)^{-1}]^n\phi\|\leqq C^{n+1}n!. \tag{A.4}$$

Fix a and view $(H_0-\lambda)^{-1}$ as a map from $\mathfrak{H}_{ak/n}$ to itself and W as a map from $\mathfrak{H}_{ak/n}$ to $\mathfrak{H}_{a(k-1)/n}$. Then

$$\text{L.H.S. of (A.4)}\leqq\prod_{j=1}^{n}[\|W\|_{j,j-1}\sup_{|\lambda-E_0|=\varepsilon}\|(H_0-\lambda)^{-1}\|_{j,j}]\|\phi\|_{\mathfrak{H}_a},$$

where $\|B\|_{j,\ell}$ is the norm of B as a map from $\mathfrak{H}_{ja/n}$ to $\mathfrak{H}_{\ell a/n}$. Now

$$\|W_{j,j-1}\|\leqq c\|xe^{-|x|a/n}\|_\infty=\tilde{c}n.$$

Also, by interpolation, the $\|\cdot\|_{j,j}$ norm is bounded by the maximum of the $\|\cdot\|_{0,0}$ and the $\mathfrak{H}_a$ to $\mathfrak{H}_a$ norm. Thus since $n^n\leqq n!B^n$, and $\phi\in\mathfrak{H}_a$ for a suitable (A.4) holds if we show that for ε fixed and small:

$$\sup_{|\lambda-E_0|=\varepsilon}\|(H_0-\lambda)^{-1}\|_{a,a}<\infty. \tag{A.5}$$

(A.5) is an estimate of Combes and Thomas [9] similar to that we used in Sect. 5.

This describes the proof of (4.4) in the non-degenerate case. We describe the extension to the case where degeneracy is removed to first order as a series of steps (we take $Q_1=I$ without loss of generality):

1. We begin with the standard method [18] of reduction to a non-degenerate problem. Add a constant to W so that the eigenvalues of P_0WP_0 $\left(P_0=(2\pi i)^{-1}\right.$ $\left.=(2\pi i)^{-1}\oint_{|z-E_0|=\varepsilon}(H_0-z)^{-1}dz\right)$

are all non-zero. This will not affect the errors for $n \geqq 2$. Define

$$\hat{H}(f) = P(f)[H(f) - E_0]P(f)/f \equiv \hat{H}(0) + V(f).$$

Then $\hat{H}(0) = P_0 W P_0$ has non-degenerate θ-independent spectrum away from 0 and, if the non-zero eigenvalues of $\hat{H}(f)$ are $\lambda_1(f), \ldots, \lambda_1(f)$; then the eigenvalues of $H(f)$ near E_0 are just $E_0 + f\lambda_i(f)$.

2a. For some $a > 0$, $\|(z - H(f))^{-1}\|_{b,b} \leqq$ const uniformly in $0 \leqq b \leqq a$, (η, θ) in $\mathscr{R}_1$ ($\mathscr{R}_1 \subsetneqq \mathscr{R}_0$ and compact), $|f|$ small and $|z - E_0| = \varepsilon$ for some $\varepsilon > 0$ which is as small as desired. To prove this we use the stability argument of Sect. 3 for $H(f, \theta, \boldsymbol{\alpha}) = e^{\boldsymbol{\alpha}\cdot\mathbf{r}} H(f, \theta)\, e^{-\boldsymbol{\alpha}\cdot\mathbf{r}}$ to obtain a bound on $(z - H_D(f, \theta, \boldsymbol{\alpha}))^{-1}$ uniform in small $|f|$, (η, θ) in $\mathscr{R}_1$, $|\boldsymbol{\alpha}|$ small and z near E_0. Here D is a non-trivial cluster decomposition. The argument in the proof of Theorem 4.1 using the Weinberg–van Winter equation then leads to the desired result.

2b. For some $a > 0$, $\|(z - \hat{H}(0))^{-1}\|_{b,b} \leqq$ const uniformly in $0 \leqq b \leqq a$, θ in compacts of $\{\theta : |\mathrm{Im}\,\theta| < \phi\}$ $|z - \mu_i| = \varepsilon$ where μ_i are the eigenvalues of $P_0 W P_0 = \hat{H}(0)$. This is easily seen from the relation $(z - \hat{H}(0))^{-1} = z^{-1}(1 - P_0) + P_0(z - \hat{H}(0))^{-1} P_0$, the fact that $\|(z - \hat{H}(0))^{-1}\|$ is uniformly bounded for the above z and the bound $\|e^{a|\mathbf{r}|} P_0\| \leqq$ const uniformly in θ for the above θ.

2c. $\|(z - \hat{H}(f))^{-1}\|$ is uniformly bounded in f, θ, z for (θ, η) in $\mathscr{R}_1$, $|f|$ small and $|z - \mu_j| = \varepsilon$ with $\varepsilon > 0$ and small. To see this we write

$$(z - \hat{H}(f))^{-1} = z^{-1}(1 - P(f)) + \sum_i (z - \lambda_i(f))^{-1} P_i(f)$$

where $P_i(f)$ is the spectral projection associated with $\hat{H}(f)$ for eigenvalue $\lambda_i(f)$, defined by a contour integral. We have already seen that $\lambda_i(f) \to \mu_i$ uniformly for f in the relevant region. By an argument of Kato [18, p. 446] easily made uniform in the surpressed variable θ, $P_i(f) \xrightarrow{\|\cdot\|} P_i(0)$ where $P_i(0)$ is the spectral projection for eigenvalue μ_i associated with $\hat{H}(0) = P_0 W P_0$.

3. Using $P(f) = (2\pi i)^{-1} \oint_{|z - E_0| = \varepsilon} (z - H(f))^{-1} dz$ in the definition of $\hat{H}(f)$, and the expansion

$$(z - H(f))^{-1} = \sum_{k=0}^{n} (z - H_0)^{-1} (W(z - H_0)^{-1})^k f^k + f^{n+1} (z - H(f))^{-1} (W(z - H_0)^{-1})^{n+1}$$

one can write

$$V(f) = \sum_{n=1}^{m} A_n f^n + R_{m+1}(f),$$

where A_n is a sum of $0(n)$ integrals of products of $(z - H_0)^{-1}$'s and W's, each term containing exactly $n + 1\,W$'s while R_{m+1} has $O(m)$ terms, each term with $m + 2\,W$'s but in addition to $(z - H_0)^{-1}$'s each term will contain some $(z - H(f))^{-1}$'s.

The perturbation series is obtained writing $\lambda_j(f)$ as a ratio of

$$\left(\phi_j, \int_{|z - \mu_j| = \varepsilon} dz (z - \hat{H}(f))^{-1} z \phi_j\right) \quad \text{to} \quad \left(\phi_j, \int_{|z - \mu_j| = \varepsilon} dz (z - \hat{H}(f))^{-1} \phi_j\right)$$

where ϕ_j is the eigenvector of $P_0 W P_0$ of eigenvalue μ_j $(\lambda_j(f) \to \mu_j)$ and then expanding $(z - \hat{H}(f))^{-1} = (z - \hat{H}(0) - V(f))^{-1}$ to n^{th} order in $V(f)$ and collecting terms of order $\leqq n$ in f. As in the non-degenerate case, we need only prove the $n!$ bound on the remainder for numerator and denominator separately. The error is a sum of terms which we obtain as follows: In k^{th} order in $V(f)$ we have $(\hat{H}(0)-z)^{-1}(V(f)\,(\hat{H}(0)-z)^{-1})^k$. Write the first $V(f)$ as $\sum_{\ell=1}^{n-k+1} A_\ell f^\ell + R_{n-k+2}(f)$. The term

$$(\hat{H}(0)-z)^{-1}R_{n-k+2}(f)\,(\hat{H}(0)-z)^{-1}(V(f)\,(\hat{H}(0)-z)^{-1})^{k-1}$$

goes into the error bin. In the term involving $A_j f^j$ we expand the second $V(f)$ to order $n-j-k+2$ and throw

$$(\hat{H}(0)-z)^{-1}A_j f^j(\hat{H}(0)-z)^{-1}R_{n-j-k+3}(f)\,(\hat{H}(0)-z)^{-1}(V(f)\,(\hat{H}(0)-z)^{-1})^{k-2}$$

into the error bin, etc. In the error terms we now expand A_j and R_k as sums of products of resolvents and W's [$V(f)$ can be considered as R_1 in this procedure]. In the error resulting from truncating the geometric series for $(z-\hat{H}(f))^{-1}$, namely

$$(\hat{H}(f)-z)^{-1}(V(f)\,(\hat{H}(0)-z)^{-1})^{k+1}$$

we use the bound

$$\|(\hat{H}(f)-z)^{-1}(V(f)\,(\hat{H}(0)-z)^{-1})^{n+1}\phi_j\| \leqq c\|(V(f)\,(\hat{H}(0)-z)^{-1})^{n+1}\phi_j\|$$

which follows from (2c). Thus this error is of the same form as the ones already considered. We estimate the total error by the product of the number of terms times an upper bound on the size of any term.

4. Let us show that the total number of terms is $O(c^n)$ for some $c>1$. We first consider the error from $(\hat{H}(0)-z)^{-1}(V(f)\,(\hat{H}(0)-z)^{-1})^k$. From expanding the first $V(f)$ our error is schematically (leaving out all resolvents)

$$= R_{n-k+2}R_1^{k-1}$$

which has $O(n-k+2)O(c^{k-1}) \leqq O(n+1)c^k$ terms. From expanding the second V we get an error of the form

$$\sum_{\substack{j_1, j_2 \geqq 1 \\ j_1+j_2=n+1-(k-2)}} A_{j_1} f^{j_1} R_{j_2}(R_1)^{k-2}$$

which has on the order of $c^k \sum_{\substack{j_1, j_2 \geqq 1 \\ j_1+j_2=n+1-(k-2)}} j_1 j_2 \leqq c^k \sum_{\substack{j_1, j_2 \geqq 1 \\ j+j_2=n+1}} j_1 j_2$ terms.

Continuing in this way we see that the error resulting from $(\hat{H}(0)-z)^{-1}(V(f)\,(\hat{H}(0)-z)^{-1})^k$ has at most N_k terms with

$$N_k = c^k\left(\sum_{j_1=1}^{n+1} j_1 + \sum_{j_1+j_2=n+1} j_1 j_2 + \ldots + \sum_{j_1+\ldots+j_k=n+1} j_1 j_2 \ldots j_k\right).$$

Now subject to $\sum_{i=1}^{\ell} x_i = 1$, $x_i \geqq 0$, $x_1 \ldots x_\ell$ attains its maximum with all x_i's equal so $\sum_{j_1+\ldots+j_\ell=n+1} j_1 \ldots j_\ell \leqq \left(\frac{n+1}{\ell}\right)^\ell \cdot$(number of decompositions $j_1+\ldots+j_\ell=n+1$ with $j_i \geqq 1) = \left(\frac{n+1}{\ell}\right)^\ell \binom{n}{\ell-1} \leqq \left(1+\frac{n+1}{\ell}\right)^\ell \binom{n}{\ell-1} \leqq e^{n+1}\binom{n}{\ell-1}$. Thus

$$N_k \leqq c^k \sum_{\ell=1}^{k} \binom{n}{\ell-1} e^{n+1} \leqq c^n 2^n e^{n+1}.$$

Adding all terms from $k=1, \ldots, n$ and the geometric series truncation error terms we have $O(c^n)$ terms for some $c>1$.

5. Each term that we must bound is an integral of a product of resolvents and at most $n+2W$'s and thus by the method used in the non-degenerate case is bounded by $c^n n!|f|^{n+1}$ for some c.

This completes the sketch of the proof of (4.4) when the eigenvalue is degenerate but degeneracy is removed to first order. □

Appendix B. Essential Spectrum in the Ineffective Case

Our goal in this section is to identify the essential spectrum in case the charges and masses are ineffective. We let $\mathfrak{J}$ be the family of all decompositions $D=\{C_1, \ldots, C_k\}$ with $Q_i/M_i=Q_j/M_j$ for all i, j, and $k>1$. Recall (Sect. 2) that ineffective charges and masses are precisely those with $\mathfrak{J} \neq \emptyset$. We will prove:

Theorem B.1. *Let*

$$\tilde{H}(\theta)=-\sum_{i=1}^{N}(2m_i)^{-1}\Delta_i e^{-2\theta}+\sum_{i<j}V_{ij}(e^{\theta}(\mathbf{x}_i-\mathbf{x}_j)) + e^{\theta}\hat{e}\cdot\sum_i q_i\mathbf{x}_i \tag{B.1}$$

and let $H(\theta)$ be the corresponding operator with center of mass removed. Suppose all $V_{ij}\in C_\phi^M$, the Combes class. Then for all θ with $0<|\mathrm{Im}\,\theta|<\min(\phi, \frac{\pi}{3})$, we have that

$$\sigma_{ess}(H(\theta))=\bigcup_{\{C_1,\ldots,C_k\}\in\mathfrak{J}}\{\mu_{\alpha_1}+\ldots+\mu_{\alpha_k}+\lambda e^{-2\theta}|\lambda\geq 0, \mu_{\alpha_i}\in\sigma(H_{C_i}(\theta))\}. \tag{B.2}$$

Given this result, one easily sees inductively that

Corollary B.2. *Under the above conditions*

$$\sigma_{ess}(H(\theta))=\bigcup_{\{C_1,\ldots,C_k\}\in\mathfrak{J}}\{\mu_{\alpha_1}+\ldots+\mu_{\alpha_k}+\lambda e^{-2\theta}|\lambda\geqq 0, \mu_{\alpha_i}\in\sigma_{\mathrm{disc}}(H_{C_i}(\theta))\}. \tag{B.3}$$

The proof of this corollary depends on the observation that if $D=\{C_1, \ldots, C_k\}$ is an ineffective decomposition and if $D_1=\{C_{11}, \ldots, C_{1k_1}\}, \ldots$ are ineffective decompositions of $C_1, \ldots$, then $D_1 \ldots D_k$ is an ineffective decomposition.

By the standard argument of Hunziker [38], the RHS of (B.3) is contained in $\sigma(H(\theta))$, so again using induction, to prove Theorem B.1 we need only show that the RHS of (B.2) contains $\sigma_{ess}(H(\theta))$.

Henceforth we suppose that $\mathrm{Arg}\,\theta\in(0,\pi/3)$ so that $\mathrm{Re}(ie^{-3\theta})>0$.

Mainly we will require a strong form of Ichinose's Lemma. The following result of Herbst [39] uses in part ideas of Gearhart [40].

Definition. The generator, A, of a strongly continuous semigroup e^{-tA} $(t\geqq 0)$ on a separable Hilbert space, $\mathfrak{H}$, is said to have *contained spectrum* if and only if $\|e^{-tA}\| \leqq e^{t\omega}$ for some ω and if

(a) For all E, there is a y_E so that

$$\sigma(A)\cap\{z|\mathrm{Re}\,z\leqq E\}\subsetneqq\{z|\,|\mathrm{Im}\,z|<y_E\}.$$

(b) $\sup_{\substack{\mathrm{Re}\,z<E\\|\mathrm{Im}\,z|\geqq y_E}}\|(z-A)^{-1}\|<\infty$.

The following follows from [40]:

Lemma B.3. *If A has contained spectrum, then*

$$\sigma(e^{-tA})\backslash\{0\} = e^{-t\sigma(A)}$$

for all $t>0$.

Lemma B.4 [39]. *Let* $A_1, \ldots, A_k$ *be operators with contained spectrum on Hilbert spaces* $\mathfrak{H}_1, \ldots, \mathfrak{H}_k$. *Let A be the operator on* $\mathfrak{H}_1 \otimes \ldots \otimes \mathfrak{H}_k$ *which generates the semigroup* $e^{-tA_1} \otimes \ldots \otimes e^{-tA_k}$ *so that formally*

$$A = A_1 \otimes I \otimes \ldots \otimes I + I \otimes A_2 \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes A_k.$$

Then A has contained spectrum and

$$\sigma(A) = \sum_{i=1}^{k} \sigma(A_i).$$

In addition to these results we will need the following result of Phragmén–Lindelöf type whose proof is standard:

Lemma B.5. *Let* $F(z)$ *be an operator valued analytic function in a neighborhood of* $S \equiv \{z | 0 \leqq \operatorname{Re} z \leqq 1,\ \operatorname{Im} z \geqq 0\}$. *If* $A \equiv \sup_{z \in S} \|F(z)\| < \infty$ *and* $\lim_{y \to \infty} \|F(iy)\| = 0$, *then for any* $\theta < 1$:

$$\lim_{y \to \infty} \sup_{0 \leqq x \leqq \theta} \|F(x+iy)\| = 0.$$

Proof. Apply the maximum principle to the function

$$e^{+(z-iy)^2} e^{-Bz} F(z); \qquad B>0$$

in the region $0 \leqq \operatorname{Re} z \leqq 1$, $\frac{y}{2} \leqq \operatorname{Im} z \leqq \frac{3y}{2}$ to find that

$$\sup_{0 \leqq x \leqq \theta} \|F(x+iy)\| \leqq c e^{-y^2/4} e^{B\theta} + c e^{-B(1-\theta)} + e^{1+B\theta} \sup_{a \geqq y/2} \|F(ia)\|.$$

Choosing B suitably the result follows.

Finally we need the following:

Lemma B.6. *Let* $H_0(\theta)$ *be the operator* $H(\theta)$ *when all* $V_{ij}=0$, *let* $L_0 = ie^{-\theta} H(\theta)$ *and let* $V_1, \ldots, V_m$ *be (not necessarily distinct), two body potentials so that*

$$F(y) \equiv (E_0 + L_0 + iy)^{-1} V_1 (E_0 + L_0 + iy)^{-1} \ldots (E_0 + L_0 + iy)^{-1} V_m$$

is compact (i.e., the V's determine a connected diagram). Then for each E_0,

$$\lim_{|y| \to \infty} \|F(y)\| = 0.$$

Proof. By a limiting argument, it suffices to consider the case where each V_{ij} is in C_0^∞ (as a function of $\mathbf{r}_i - \mathbf{r}_j$). Then

$$F(y) = \int_0^\infty dt_1 \ldots dt_m g(t_1, \ldots, t_m) e^{-iy(t_1 + \ldots + t_m)}$$

with

$$g(t_1, \ldots, t_m) = e^{-t_1 L_0} V_1 \ldots e^{-t_m L_0} V_m e^{-(t_1 + \ldots + t_m) E_0}.$$

Now g is compact; indeed it is Hilbert–Schmidt as can be seen by using the explicit integral kernel of e^{-tL_0}, so that g takes values in a separable space; clearly g is measurable and since $\|e^{-sL_0}\| \leqq Ce^{-Ds^3}$, $\|g\|$ is in L^1. Thus, by the Riemann–Lebesgue lemma, $\|F(y)\| \to 0$ as $y \to \infty$. □

Proof of Theorem B.1. As usual let $L^C(\theta) = ie^{-\theta} H^C(\theta)$, $L(\theta) = ie^{-\theta} H(\theta)$ and $L_D(\theta) = ie^{-\theta} H_D(\theta)$. We will prove inductively that each $L^C(\theta)$ has contained spectrum and that

$$\sigma_{ess}(L(\theta)) \subseteqq \bigcup_D \sigma(L_D(\theta)). \tag{B.4}$$

If $D = \{C_1, \ldots, C_k\}$, then

$$L_D(\theta) = L^{C_1} \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes L^{C_k} \otimes I + I \otimes \ldots \otimes \tilde{L} \tag{B.5}$$

with $\sigma(\tilde{L}) = \emptyset$ for effective D and $\sigma(\tilde{L}) = \{ie^{-3\theta}\lambda | \lambda \geqq 0\}$ if D is ineffective. Thus (B.4), the result on contained spectrum which we will prove and Lemma B.4 complete the proof of Theorem B.1.

We begin the induction by noting that in case $N = 1$, $L(\theta)$ is easily seen to have contained spectrum. Thus suppose we know every $L^C(\theta)$ with $C \subsetneqq \{1, \ldots, N\}$ has contained spectrum. By (B.5) and Lemma (B.4) each $L_D(\theta)$ has contained spectrum, so using the Weinberg–van Winter equation

$$(L(\theta) - z)^{-1} = D(\theta; z) + I(\theta; z)(L(\theta) - z)^{-1}. \tag{B.6}$$

we see that D and I are bounded as $|\mathrm{Im}\, z| \to \infty$ with $\mathrm{Re}\, z$ bounded from above. Expanding I in diagrams when $\mathrm{Re}\, z$ is very negative and using Lemma B.6, we see that $\|I(\theta, z)\| \to 0$ as $|\mathrm{Im}\, z| \to \infty$ with $\mathrm{Re}\, z$ very negative. But then exploiting Lemma B.5:

$$\lim_{\substack{|\mathrm{Im}\, z| \to \infty \\ \mathrm{Re}\, z \leqq E}} \|I(\theta, z)\| = 0.$$

By the Weinberg–van Winter equation (B.5) we conclude that $L(\theta)$ has contained spectrum.

(B.4) follows in the usual way from the Weinberg–van Winter equation if we note that by inductively proving (B.2) we know that $\{z | \mathrm{Re}\, z < E\} \setminus \bigcup_D \sigma(L_D(\theta))$ is connected. □

Appendix C. Some Estimates

The object of this appendix is to prove certain estimates which one of us has used elsewhere [28].

Theorem C.1. *Let $h_0 = -\Delta + f x_1$, $f > 0$. Then if $\chi_{[0,\infty)}$ is the characteristic function of $[0, \infty)$*

$$x_1 \chi_{[0,\infty)}(x_1)(h_0 - z)^{-1},\ \sqrt{x_1}\, \chi_{[0,\infty)}(x_1) \partial_j (h_0 - z)^{-1},\ \chi_{[0,\infty)}(x_1) \Delta (h_0 - z)^{-1}$$

are bounded operators for $z \notin \sigma(h_0)$.

Proof of Theorem C.1. By first making a scale transformation we see that f can be set equal to 1 without loss of generality. Let $p_j = -i\partial_j$, $p^2 = -\Delta$ and let θ be a non-negative function in $C^\infty(\mathbb{R})$ with $\theta(x)=1$ if $x \geqq 1$, $\theta(x)=0$ if $x \leqq 0$. Let $A=\theta(x_1)(p^2+x_1)$ with domain C_0^∞. We compute

$$\begin{aligned} A^*A &= p^2\theta^2p^2 + x_1^2\theta^2 + x_1\theta^2p^2 + p^2\theta^2x_1 \\ &= p^2\theta^2p^2 + x_1^2\theta^2 + 2\sum_j p_jx_1\theta^2p_j + [p_1,[p_1,x_1\theta^2]]. \end{aligned}$$

Since $[p_1,[p_1,x_1\theta^2]] = -\dfrac{d^2}{dx_1^2}(x_1\theta^2) \geqq -c$ we have for $\psi \in C_0^\infty$

$$\|A\psi\|^2 + c\|\psi\|^2 \geqq \|\theta(x_1)p^2\psi\|^2 + \|x_1\theta(x_1)\psi\|^2 + 2\sum_j \|\sqrt{x_1}\,\theta(x_1)p_j\psi\|^2. \quad \text{(C.1)}$$

Using the fact that C_0^∞ is a core for p^2+x_1 and that for $\mathrm{Im}\, z \neq 0$, $A(p^2+x_1-z)^{-1}$ is bounded we conclude from (C.1) that $\theta(x_1)p^2(h_0-z)^{-1}$, $x_1\theta(x_1)\,(h_0-z)^{-1}$ and $\sqrt{x_1}\,\theta(x_1)p_j(h_0-z)^{-1}$ are bounded. Translating x_1 by 1 we see that the above operators are bounded when x_1 is replaced by x_1+1 and $\theta(x_1)$ by $\theta(x_1+1)$. Multiplying by $\chi_{[0,\infty)}(x_1)$ from the left leads to the desired result. □

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Communicated by J. Ginibre

Received October 6, 1980