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PHASE SPACE ANALYSIS OF SIMPLE SCATTERING SYSTEMS: EXTENSIONS OF SOME WORK OF ENSS

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Contents

§1. **Introduction.** In recent years, the spectral and scattering theory of partial differential operators on $\mathbb{R}^\nu$ has been an extensively studied subject, especially for operators which resemble Schrödinger operators; see [28, 29] for a comprehensive review. Although scattering is basically a time-dependent phenomenon, very few results have been obtained with time-dependent methods. Indeed, the main time-dependent technique is Cook's method [28, §XI.3] which traditionally yields only existence of wave operators. There are certain methods, most notably the Kato-Birman (trace class) theory [28, §XI.3] and the theory of smooth perturbations [29, §XIII.7] which have both time-dependent and time-independent versions but the sharpest results have seemed to require time-independent methods, most notably the Agmon-Kuroda analysis of weighted L^2 estimates [29, §XIII.8].

This situation has been dramatically changed by an exciting and beautiful paper of Enss [13] who uses purely time-dependent methods to obtain virtually identical results to those of the Agmon-Kuroda theory in the case $H = -\Delta + V$ with V a multiplication operator. Particularly exciting developments suggested by Enss' tour de force involve the inclusion of Coulomb potentials and the extension to multiparticle systems. Enss [14] has solved the first of these and has made substantial progress on the second! In this paper, we want to explore the more straightforward extension of studying $H = H_0 + V$ for more general H_0 and V where V is still "localized" in a bounded region of space and H_0 is still an operator with "no scattering" in a geometric sense. It is hoped that these

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generalizations will help illustrate what is going on in Enss' paper. In addition, they show that Enss' method is capable of recovering virtually all the known results on H's of the above form.

In fact, it is already clear from Enss' original paper that his methods have striking applicability. In the first place, he only needs that V be "localized" in the sense that there is some falloff at infinity. He does not require that V be "local," i.e., a multiplication operator, or even "pseudo-local," i.e., that $(f, Vg) = 0$ if f, g are in C_0^∞ with disjoint supports. (Note that we distinguish between "local" and "localized.") That localization suffices for Cook's method is a result that goes back at least to Jörgens-Weidmann [21]. On the other hand, pseudo-locality is often used in the Agmon-Kuroda theory; we note that Jensen [20] has quoted some lectures of Kuroda where non-local potentials are treated by the Agmon-Kuroda method. Secondly, as Enss points out, the relative compactness of V plays no role in his work. In fact, as we shall see below, even relative boundedness is not needed! Rather, one is close to merely requiring mutual subordinateness of H and H_0 in the sense of Birman [5] (see §2).

In our extensions and simplifications below, we rely on three technical devices which go beyond those that Enss used in [13]:

(1) *Decompositions in phase space.* In [13], Enss decomposes phase space into products of cones in x and k. He then reduces his analysis to a one dimensional analysis which in essence has the effect of slicing his x-space cones with an infinite number of planes parallel to each other but not perpendicular to the axis of the cone. For non-isotropic H_0 this is especially inconvenient. Instead, our decomposition will be into cubes in x-space. Enss has informed me that he used a similar kind of decomposition in his analysis of the Coulomb problem [14].

(2) *Integration by Parts Machine.* Enss exploits the explicit Gaussian kernel of $e^{t\Delta}$. Here we use an integration by parts machine systematized by Hörmander [19] to treat general e^{-tH_0}. The point is that if ϕ is localized near x_0 in x-space and *strictly* localized in a set K in k-space, then $(e^{-tH_0}\phi)(x)$ is very small outside of the classically allowed region, $\{x_0 + tv \mid v = \partial P/\partial k \text{ for } k \in K\}$ when $H_0 = P(k)$.

(3) *Asymptotic Equality of H and H_0.* In the Enss analysis, it is important to prove that $\|[\Phi(H) - \Phi(H_0)]\eta_n\| \to 0$ for a certain set of Φs and η_n. Enss relies on Cook's method and needs to control $e^{-itH}\eta_n$ for $t \in [-T_n, T_n]$ with $T_n \to \infty$. Following a suggestion of Hunziker, we rely instead on a technique from Simon [36].

What comes out of the presentation in §2 is that the Enss analysis depends on four conceptual ideas:

(a) One lets the dynamics do the hard work, i.e., move the wave function far from the scatterer.

(b) Apply Cook's method. The deep discovery of Enss is that Cook's method can be a useful tool in proving completeness and the absence of singular spectrum.

(c) Control $(e^{-itH_0}\phi)(x)$ by exploiting the fact that once one is far from the scatterer, one can work on a scale where quantum and classical mechanics are essentially identical!

(d) Make a suitable joint decomposition in x and k-space to accomplish (c). In this decomposition, it is critical that one localize strictly in k-space but only weakly in x-space. In fact the x-space decomposition can be done in a way that does not destroy the strict k-space localization. It is this phase space decomposition that I find most attractive and characteristic of the method which leads to my proposed name of "phase space analysis." It suggests that the cotangent bundle is the right place to do scattering theory! Indeed, it may be possible to treat localized perturbations of Laplace-Beltrami operators on manifolds by the Enss method; one needs to prove that the quantum notion is almost that under the geodesic flow.

We should mention a number of examples that we treat in the text below which illustrate the scope of the extensions.

Example 1.1. (Dirac operators) It is easy to accomodate H_0's which are elliptic systems. See §2.

Example 1.2. (Optical and Acoustical scattering) Acoustical scattering is easy since it is elliptic as a second order system and optical scattering can be accomodated by the trick [30] of adding fictitious degrees of freedom. We treat both inhomogeneities (§2) and obstacles (§6).

Example 1.3. (Higher order perturbations) Consider

$$H = -\Delta + \Delta\left((1 + |x|^2)^{-\alpha/2}\right)\Delta$$

where the perturbation is of higher order and thus probably not amenable to any technique depending on inverting $1 + V(H_0 - z)^{-1}$. For $\alpha > \nu$ (= no. of space dimensions), Birman's theorem [5] shows that $\Omega^{\pm}(H, H_0 = -\Delta)$ exist and are complete. In §2, we see that this is true for $\alpha > 1$ and moreover that there is no singular spectrum. This result is new.

Example 1.4. (Positive singular perturbations) In §4, we will show that if V is a non-negative function with

$$\sup_y \int_{|x-y|\leq 1} (1 + |x|)^{2+\epsilon} V(x)\,dx < \infty,$$

then $H = -\Delta + V$ has no singular spectrum and complete scattering. Presumably, $2 + \epsilon$ can be replaced with $(1 + \epsilon)$ but in that case, we need

$$\sup_y \int_{|x-y|\leq 1} (1 + |x|)^{1+\epsilon} |V(x)|^p\,dx < \infty \qquad \text{where} \quad p = 2\nu/(2 + \nu)$$

if $\nu > 2$, $p = 1$ if $\nu = 1$ and $p > 1$ if $\nu = 2$. These results are new for $\nu > 1$.

Example 1.5. (Half-Solid) Let W be a bounded periodic function on $(-\infty, \infty)$ and let $V(x) = W(x)$ (resp. 0) for $x > 0$ (resp. $x \leqslant 0$). Let $H = -d^2/dx^2 + V$. Recently Davies and Simon [10] showed that H has no singular spectrum by using a "twisting trick" which allowed one to exploit time-independent methods. With the Enss method, one does not need such cleverness! See §6.

Example 1.6. (Constant Electric or Magnetic Fields) Recent results [2, 3, 17, 41] on scattering in constant electric or magnetic field are recovered. Here, as always, the Enss method depends on directly analyzing the basic physics!

Example 1.7. (Schrödinger Equation with absorbtion) Let $H = -\Delta + V$ where V is not self-adjoint, but rather $i(V - V^*) \geqslant 0$. Then if V is $-\Delta$-bounded with relative bound smaller than 1, e^{-itH} will be a contraction semigroup for $t > 0$. Such semigroups are approximations which arise in nuclear physics under the rubric "optical" model. In a recent paper, Davies [9] has advocated their study and showed how to use the Kato-Birman theory to analyze them. The Enss method can be used to analyze such semigroups under fairly general circumstances. (See §9).

Finally, we sketch the contents of the paper. §2 is the central section of the paper where we abstract [13]. In that paper, Enss only recovered the Agmon-Kuroda results on the absolutely continuous and singular continuous spectrum. He did not bother to prove that only 0 can be an accumulation point of point spectrum. We prove this result, which should be significant in multiparticle systems, in §3. In §4, we allow V to be a form perturbation. Given the form version of Cook's method [35], this is quite easy. In §5, we consider H_0's which are not constant coefficient partial pseudo-differential operators but which possess nice eigenfunction expansions. In particular, we consider $-\Delta + W$ with W periodic (a solid) and scattering in the zero temperature Heisenberg ferromagnet where H_0 is a finite difference operator. In §6, we use the device of looking at $e^{+itH}Je^{-itH_0}$ for a J with $1 - J$ relatively compact to recover known results where V has local singularities. In §7, 8 we discuss constant electric or magnetic fields and in §9, Schrödinger operators with absorbtion.

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§2. General Pseudo-Differential Operators. This is the central section of this paper. Our strategy is that of Enss [13] but there are some differences in tactics. Normally, we work on $L^2(\mathbf{R}^\nu)$. k stands for the ν-tuple of differential operators $i^{-1}\nabla$ and also for the Fourier transform variable. The symbols $F(x \in X)$ and $F(k \in K)$ denote respectively multiplication by the characteristic function of X and the spectral projection for the ν-tuple k (characteristic function on Fourier space). We consider operators $H = H_0 + V$ where H_0 is a "pseudo-differential" operator. We intend this last phrase in the weak sense that H_0 is a *continuous* function, $P(k)$, of k; we do *not* assume any estimates of the form $|D^\alpha P(k)| \leqslant C_\alpha(1+|k|)^{n-|\alpha|}$. We will need some weak conditions on P which require it to be "essentially elliptic."

Definition. k_0 is called a *singular point* of P if P is not C^∞ in a neighborhood of k_0. k_0 is called a *critical point* if it is not a singular point and $\nabla P(k_0) = 0$. The values of P at the singular (resp. critical) points are called *singular* (resp. *critical*) *values*. The family of singular points, singular values, critical points and critical values will be denoted by S_p, S_v, C_p, C_v respectively.

Definition. A continuous function P is called *vaguely elliptic* if and only if:
(A) $S_v \cup C_v$ has a countable closure
(B) $P(k) \to \infty$ as $|k| \to \infty$.
(C) $P^* = P$.

Definition. A symmetric operator, V, and self-adjoint operator, H is called a *regular perturbation* of $H_0 = P(k)$ if and only if
(i) For some N, $D(V) \supset D(|k|^{2N})$ and $h(R) \equiv \| V(|k|^{2N}+1)^{-1}F(|x| \geqslant R)\|$ obeys

$$\int_0^\infty h(x)dx < \infty; \qquad h(0) < \infty \tag{2.1}$$

(ii) H is a self-adjoint extension of $(H_0 + V)\restriction D(H_0) \cap D(|k|^{2N})$
(iii) For all a, b in $\mathbb{R}$, there is a positive continuous function Q going to infinity at infinity so that

$$Q(k)E_{(a,b)}(H)$$

is bounded.

We will call (2.1) the *Enss condition*. Since h is monotone decreasing, (2.1) implies that $h(x) \to 0$ at infinity; indeed, $h(x) \leqslant |x|^{-1}\int_0^\infty h(y)dy$. The basic theorem is

THEOREM 2.1. *Let $H_0 = P(k)$ with P vaguely elliptic. Let $H = H_0 + V$ be a regular perturbation. Then*:
(a) $\Omega^\pm(H, H_0) = s\text{-}\lim_{t\to\mp\infty} e^{itH}e^{-itH_0}P_{ac}(H_0)$ *exist*
(b) *H has no singular continuous spectrum*
(c) $\operatorname{Ran}\Omega^+ = \operatorname{Ran}\Omega^-$ *is the absolutely continuous space for H*
(d) *The only possible (finite) limit points for the point spectrum of H are in $\overline{C_v \cup S_v}$. Any eigenvalue not in $\overline{C_v \cup S_v}$ has finite multiplicity.*
(e) $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) = \overline{\{P(k) \mid k \in \mathbb{R}^\nu\}}$.

We will prove (a)–(c) below. (d) (which is not a result of Enss in his case) will be proven in §3. (e) follows easily from (a)–(d).

Remarks 1. It is easy to accomodate multicomponent systems within this framework. $\mathcal{H}$ is then $\bigoplus_{i=1}^m L^2(\mathbb{R}^\nu)$. P is now a matrix $P_{ij}(k)$; $1 \leqslant i, j \leqslant n$ and condition (C) is that $\overline{P(k)}$ is a Hermitian symmetric matrix. One then writes $P_{ij}(k) = \sum_{l=1}^{m} \mu_l(k) \bar{e}_i^{(l)}(k) e_j^{(l)}(k)$ where $\{e^{(l)}(k)\}$ is an orthonormal basis for C^m. Singular points are now points where either some μ_l or some $e^{(l)}$ is singular, and critical points are points where some $\nabla \mu_l$ is zero. Singular values are all the $\mu_l(k_0)$ for k_0 singular. Critical values are the values of $\mu_l(k_0)$ for those l with $\nabla \mu_l(k_0) = 0$. With these changes, Theorem 2.1 extends with minor changes in the proof. One can accommodate the case where the $e^{(l)}$'s can only be chosen locally away from S_p rather then globally, as might happen with an isolated eigenvalue degeneracy. One can also deal with permanent degeneracies (i.e., some eigenvalue doubly degenerate for all k as in the Dirac equation) by allowing the rank one projection $\bar{e}_i^{(l)} e_j^{(l)}$ to be replaced by a C^∞ finite rank projection. It was with systems in mind that we wrote (B) as $|P| \to \infty$ rather than $P \to \infty$. If $\nu > 1$, $|P| \to \infty$ implies that $\pm P \to \infty$ for a single P. For systems, the condition is $|\mu^{(l)}| \to \infty$ for each l, and different signs can occur as in the Dirac equation.

2. We allow the possibility of singular points to include an example like $P(k) = |k|$. We could allow P to go to infinity at isolated points or submanifolds without any trouble. Moreover, C^∞ on $\mathbf{R}^\nu \backslash S_p$ is not essential; some finite computable number of derivatives would do. Note that only S_v and C_v aren't allowed to be "fat"; S_p or C_p could be. For example, P can be constant on an open set without necessarily violating (A)–(C).

3. Recall the notion of Birman [5]: Given two operators A and B we say that A is *subordinate* to B, if and only if $\| f(A)(g(B) + 1)^{-1} \| < \infty$ for some continuous, positive functions, f, g, on $\mathbf{R}$, with f going to infinity at infinity. It is not hard to see that hypothesis (iii) is equivalent to saying that H_0 is subordinate to H. (i) implies that H is subordinate to H_0 but is considerably stronger in that the $f(A)$ can be chosen to be A. (iii) can be weakened to require that only intervals (a, b) which avoid some closed countable set have the necessary property (but then this set may also include limit points of eigenvalues).

4. Let $h_1(R) = \| V(k^{2N} + 1)^{-1} j_{\geqslant R} \|$ where $j_{\geqslant R}(x) = \phi(x/R)$ with $1 - \phi \in C_0^\infty$ and $\phi(y) = 0$ (resp. 1) for $|y| \leqslant 1$, (resp. $\geqslant 2$). Then $h_1(R) \leqslant h(R) \leqslant h_1(R/2)$ so $h \in L^1(0, \infty)$ if and only if h_1 is. Let $h_2(R) = \| V j_{\geqslant R}(k^{2N} + 1)^{-1} \|$. Then using $[j_{\geqslant R}, (k^{2N} + 1)^{-1}] = (k^{2N} + 1)^{-1} j_{\geqslant R/2} [k^{2N}, j_{\geqslant R}] (k^{2N} + 1)^{-1}$, one sees that $|h_2(R) - h_1(R)| \leqslant CR^{-1} h_1(R/2)$. From this we see that when $h(0) < \infty$, $\int_0^\infty h(R)\,dR < \infty$ if and only if $\int_1^\infty h_2(R)\,dR < \infty$.

5. The use of k^{2N} in (i) is not essential; any function of k which is continuous and divergent at infinity will do.

6. Let $P(k)$ be a real analytic function on $\mathbf{R}^\nu$ going to infinity at infinity. Then the critical values of P are discrete, i.e., there are only finitely many in any compact subset of $\mathbf{R}^\nu$. To see this, we note that since $P \to \infty$ at infinity, it suffices to show that P has only finitely many critical values when restricted to a compact K of $\mathbf{R}^\nu$. The functions $f_i(k) = \partial P / \partial k_i$ define ν functions analytic in a complex neighborhood, N, of K. Since K is compact, we can shrink the neighborhood, so that the variety $\{f_i(k) = 0\}$ has only finitely many connected components in N. Each component has the property that its non-manifold

points do not disconnect it [16] so that P is constant on each component since any two points can be joined by a curve, γ, along which $(\nabla P)\cdot\gamma = 0$.

Before proving the main theorem, we give a number of examples which illustrate the conditions. We will not be explicit about giving all the details. In addition, we make no attempt to give reference to earlier work on these models; a fairly comprehensive bibliography can be found in [28].

Example 2.1. (Local potentials and differential operators). Let Δ_α, $\alpha \in \mathbf{Z}^\nu$ be a covering of $\mathbf{R}^\nu$ by unit cubes with center at α. For a function W, let $w_\alpha = [\int_{\Delta_\alpha} |W(x)|^2]^{1/2}$. Then, Strichartz [40] has shown that for $m > \nu$, and W a multiplication operator, $W(|k|^m + 1)^{-1}$ is bounded if and only if $\sup_\alpha w_\alpha < \infty$ and this sup defines a norm equivalent to $\| W(|k|^m + 1)^{-1}\|$. Using remark 4 above, we see that a multiplication operator W obeys (2.1) for some N if and only if

$$\int_0^\infty \Big[\sup_{|\alpha| \geq R} w_\alpha \Big] dR < \infty. \tag{2.2}$$

(2.2) should be compared with the condition for $W(|k|^m + 1)^{-1}$ to be trace class for some m, viz. $\sum w_\alpha < \infty$ (see e.g., [38] for a proof of this fact). More generally, if $V = \sum W^{(\beta)} D^\beta$ with $W^{(\beta)}$ a multiplication operator and V formally symmetric, then (2.1) holds if each $W^{(\beta)}$ obeys (2.2). To be sure that Theorem 2.1 is applicable, we need for (iii) to be true. This will happen, e.g., if H and H_0 are semi-bounded and $Q(H) = Q(H_0)$.

Example 2.2. (Higher order operators $\equiv$ Ex. 1.3) If $H_0 \geq 0$ and $V = \sum W^\beta D^\beta$ is positive on $D(k^{2N}) \times D(k^{2N})$, then the form sum $H_0 + V$ will have $Q(H) \subset Q(H_0)$ so (iii) holds. Thus we can completely analyze H if (2.1) also holds (see the discussion above for this condition). There is no restriction on the degree of V relative to that of H_0. For example, if $H = -\Delta + \Delta f \Delta$ with f positive and $(1 + |x|)^{1+\epsilon} g$ in $L^2_u = \{ W \mid \sup_\alpha w_\alpha < \infty \}$ for $g = f, \partial_i f, \Delta f$, then H obeys all the conclusions of Theorem 2.1.

Example 2.3. (Dirac operators $\equiv$ Ex. 1.1). Let $H_0 = k\beta + m\alpha$ where α, β_1, β_2, β_3 are Dirac matrices and H_0 is an operator on $L^2(\mathbf{R}^3, \mathbf{C}^4)$. H_0 is a vaguely elliptic system; the eigenvalues $\pm\sqrt{k^2 + m^2}$ go to infinity at infinity. One can get smooth (indeed analytic) eigenvalues, by picking an eigenbasis $\{u^{(l)}\}_{l=1}^4$ and letting $e^{(l)}(\vec{k}) = S(\vec{k})u^{(l)}$ where $S(\vec{k})$ implements the unique Lorentz transformation which takes $(m, 0, 0, 0)$ to $(\sqrt{k^2 + m^2}, \vec{k})$ and leaves $(0, \vec{a})$ invariant if $\vec{a}\cdot\vec{k} = 0$. (pure boost). A 4×4 matrix of multiplication operators will obey (2.1) if and only if each element obeys (2.2). Under suitable conditions, e.g., $H_0 + V$ self-adjoint on $D(H_0)$, (iii) will hold.

Example 2.4. (Acoustical scattering = Ex. 1.2) Here we consider the case of scattering from inhomogeneities and in §6, the case of scattering from obstacles. This does not fit the framework given above since the basic generators A and A_0 are self-adjoint in distinct inner products. We sketch the modifications needed.

Here $W_0(t)$ solves $\ddot{u} = c_0^2 \Delta u$ and $W(t)$ solves $\ddot{u} = c^2 \rho \nabla \cdot \rho^{-1} \nabla u$ where c, ρ are functions going to constants ρ_0, c_0 at infinity. The basic Hilbert space is $L^2 \oplus \overline{D(\nabla)}$ where $\overline{D(\nabla)}$ denotes the completion of $\mathcal{S}$ in $\|\nabla f\|$ − norm. $W(t) = e^{-iAt}$ and $W_0(t) \equiv e^{-iA_0 t}$ are unitary but in different inner products each equivalent to the usual inner product. Thus both are uniformly bounded families of operators in the usual inner product; see [28, 30] for further discussion of the basic framework. We claim that all the arguments below easily extend to this case: e.g., the Hörmander type estimates (lemma 1) are applicable if one writes the matrix elements of $W_0(t)$ i.e., $\cos|k|c_0 t$ and $\mp|k|^{\pm 1} c_0^{\pm 1} \sin(|k|c_0 t)$ in terms of complex exponentials, or, in proving lemma 4, we use the fact that $\Phi(A)$ (defined using the inner product in which A is self-adjoint) is a limit of polynomials in the energy norm for A, so in the equivalent norm which is the usual norm on $L^2 \oplus \overline{D(\nabla)}$. An additional complication is that $F(|x| \leqslant R)$ is not bounded on $\mathcal{H}$ because of the derivative in the norm. One uses $j_{\leqslant R} = 1 - j_{\geqslant R}$ with $j_{\geqslant R}$ given in remark 4 above, in place of F. The hypothesis needed on ρ, c comes from the requirements that $V = A - A_0$ obeys (2.1) (with F replaced by j) and that $j_{\leqslant R} E_{(a,b)}(A)$ be compact (local compactness).

Example 2.5. (Optical Scattering from Inhomogeneous Media) We have a structure very similar to that in the last example. We want to solve $\ddot{E} = -\epsilon^{-1} \nabla \times (\mu^{-1}(\nabla \times E))$ where E is a vector valued function on $\mathbf{R}^3$ and ϵ and μ are x dependent positive definite matrices approaching fixed matrices ϵ_0 and μ_0 as $x \to \infty$. The extra problem is that the system replacing H_0 is $P(k)v = \epsilon_0^{-1/2}[k \times (\mu_0^{-1} k \times \epsilon_0^{-1/2} v)]$ so that $v = \epsilon_0^{1/2} k$ is a zero eigenvector. Thus every point is a critical point and the eigenvalues don't go to infinity! If we borrow a trick from [30], this difficulty can easily be overcome. The trick is to note that $P(k)v$ is always orthogonal to $\epsilon_0^{1/2} k$ so that the bad modes decouple from the good modes; put more prosaically, if E obeys $\nabla \cdot (\epsilon_0 E) = 0$ $\nabla \cdot (\epsilon_0 \partial E / \partial t) = 0$ initially (and these follow from two of Maxwell's equations since $\epsilon_0 \partial E / \partial t = \nabla \times H$), then it does for all time. So one introduces a new P by

$$\tilde{P}(k)v = e_0^{-1/2} k \times \left(\mu_0^{-1} k \times \epsilon_0^{-1/2} v \right) + \epsilon_0^{1/2} k \left(\epsilon_0^{1/2} k \cdot v \right)$$

which is vaguely elliptic. One then solves the equation

$$\ddot{E} = -\epsilon^{-1} \nabla \times \left(\mu^{-1} (\nabla \times E) \right) + \operatorname{div}(\nabla \cdot \epsilon E)$$

and studies its spectral and scattering properties. Restricted to the subspace with $\nabla \cdot \epsilon E = \nabla \cdot \epsilon \partial E / \partial t = 0$, the equations and scattering, etc. are identical; see [30] for the details when the Kato-Birman theory is used in place of the Enss method. To apply the methods here, the V that is needed in (2.1) is a two-by-two matrix with only one non-zero entry, namely $H - H_0$ with (Note: H = Hamiltonian, not a magnetic field!)

$$H = -\epsilon^{-1} \operatorname{curl}(\mu^{-1} \operatorname{curl} \cdot) + \operatorname{div}(\operatorname{grad} \epsilon \cdot)$$

and

$$H_0 = -\epsilon_0^{-1}\mathrm{curl}(\mu_0^{-1}\mathrm{curl}\,\cdot\,) + \mathrm{div}(\mathrm{grad}\;\epsilon_0\,\cdot\,).$$

The constructions are always such that $D(A) = D(A_0)$ so that (iii) is easy because $\tilde{P}$ which is used to construct A_0 is elliptic.

As a preliminary to the proof, we note that since $F(|x| \leqslant R)(Q(k)+1)^{-1}$ is compact for any $Q \geqslant 0$ going to infinity at infinity, (iii) implies

(iii′) $F(|x| \leqslant R)E_{(a,b)}(H)$ is compact for all a, b in $\mathbf{R}$.

(iii′) is often called local compactness.

We abstract the basic device from Enss [13] in a theorem. To state it, we suppose existence (conclusion (a) of Theorem 2.1) is known; we prove that below.

THEOREM 2.2. (Enss Decomposition Principle) *Let H obey the hypotheses of Theorem* 2.1. *Let ϕ_n be a sequence of unit vectors with*

(a) $\|F(|x| \leqslant n)\phi_n\| \to 0$ *as* $n \to \infty$

(b) *For some* $[a, b]$ *disjoint from* $\overline{S_v \cup C_v}$, $E_{[a,b]}(H)\phi_n = \phi_n$ *for all* n.

Then, one can decompose $\phi_n = \phi_{n,\,\mathrm{in}} + \phi_{n,\,\mathrm{out}} + \phi_{n,\,w}$ *so that*:

(1) $\|\phi_{n,\,w}\| \to 0$ *as* $n \to \infty$.

(2) $\|(\Omega^+ - 1)\phi_{n,\,\mathrm{in}}\| \to 0$ *and* $\|(\Omega^- - 1)\phi_{n,\,\mathrm{out}}\| \to 0$ *as* $n \to \infty$.

(3) $\sup_n\|\phi_{n,\,\mathrm{in}}\| < \infty$, $\sup_n\|\phi_{n,\,\mathrm{out}}\| < \infty$.

(4) $\lim_{n\to\infty}[\sup_{t<0}\|F(|x| \leqslant \delta n)e^{-itH_0}\phi_{n,\,\mathrm{in}}\|] = 0$ for some $\delta > 0$ (only depending on a, b).

Basically, $\phi_{n,\,\mathrm{in}}$ is the part of ϕ_n with velocities ($\equiv \partial P/\partial t$) pointing $\binom{\text{inwards}}{\text{outwards}}$. $\phi_{n,\,w}$ is a number of pieces which we toss into the wastebasket along the way.

The following is virtually identical to some arguments of Enss [13].

Proof of Theorem 2.1(b), (c) *given Theorem* 2.2. Let $\mathcal{H}_{\mathrm{sing}}$ denote the singular continuous subspace of H. Since $\mathcal{H}_{\mathrm{sing}}$ reduces H and since $\overline{S_v \cup C_v}$ cannot support a singular continuous measure, if $\mathcal{H}_{\mathrm{sing}} \neq \{0\}$, we can find $\phi \in \mathcal{H}_{\mathrm{sing}}$, $\phi \neq 0$ with $E_{[a,b]}\phi = \phi$ for some $[a, b]$ disjoint from $\overline{S_v \cup C_v}$. By Wiener's theorem on the Fourier transforms of continuous measures, (iii′) implies that $\|F(|x| \leqslant n)e^{-itH}\phi\|$ goes to zero in L^2-mean sense ([31, 1, 13, 28]). In particular, we can find t_n inductively with $t_n \geqslant \max(n, t_{n-1})$ so that $\|F(|x| \leqslant n)e^{-it_nH}\phi\| \leqslant 1/n$. Let $\phi_n = e^{-it_nH}\phi$. Then ϕ_n obeys the hypothesis of Theorem 2.2. Thus, by (1) and (2)

$$\|\phi_n - \Omega^+\phi_{n,\,\mathrm{in}} - \Omega^-\phi_{n,\,\mathrm{out}}\| \to 0 \tag{2.3}$$

Thus

$$\phi = \lim_{n\to\infty} e^{it_nH}\left[\Omega^+\phi_{n,\,\mathrm{in}} + \Omega^-\phi_{n,\,\mathrm{out}}\right]$$

which means that ϕ is in the absolutely continuous space for H. This contradiction implies that $\mathcal{H}_{\mathrm{sing}} = \{0\}$.

Similarly, since $\mathcal{H}_{ac} \cap (\text{Ran } \Omega^-)^\perp$ reduces H, if this space is non-zero, we can proceed as above and find $\phi \in (\text{Ran } \Omega^-)^\perp \cap \mathcal{H}_{ac}$, so that $\phi \neq 0$ and so that (2.3) holds for $\phi_n = e^{-it_n H}\phi$. But

$$|(\phi_n, \Omega^+ \phi_{n,\,\text{in}})| = |((\Omega^+)^*\phi, e^{+it_n H_0}\phi_{n,\,\text{in}})| \leqslant \alpha + \beta$$

where

$$\alpha \leqslant \|F(|x| \geqslant \delta n)(\Omega^+)^*\phi\| \, \|\phi_{n,\,\text{in}}\|$$

goes to zero by (3) of Theorem 2.2 and

$$\beta = \|(\Omega^+)^*\phi\| \, \|F(|x| \leqslant \delta n)e^{it_n H_0}\phi_{n,\,\text{in}}\|$$

goes to zero by conclusion (4) of Theorem 2.2. Thus, by (2.3)

$$(\phi, \phi) = (\phi_n, \phi_n) = \lim[(\phi_n, \Omega^+ \phi_{n,\,\text{in}}) + (\phi_n, \Omega^- \phi_{n,\,\text{out}})] = 0$$

since $\phi_n \in (\text{Ran } \Omega^-)^\perp$ by hypothesis. □

Remark. Enss [14] has recently found a beautiful argument for directly showing that for $\phi_n = e^{-it_n H}\phi$ with $t_n \to \infty$, one has $\phi_{n,\,\text{in}} \to 0$. This can replace the last paragraph in the above proof; see §9.

We now turn to the proof of Theorem 2.2. The basic estimate that tells us that quantum paths are essentially classical is:

LEMMA 1. *Fix K compact and disjoint from $\overline{S_p \cup C_p}$. Let $\mathcal{O}$ be a fixed open neighborhood of $\{(\partial P/\partial k)(k_0) \mid k_0 \in K\}$. Then for all n, there exists a constant C depending only on K, $\mathcal{O}$, P and n so that*

$$|(e^{-itH_0}\phi)(x)| \leqslant C(1 + |x - x_0| + |t|)^{-n} \, \|(1 + |x - x_0|^n)\phi\| \qquad (2.4)$$

for all ϕ with supp $\hat{\phi} \subset K$, *all x_0 and all x, t with $(x - x_0)/t \notin \mathcal{O}$.*

Remarks 1. (2.4) is intended in the sense of holding for ϕ's in $\mathcal{S}$ with supp $\hat{\phi} \subset K$. It then extends to any ϕ with the norm on the right finite so long as supp $\hat{\phi} \subset K$. For, we need only slightly increase K so that we can modify any $\hat{\phi}$ to yield a ϕ in $\mathcal{S}$.

2. Classically, a particle moving under the Hamiltonian $P(k)$ with $k(t = 0) = k_0$, and $x(t = 0) = x_0$ will follow the orbit $x_0 + (\partial P/\partial k)t$. (2.4) says that if ϕ is originally "localized near x_0 in x-space and strictly in K in k-space," then $e^{-itH_0}\phi$ is localized near the classical orbits associated to these initial conditions.

Proof. If (2.4) is proven for $x_0 = 0$, it follows for all x since e^{-itH_0} commutes with translations in x. In [19], Hörmander proved that for $x/t \notin \mathcal{O}$ and supp $u \subset K$:

$$\left|\int e^{-itP(k) - ik\cdot x}u(k)dk\right| \leqslant C(1 + |x| + |t|)^{-n} \sum_{|\alpha| \leqslant n} \int |D^\alpha u(k)|dk. \qquad (2.5)$$

((2.5) is proven by repeated integration by parts. $x/t \notin \mathcal{O}$, means that $[e^{-itP(k)-ik\cdot x}]$ has no points of stationary phase.) Since K is bounded $\int |D^\alpha u| dk \leqslant C(\int |D^\alpha u|^2 dk)^{1/2}$ so that

$$\sum_{|\alpha| \leqslant n} \int |D^\alpha u(k)| dk \leqslant C\Big(\sum_{|\alpha| \leqslant n} \int |D^\alpha u|^2\Big)^{1/2}$$
$$= C \sum_{|\alpha| \leqslant n} \int |x^\alpha \check{u}|^2 dx$$

Taking $u = \hat{\phi}$, we have proven (2.4). □

Proof of Theorem 2.1(a). By Cook's argument, it suffices to show that $\| Ve^{-itH_0}\phi \| \equiv f(t)$ is in $L^1(-\infty, \infty)$ for any $\phi \in \mathcal{S}$ with $\hat{\phi}$ having compact support disjoint from $\overline{C_p \cup S_p}$. We write

$$f(t) \leqslant \| V(k^{2N}+1)^{-1} e^{-itH_0}(k^{2N}+1)\phi \| = A + B$$

where

$$A \leqslant \| V(k^{2N}+1)^{-1} \| \, \| F(|x| \leqslant \delta t) e^{-itH_0}(k^{2N}+1)\phi \|$$
$$B \leqslant \| V(k^{2N}+1)^{-1} F(|x| \geqslant \delta t) \| \, \| (k^{2N}+1)\phi \|.$$

By lemma 1, $A \in L^1$, if we choose $\delta > 0$ smaller than the smallest velocity in supp $\hat{\phi}$. By the Enss condition (2.1), $B \in L^1$. □

To apply Lemma 1, we need an efficient way of localizing in x and k-space simultaneously. Since we need strict localization in k, we can't strictly localize in x. Thus, let χ_α be the characteristic function of the unit cube centered at $\alpha \in \mathbf{Z}^\nu$. Pick $f \in \mathcal{S}$ and let

$$f_\alpha = f * \chi_\alpha \tag{2.6}$$

If $\int f = 1$, then

$$\sum_\alpha f_\alpha(x) = 1 \tag{2.7}$$

and since $f \in \mathcal{S}$:

$$\sup_\alpha \| |x - \alpha|^n f_\alpha \| < \infty \tag{2.8}$$

for each n. Thus, for any u, $u = \sum f_\alpha u$ decomposes u as a sum of pieces localized about different points in Z^ν. The point is that $\operatorname{supp}(\widehat{f_\alpha u}) = \operatorname{supp}(\hat{f}_\alpha * \hat{u}) \subset \operatorname{supp} \hat{u} + \operatorname{supp} \hat{f}_\alpha \subset \operatorname{supp} \hat{u} + \operatorname{supp} \hat{f}$ so, by taking $\hat{f}$ with very small support we can localize in x without losing much strict localization in k. Once we have localized near α, we will want to split into those velocities pointing "away" from $x = 0$ and those pointing "towards" $x = 0$. Thus, we want to follow by an α-*dependent* k-space localization. For this reason, we will need:

LEMMA 2. *Let $f \in \mathcal{S}$ with $f \geqslant 0$, $\int f\, dx = 1$. Let f_α be given by* (2.6). *For each α, let g_α be given with*

$$\sup_\alpha \| (1-\Delta)^\nu g_\alpha \|_2 < \infty.$$

For $h \in \mathcal{S}$, define

$$(\mathrm{Th})(x) = \sum_{\alpha} g_\alpha(k) f_\alpha(x) h(x). \tag{2.9}$$

Then

$$\|\mathrm{Th}\|_2 \leqslant C\|h\|_2 .$$

Remark. In (2.9) $g_\alpha(k)$ denotes the function of $-i\partial/\partial x$ as per our standing convention.

Proof. $\|\mathrm{Th}\|^2 \leqslant \sum_{\alpha,\beta} \int f_\alpha(x)|h(x)|\,|H_{\alpha\beta}(x-y)|f_\beta(y)|h(y)|dx\,dy$ where $H_{\alpha\beta}$ is up to factors of 2π, the Fourier transform of $\bar{g}_\alpha g_\beta$. By Lebnitz' rule and the hypothesis on g_α, we have a uniform bound on the L^1 norm of $(1-\Delta)^\nu(\bar{g}_\alpha g_\beta)$ and thus

$$|H_{\alpha\beta}(z)| \leqslant C_1(1+|z|^2)^{-\nu}.$$

Thus

$$\begin{aligned}\|\mathrm{Th}\|^2 &\leqslant C_1 \sum_{\alpha,\beta} \int f_\alpha(x)|h(x)|\,|h(y)|f_\beta(y)(1+|x-y|^2)^{-\nu} \\ &= C_1 \int |h(x)|(1+|x-y|^2)^{-\nu}|h(y)| \leqslant C\|h\|_2^2\end{aligned}$$

where we use (2.7) in the first inequality and Young's inequality in the last step. □

LEMMA 3. *Let $f, g \in C_\infty(\mathbf{R}^\nu)$, the continuous functions vanishing at infinity. Let $g_R(x) = g(x/R)$. Then*

$$\|[f(k), g_R(x)]\| \to 0$$

as $R \to \infty$. In particular, if $j_{\geqslant R}$ is the function in the fourth remark after Theorem 2.1, and z is not real, then

$$\|[(H_0 - z)^{-1}, j_{\geqslant R}]\| \to 0 \qquad \text{as} \quad R \to \infty.$$

Proof. Since $\|[f(k), g_R(x)]\| \leqslant 2\|f\|_\infty \|g\|_\infty$, it suffices to prove the result for a dense set of f's and g's. We thus suppose that $g \in \mathcal{S}$ and f is a polynomial in $(k^2+1)^{-1}$ and $k_i(k^2+1)^{-1}$ (such polynomials are dense by the Stone-Weierstrass theorem). Using $[AB, C] = A[B, C] + [A, C]B$ we are easily reduced to the cases $f(k) = (k^2+1)^{-1}$ and $f(k) = k_i(k^2+1)^{-1}$ each of which is easy since $\|[k_i, g_R]\| \leqslant R^{-1}\|\nabla g\|_\infty$. □

Remark. An estimate on $[f(k), g(x)]$ can be found in the appendix. This estimate and the density argument in the first sentence of the proof provide an alternate proof.

LEMMA 4. *Let $j_{\geqslant R}$ be the function in the fourth remark after Theorem 2.1. Let H, H_0 obey the hypothesis of Theorem 2.1 and let $\Phi \in C_\infty(\mathbf{R})$, the continuous functions vanishing at infinity. Then*

$$\|(\Phi(H) - \Phi(H_0))(|k|^{2N}+1)^{-1} j_{\geqslant R}\| \to 0$$

as $R \to \infty$.

Proof. (following ideas in [36]). We first claim it suffices to consider the case $\Phi_z(x) = (x - z)^{-1}$ with z non-real. For then using $(z - w)^{-1}(\Phi_z - \Phi_w) = (x - z)^{-1}(x - w)^{-1} \equiv \Phi_{z,w}(x)$ we obtain convergence for $\Phi_{z,w}$. By the Vitali convergence theorem and the analyticity in z, w, we have convergence of the derivatives in z, w and thus for the case $\Phi(x) = (x - i)^{-m}(x + i)^{-k}$. By the Stone-Weierstass theorem, polynomials in $(x + i)^{-1}$ and $(x - i)^{-1}$ are dense in $C_\infty(R)$, so the general Φ is accommodated.

Since $H = H_0 + V$ on $\mathrm{Ran}[(H_0 - z)^{-1}(k^{2N} + 1)^{-1}]$ we have that

$$[(H - z)^{-1} - (H_0 - z)^{-1}](k^{2N} + 1)^{-1} j_{\geqslant R}$$
$$= (H - z)^{-1} V (k^{2N} + 1)^{-1} (H_0 - z)^{-1} j_{\geqslant R} = A + B$$

where $\|A\| = \|(H - z)^{-1} V (k^{2N} + 1)^{-1} j_{\geqslant R} (H_0 - z)^{-1}\| \leqslant |\mathrm{Im}\, z|^{-w} h(R)$ goes to zero at infinity by the basic hypothesis (2.1) on h and $\|B\| \leqslant \|(H - z)^{-1}\|\, \|V(k^{2N} + 1)^{-1}\|\, \|[j_{\geqslant R}, (H_0 - z)^{-1}]\|$ goes to zero by lemma 3. □

LEMMA 5. *Under the notation of the last lemma,*

$$\|[\Phi(H) - \Phi(H_0)] j_{\geqslant R} E_{(a,b)}(H)\| \to 0$$

as $R \to \infty$.

Proof. Since lemma 3 implies that $\|[(|k|^{2N} + 1)^{-1}, j_{\geqslant R}]\| \to 0$, we conclude that $\|[\Phi(H) - \Phi(H_0)] j_{\geqslant R} (|k|^{2N} + 1)^{-1}\| \to 0$. Let $Q(k)$ be a positive function going to infinity at infinity. Then for any ϵ,

$$(Q(k) + 1)^{-1} \leqslant C_\epsilon (|k|^{2N} + 1)^{-1} + \epsilon$$

from which we conclude that $\|[\Phi(H) - \Phi(H_0)] j_{\geqslant R} (Q(k) + 1)^{-1}\| \to 0$. Since $(Q(k) + 1) E_{(a,b)}(H)$ is bounded for suitable Q, the result follows. □

Proof of Theorem 2.2. First pick a', b' so that $[a, b] \subset (a', b') \subset [a', b']$ is disjoint from $\overline{S_v \cup C_v}$. Let Φ be a C^∞ function which is 0 off (a', b'), 1 on $[a, b]$ with $0 \leqslant \Phi \leqslant 1$. By hypothesis (b) and lemma 5:

$$\|[\Phi(H) - \Phi(H_0)] j_{\geqslant n/2} \phi_n\| \to 0$$

as $n \to \infty$. By hypothesis (a), the same is true if $j_{\geqslant n/2}$ is replaced by $j_{\leqslant n/2}$. Since $\Phi(H)\phi_n = \phi_n$, we have that

$$\|\phi_n - \tilde{\phi}_n\| \to 0; \qquad \tilde{\phi}_n = \Phi(H_0)\phi_n \tag{2.10}$$

Since Φ is supported away from the singular values of P and $P \to \infty$ at infinity, $\Phi(P(k))$ is a C_0^∞ function of k and thus convolution with a function in $\mathcal{S}$. It follows that

$$\|F(|x| \leqslant \tfrac{1}{2} n)\tilde{\phi}_n\| \to 0 \tag{2.11}$$

as $n \to \infty$.

Now, let $L = P^{-1}[a', b']$. By the hypothesis on P, L is a compact set disjoint from $\overline{S_p \cup C_p}$. Thus we can find, a bounded open set $\mathcal{O}$ and an ϵ, so that

$L + \bar{B}_\epsilon \subset \mathcal{O} \subset \bar{\mathcal{O}} \subset \mathbb{R}^\nu \backslash \overline{S_p \cup C_p}$, where $\bar{B}_\epsilon = \{|k|,\ |k| \leqslant \epsilon\}$. Now let f be a positive function in $\mathcal{S}$ with supp $\hat{f} \subset \bar{B}_\epsilon$ and let f_α be given by 2.6. We will take

$$\phi_{n,w} = \phi_n - \tilde{\phi}_n + \sum_{|\alpha| \leqslant (1/3)n} f_\alpha(x)\tilde{\phi}_n .$$

By (2.10) and (2.11), we have $\|\phi_{n,w}\| \to 0$ which is conclusion (2) of the theorem.

Next let $v(k) \equiv \partial P/\partial k$. Since $\bar{\mathcal{O}}$ is bounded and disjoint from C_p, $\{v(k) \mid k \in \mathcal{O}\}$ is contained in some set $\{v \mid A < |v| < B\}$ for $A > 0$. Pick two functions G_{in}, G_{out} in $C_0^\infty(R^\nu)$ so that

(a) $G_{\text{in}}(v) + G_{\text{out}}(v) = 1$ if $A < |v| < B$.

(b) $G_{\text{in}}(v) = 0$ if $A < |v| < B$ and the angle between v and $(1, 0, \ldots, 0)$ is smaller than 45°.

(c) $G_{\text{out}}(v) = 0$ if $A < |v| < B$ and the angle between v and $(-1, 0, \ldots, 0)$ is smaller than 45°.

(Note: 45° plays no special role; any angle strictly less than 90° but larger than 0° will do). For $\alpha \in Z^\nu$, let R_α be a rotation taking α to $(|\alpha|, 0, \ldots, 0)$ and finally, let

$$g_\alpha^{\text{in}}(k) = G_{\text{in}}(R_\alpha v(k)); \qquad g_\alpha^{\text{out}}(k) = G_{\text{out}}(R_\alpha v(k)).$$

Then, g_α^{in} and g_α^{out} obey the hypotheses of Lemma 2 so that

$$\phi_{n,\text{in}} = \sum_{|\alpha| > (1/3)n} g_\alpha^{\text{in}}(k) f_\alpha(x)\tilde{\phi}_n$$

$$\phi_{n,\text{out}} = \sum_{|\alpha| > (1/3)n} g_\alpha^{\text{out}}(k) f_\alpha(x)\tilde{\phi}_n$$

obey conclusion (3) of Theorem 2.2.

Moreover, since supp $\hat{f}_\alpha \subset \bar{B}_\epsilon$, $\text{supp}(f_\alpha\tilde{\phi}_n)\hat{} \subset \mathcal{O}$, so $(g_\alpha^{\text{in}} + g_\alpha^{\text{out}})(f_\alpha\tilde{\phi}_n)\hat{} = (f_\alpha\tilde{\phi}_n)\hat{}$ and hence $\phi_n = \phi_{n,\text{in}} + \phi_{n,\text{out}} + \phi_{n,w}$.

Now, let $\phi_{n;\alpha\,\text{in}} \equiv g_\alpha^{\text{in}}(k) f_\alpha(x)\tilde{\phi}_n$. Since one has uniform bounds on derivatives of g_α^{in}, and $f \in \mathcal{S}$, we have bounds on $\|\,|x - \alpha|^n \phi_{n;\alpha;\text{in}}\|$, uniform in n and α. Thus, using lemma 1, and some geometry, we have that

$$|e^{-itH_0}\phi_{n;\alpha,\text{in}}(x)| \leqslant C_m(1 + |\alpha| + |t|)^{-m}$$

for all x with $|x| \leqslant \delta(n + |t|)$, all $t < 0$ and all $\alpha > (1/3)n$. (This follows from the fact that no velocities in supp $\hat{\phi}_{n;\alpha,\text{in}}$ point within 45° of α. Here δ is some number depending on A, the minimal velocity). Using the $|\alpha|$ falloff to sum on α, the $|t| + |\alpha|$ falloff to sum on $|x|$ and the $|\alpha|$ falloff to translate into n falloff, we see that for $t \leqslant 0$

$$\|F(|x| < \delta(n + |t|))e^{-itH_0}\phi_{n,\text{in}}\| \leqslant C'_m(1 + n + |t|)^{-m}$$

which proves conclusion (4). Similarly, we see that

$$\|F(|x| < \delta(n + |t|))e^{-itH_0}(|k|^{2N} + 1)\phi_{n,\text{in}}\| \leqslant C'_m(1 + n + |t|)^{-m}$$

for $t \leqslant 0$ and analogously with in replaced by out and $t \leqslant 0$ by $t \geqslant 0$. These estimates and the argument following lemma 1, prove conclusion (3). □

§3. Bound States. In this section, we wish to make a few remarks about the connection between bound states and phase space analysis. We expect that there is much more to be said about this subject than we do here! We begin with the proof of Theorem 2.1(d). We remind the reader that for local or pseudo-local V's, this kind of result is a consequence of the Agmon-Kuroda method.

Proof of Theorem 2.1(d). Suppose to the contrary. Then we can find an orthonormal family ϕ_n with $H\phi_n = E_n\phi_n$ and $E_n \to E \notin C_v \cup S_v$. By throwing out finitely many ϕ_n's we can suppose that each $E_n \in [a, b]$, an interval disjoint from $C_v \cup S_v$. Thus $E_{[a,b]}(H)\phi_n = \phi_n$. Moreover, since $F(|x| \leqslant R)(H+i)^{-1}$ is compact by (iii), and $\phi_n \to 0$ weakly, we have that $\|F(|x| \leqslant R)\phi_n\| = |E_n + i|\,\|F(|x| \leqslant R)(H+i)^{-1}\phi_n\| \to 0$ so that, by passing to a subsequence we can suppose that $\|F(|x| \leqslant n)\phi_n\| \to 0$. Thus the Enss Decomposition Principle (Theorem 2.2) is applicable so $\phi_n - \Omega^+\phi_{n,\text{in}} - \Omega^-\phi_{n,\text{out}} \to 0$. Since ϕ_n, as an eigenfunction, is orthogonal to Ran $\Omega^+ \cup$ Ran Ω^-, this is impossible. □

Results related to Theorem 2.1(d) will be of importance in analyzing n-body systems since it will be important that scattering thresholds have a countable closure. Our other result on bound states is:

THEOREM 3.1. *Suppose that H_0 obeys $\|[H_0, j_{\geqslant R}](H_0+i)^{-1}\| \leqslant cR^{-1}$ for $R \geqslant 1$ (e.g. if H_0 is an elliptic polynomial) and that*

$$\int_0^\infty \|V(H_0+i)^{-1}F(|x| \geqslant R)\|\,dR < \infty$$

and that $D(H) = D(H_0)$. Let $H\phi = E\phi$ with $E \notin \sigma(H_0)$. Then

$$\int_0^\infty \|F(|x| \geqslant R)\phi\|\,dR < \infty. \tag{3.1}$$

Proof. By the arguments in the fourth remark following Theorem 2.1, the commutator estimate and L^1 condition on $\|V(H_0+i)^{-1}F(|x| \geqslant R)\|$ imply that $\int_0^\infty \|Vj_{\geqslant R}(H_0-E)^{-1}\|\,dR < \infty$, so running the argument back:

$$\int_0^\infty \|V(H_0-E)^{-1}F(|x| \geqslant R)\|\,dR < \infty. \tag{3.2}$$

For later purposes, we note that the only E-dependence in the estimate in (3.2) comes from estimating $(H_0-E)^{-1}(H_0+i)$ and so the integral in (3.2) is uniformly bounded for $\{E \mid \text{dist}(E, \sigma(H_0)) \geqslant \epsilon > 0\}$. Now, since V is H_0-operator bounded, $H\phi = E\phi$ implies that

$$\phi = -(H_0-E)^{-1}V\phi$$

so that

$$\|F(|x| \geqslant R)\phi\| \leqslant \|F(|x| \geqslant R)(H_0-E)^{-1}V\|\,\|\phi\|.$$

Thus (3.2) yields (3.1). □

Theorem 3.1 is of interest because it implies that "effective potentials" in multiparticle systems will obey (2.1)-type conditions (See [28, 29] for multiparticle notation):

THEOREM 3.2. *Let* $h_0 = -\Delta$ *on* $L^2(\mathbb{R}^\nu)$. *For each* i, j, $1 \leqslant i < j \leqslant N$, *suppose that* $\int_0^\infty \| v_{ij}(h_0 + i)^{-1} F(|x| \geqslant R) \| dR < \infty$. *Let* H_0 *be the operator on* $L^2(\mathbb{R}^{\nu(N-1)})$ *obtained by removing the center of mass from* $\sum(-2\mu_i)^{-1}\Delta_i$ *and let* $H = H_0 + \sum V_{ij}$ *where* $V_{ij} = v_{ij} \otimes 1$ *according to the coordinate decomposition* ($r_i - r_j$, *orthogonal coordinates*). *Let* D *be a cluster decomposition* $D = \{C_1, \ldots, C_l\}$, *let* $\alpha \equiv (\eta_1, \ldots, \eta_l)$ *be a family of bound states* $H(C_i)\eta_i = E_i \eta_i$ *and let* P_α *be the corresponding channel projection operator onto* $\{(\pi\eta_i)\phi \mid \phi$ *a function of differences of cluster c.m.*$\}$. *Let* $I_D = \sum_{\sim iD_j} V_{ij}$ *be the intercluster potential and let* R_{ij} = *difference between the center of masses of clusters i and j. Suppose that* $\int_0^\infty \| F(|\zeta_i| \geqslant r)\eta_i(\zeta_i) \| dr < \infty$ *where* ζ_i *is the internal coordinates of* C_i. *Then*

$$\int_0^\infty \| F(|R_{ij}| \geqslant R, \text{ all } i,j) P_\alpha (H_0 + i)^{-1} I_D \| dR < \infty. \tag{3.3}$$

Proof. It suffices to consider a single V_{ij} term in I_D, say with $i \in C_1$ and $j \in C_2$. Then $r_{ij} = R_{12} + \zeta_1 + \zeta_2$ with ζ_1, ζ_2 internal coordinates of C_1 and C_2 respectively. Clearly, if $|R_{12}| \geqslant R$, either $|r_{ij}| \geqslant R/3$ or $|\zeta_1| \geqslant R/3$ or $|\zeta_2| \geqslant R/3$, so

$$F(|R_{kl}| \geqslant R \text{ all } kl) \leqslant F\left(|r_{ij}| \geqslant \frac{R}{3}\right) + F\left(|\zeta_1| \geqslant \frac{R}{3}\right) + F\left(|\zeta_2| \geqslant \frac{R}{3}\right) \tag{3.4}$$

and therefore, we need only prove (3.3) with I_D replaced by V_{ij}, and F replaced by each possible F on the right of (3.4). The condition on $\| F(|\zeta_i| \geqslant r)\eta_i \|$ implies that the $F(|\zeta_j| \geqslant R/3)P_\alpha$ terms are in L^1. Writing

$$F\left(|r_{ij}| \geqslant \frac{R}{3}\right)P_\alpha = F\left(|r_{ij}| \geqslant \frac{R}{3}\right)P_\alpha \left\{ F\left(|r_{ij}| \leqslant \frac{R}{6}\right) + F\left(|r_{ij}| \geqslant \frac{R}{6}\right) \right\}$$

it suffices to prove that

$$\int_0^\infty \left\| F\left(|r_{ij}| \geqslant \frac{R}{6}\right)(H_0 + i)^{-1} V_{ij} \right\| dR < \infty \tag{3.5}$$

and

$$\int_0^\infty \left\| F\left(|r_{ij}| \geqslant \frac{R}{3}\right) P_\alpha F\left(|r_{ij}| \leqslant \frac{R}{6}\right) \right\| dR < \infty. \tag{3.6}$$

(3.5) follows from the hypothesis on $V_{ij}(h_0 + i)^{-1}F$ and the uniformity noted after (3.2).

To prove (3.6), we note that $|r_{ij}| \geqslant R/3$ (resp. $\leqslant R/6$) implies that either $|R_{12}| \geqslant R/4$ (resp. $\leqslant R/5$) or some $|\zeta_j| \geqslant R/24$ (resp. $\geqslant R/60$). The object in (3.6) is thus dominated by a sum of 9 terms. 8 of them have a $P_\alpha F(|\zeta_j| \geqslant \delta R)$ or $F(|\zeta_j| \geqslant \delta R)P_\alpha$ and so L^1 norms by the hypothesis on $F(|\zeta_i| \geqslant r)\eta_i$. The ninth term is $F(|R_{12}| \geqslant R/4)P_\alpha F(|R_{12}| \leqslant R/5)$ which is zero since R_{12} commutes with P_α. □

Example 3.1. If V is a local potential, it is well-known that $E \notin \sigma(H_0)$ implies that $\|F(|x| \geq r)\eta\|$ falls off exponentially in r for $H\eta = E\eta$. But Theorem 3.1 applies to any V including non-local V's. For example, pick any $\eta \in D(H_0)$ and $E \neq (\eta, H_0\eta)$. Let $\psi = (H_0 - E)\eta$ and

$$V\phi = -(\eta, \psi)^{-1}(\psi, \phi)\psi$$

so that $(H_0 + V)\eta = E_\eta$. The condition $\|V(H_0 - E)^{-1}F(|x| \geq R)\| \in L^1$ is equivalent to $\|F(|x| \geq R)\eta\| \in L^1$, so that there may be no more falloff then that guaranteed by Theorem 3.1.

Example 3.2. If V is a one dimensional potential which is asymptotically $\alpha(\alpha + 1)|x|^{-2}$ at infinity $(\alpha > 0)$, then a solution of $-\phi'' - V\phi = 0$ will asymptotically look like $x^{-\alpha}$ at infinity. If $1/2 < \alpha \leq 3/2$, then $\phi \in L^2$ but $F(|x| \geq R)\phi$ is not in L^1. This shows that Theorem 3.2 will not extend to general bound states at thresholds. This may produce difficulties in the multiparticle scattering theory.

We expect that Theorem 3.1 extends to any E not in $\overline{C_v \cup S_v}$. For local potentials, this is a result of Agmon-Kuroda theory.

§4. Forms. The hypotheses of Theorem 2.1 require V to be a densely defined operator. In many cases of physical or mathematical interest, V is only a quadratic form. In this section, we want to describe modifications to accommodate V's which are only forms. We suppose that V is positive. One easily accommodates relatively form bounded perturbations of H_0 which are negative.

THEOREM 4.1. *Let H_0 be a positive, vaguely elliptic operator. Let V be a positive quadratic form so that for some N*

(i) $(k^{2N} + 1)^{-1}V(k^{2N} + 1)^{-1}$ *is bounded, i.e., V is a relatively form bounded perturbation of k^{4N}.*

(ii) $H = H_0 + V$ *is a closed quadratic form on* $Q(H_0) \cap Q(V)$ *with* $Q(H_0) \cap D(|k|^{2N})$ *as form core.*

(iii) $V = W^*U$

(iv) $W(H + 1)^{-1/2}$ *is bounded.*

(v) $\int_0^\infty \|U(k^{2N} + 1)^{-1}F(|x| \geq R)\|dR < \infty$.

Then H obeys the conclusions of Theorem 2.1.

Remarks 1. $V = W^*U$ is meant in the sense that U, W are operators from $D(|k|^{2N})$ to $\mathcal{H}$ so that

$$(\phi, V\psi) \equiv (W\phi, U\psi) \tag{4.1}$$

Given (i)–(v), (4.1) then easily extends to $\phi \in Q(H)$ and $\psi \in D(k^{2N}) \cap Q(H)$.

2. U, W may be maps from L^2 to another Hilbert space K. By following with a unitary map from K to L^2, we can restate everything in terms of $K = L^2$. If we take $K = \bigoplus_{i=1}^n L^2$, we can accommodate $V = \sum W_i^* U_i$.

3. Hypotheses (i) and (ii) imply that C_0^∞ (k-space) is a form core for H. For,

given $\phi \in Q(H_0) \cap D(|k|^{2n})$, the obvious C_0^∞ (k-space) approximants converge in H-form norm.

4. Hypothesis (ii) holds if V is a form bounded perturbation of H_0 since then $Q(H_0) \cap Q(V) = Q(H_0)$ and it suffices that $Q(H_0) \cap D(|k|^{2N})$ is a form core for H_0, which is obvious since C_0^∞ (k-space) is a core for H_0.

5. If V is a closed form, then $Q(H_0) \cap Q(V)$ is a closed form. The core hypothesis does not seem to be automatic. In case $H_0 = -\Delta$ and V is in L^1_{loc}, C_0^∞ (x-space) and hence $D(H_0) \cap D(|k|^{2N})$ is a form core; see e.g., [37].

Proof of Theorem 4.1. The proof exactly follows that of Theorem 2.1 with two changes: (a) One must modify the proof of lemma 4 (b) Cook's method must be modified to accommodate forms following [35].

(a). The proof of lemma 4 estimated $[(H-z)^{-1} - (H_0 - z)^{-1}](k^{2N}+1)^{-1}j$ as $(H-z)^{-1}V(H_0-z)^{-1}(|k|^{2N}+1)^{-1}j$. If we use hypothesis (iv), it suffices to show that $\| U(H_0 - z)^{-1}(|k|^{2N}+1)^{-1}j_{\geqslant R}\| \to 0$ and this follows by the argument in lemma 4. The formula

$$\begin{aligned}&\left[(H-z)^{-1} - (H_0 - z)^{-1}\right]\left(|k|^{2N}+1\right)^{-1}\\ &= \left[W(H-z)^{-1}\right]^*\left[U(H_0-z)^{-1}\left(|k|^{2N}+1\right)^{-1}\right]\end{aligned}$$

which is required follows from the extended version of (4.1).

(b). Let $Q(s) = e^{isH}e^{-isH_0}$. Since

$$\|(Q(t) - Q(s))\phi\|^2 = (Q(t)\phi, [Q(t) - Q(s)]\phi) + (Q(s)\phi, [Q(s) - Q(t)]\phi)$$

we need only show how to estimate $(Q(t)\phi, [Q(t) - Q(s)]\phi)$. Following [35], we write for $\phi \in C_0^\infty$ (k-space)

$$\begin{aligned}(Q(t)\phi, (Q(t) - Q(s))\phi) &= \left(e^{-itH_0}\phi, e^{-i(t-\tau)H}e^{-i\tau H_0}\phi\right)\Big|_s^t\\ &\leqslant \left[\sup_{u,v} \| We^{iuH}e^{-ivH_0}\phi\|\right]\int_s^t \| Ue^{-ixH_0}\phi\| \, dx. \qquad (4.2)\end{aligned}$$

The estimates necessary to justify evaluating $d/d\tau(--)$ as $(W--, U--)$ from (4.1) are proven below.

By hypothesis, $(H+1)^{1/2}(|k|^{2N} + P(k) + 1)^{-1}$ is bounded so that

$$\begin{aligned}\| We^{ivH}e^{-iuH_0}\phi\| &\leqslant \| W(H+1)^{-1/2}\| \, \|(H+1)^{1/2}\left(|k|^{2N} + P(k) + 1\right)^{-1}\|\\ &\quad \times \|\left(k^{2N} + P(k) + 1\right)\phi\|\end{aligned}$$

is bounded uniformly in v and u. Given (4.2): the proof of Theorem 2.1(a) and Theorem 2.2 go through easily. □

In applying this theorem, we must choose a factorization of V. This must be done in such a way that $W(H+1)^{-1/2}$ is bounded. Since $Q(H) = Q(H_0) \cap$

$Q(V)$, there are two natural ways of trying to do this; either bound $W(V+1)^{-1/2}$ or $W(H_0+1)^{-1/2}$. We do this for $-\Delta + V$ in examples 1 and 2 below. Note that the first possibility allows worse local singularities at the cost of requiring more falloff.

Example 4.1. Suppose that $V \geqslant 0$, $\int_0^\infty [\sup_{|\alpha| \geqslant R} (\int_{\Delta_\alpha} |V(x)| dx)^{1/2}] dR < \infty$ (e.g., $(1+|x|)^{2+\epsilon} V \in L_u^1$, the function uniformly in L_{loc}^1) and let $H_0 = -\Delta$. If $2N > \nu$, $W(k^{2N}+1)^{-1}$ is bounded so long as $W \in L_u^2$ so (i) holds. (ii) holds according to remark 5 above. We take $W = U = |V|^{1/2}$. Then (iv) holds since $Q(H) \subset Q(V) = D(W)$. (v) holds as in Example 2.1.

Example 4.2. Suppose that $P(k) \geqslant c(|k|^l - 1)$, and that $0 \leqslant \tilde{V} \equiv (1 + |x|)^{1+\epsilon} V \in L_u^p$ where $p^{-1} = (1/2 + 2/l\nu)$ for $l\nu > 4$, $p^{-1} = 1$ if $l\nu < 4$ and $p^{-1} < 1$ if $l\nu = 4$. Then one can factor $V = W^* U$ with $U = |\tilde{V}|^{p/2}(1+|x|)^{-1-\epsilon}$ and $W = |\tilde{V}|^{2p/l\nu}$. Then $(1+|x|)^{1+\epsilon} U \in L_u^2$, so U obeys (v). By a Sobolev-type estimate [40], W obeys $\| W(H_0+1)^{-1/2} \| < \infty$ and so (iv) holds. (i) always holds and (ii) will hold under suitable circumstances.

We summarize these two examples with

THEOREM 4.2. *Let $H = -\Delta + V$ (where $V \geqslant 0$) be defined as a form sum. Suppose $V = V_1 + V_2$ with $V_i \geqslant 0$ and*

(a) $\sup_\alpha (1+|\alpha|)^{2+\epsilon} [\int_{\Delta_\alpha} |V_1(x)| dx] < \infty$.

(b) $\sup_\alpha (1+|\alpha|)^{1+\epsilon} [\int_{\Delta_\alpha} |V_2(x)|^p dx] < \infty$

with $p^{-1} = 1/2 + 1/\nu$ ($\nu > 2$), $p^{-1} = 1$ ($\nu = 1$) or $p^{-1} < 1$ ($\nu = 2$). Then H obeys the conclusions of Theorem 2.1.

This result should hold with $p = 1$ for general ν. The only previous results on completeness only supposing L_{loc}^1 conditions are those allowing purely local singularities (see §6) and those appealing to the Kato-Birman theory and thus faster falloff.

Example 4.3. One might think that a V which is singular (i.e., non-closable) form cannot be factored but this is false. For example, let $V \equiv \mu$ be a finite measure on $(-\infty, \infty)$. Then $(1-\Delta)^{-1/2} \mu (1-\Delta)^{-1/2} = A$ is bounded so we can take $U = W = A^{1/2}(1-\Delta)^{1/2}$. Since V is $-\Delta$-form bounded with relative bound zero, (ii) is obvious as is (i). (iv) holds easily and (v) will hold if μ has $|x|^{-2-\epsilon}$ falloff. Presumably $|x|^{-1-\epsilon}$ falloff can be accommodated. Higher dimensional measures of the type treated by Davies [8] can be accommodated.

This example illustrates the fact that (iii) is only a formal way of writing (4.1) for the W here is *not* closable as an operator, i.e., W^* is not densely defined.

Example 4.4. Following Schechter [32, 33] and Combescure-Ginibre [6], one can handle highly oscillatory potentials. For example if $V = \nabla W$ with $(1+|x|)^{1+\epsilon} W \in L_u^2(\mathrm{R}^3)$ (e.g., $W(x) = (1+|x|)^{-2} \cos(e^{|x|})$), then $-\Delta + V$ obeys all the conclusions of Theorem 2.1. For $V = W_1^* U_1 + W_2^* U_2$ with $W_1^* = \nabla$, $U_1 = W$, $W_2^* = (1+|x|)^{1+\epsilon} W$, $U_2 = (1+|x|)^{-1-\epsilon} \nabla$. (i), (ii), (v) are easy since U_i^2, W_i^2 are $-\Delta$ bounded. (iv) follows as in Example 2.1; see note added in proof.

§5. Other eigenfunction expansions: solids, magnons. In this section we want to consider what happens when H_0 is not a pseudo-differential operator but rather an operator with a distinct but still nice eigenfunction expansion. The two cases we have in mind are solids, i.e., $H_0 = -\Delta + W$ with W periodic and magnons, i.e., H_0 is a finite difference operator on $l^2(\mathbb{Z}^\nu)$. We will see that (generically) solids can be accommodated using a Bloch wave expansion in three or fewer dimensions but that there are difficulties, perhaps soluable, in four or more dimensions (see Example 5.3).

The precise nature of the Fourier eigenfunction expansion for the H_0 of §2 enters at four technical points:

(I) Lemma 1, "the no stationary phase" argument yielding power falloff outside of the classical region.

(II) The existence of maps f_α which sum up to 1 so that $f_\alpha \phi$ is localized near α while f_α doesn't destroy much strict localization in the eigenfunction transform.

(III) Lemma 2, the result bounding $\sum g_\alpha(k) f_\alpha(x)$.

(IV) Lemma 3–5, in proving $\|[\Phi(H) - \Phi(H_0)] j_{\geqslant R} E_{(a,b)}(H)\| \to 0$.

The fourth problem is easily solved if we don't allow extremely singular V's and require that $\|V(H+i)^{-1} F(|x| \geqslant R)\| = h(R)$ is in $L^1 \cap L^\infty$. Then $\|[(H-z)^{-1} - (H_0 - z)^{-1}] F(|x| \geqslant R)\| \to 0$ so, as in lemma 4, $\|[\Phi(H) - \Phi(H_0)] F(|x| \geqslant R)\| \to 0$ for any Φ in $C_\infty(\mathbb{R})$. The third problem is solved by using

LEMMA 2′. *Let $\{f_\alpha\}_{\alpha \in \mathbb{Z}^\nu}$ be a family of positive operators on $\mathcal{H}$ so that s-lim $\sum f_\alpha = 1$. Let $g^{(1)}, \ldots, g^{(m)}$ be a finite number of bounded operators and for each α, let g_α be one of $g^{(1)}, \ldots, g^{(m)}$. Then*

$$\mathrm{Th} = \sum g_\alpha f_\alpha h$$

is a bounded operator.

Proof. Let $A_i = \{\alpha \mid g_\alpha \equiv g^{(i)}\}$. Then

$$\|\mathrm{Th}\| \leqslant \sum_{i=1}^{m} \|g^{(i)}\| \left\| \left(\sum_{\alpha \in A_i} f_\alpha h \right) \right\| \leqslant \sum_{i=1}^{m} \|g^{(i)}\| \, \|h\|$$

since $\sum_{\alpha \in A_i} f_\alpha \leqslant 1$. □

In section 2, we picked g_α which were different for infinitely many α's but this was a luxury which was not really necessary. What was important was that the velocities in supp g_α^{out} have a gap in some cone about $-\alpha$. We arranged this by choosing the cone, C_α of open angle 45° about $-\alpha$, but clearly we could have used a finite number of cones, $K_1, \ldots, K_n$ of opening angle 60° and been sure that one contained C_α. In this way, we could have arranged to have only finitely many $g^{(i)}$'s and this is what we will do below.

This reduces the difficulties to a detailed consideration of problems (I) and (II).

Example 5.1. (One Dimensional Solids) The necessary "stationary phase" analysis of lemma 1 has already been noted by Davies-Simon [10]; we repeat their argument since we need the notation for overcoming (II): Let W be a periodic function on $(-\infty, \infty)$; say $W(x+1) = W(x)$ and suppose $\int_0^1 |W(x)|dx < \infty$. Let $H_0 = -d^2/dx^2 + W$. Then, [29], there exist functions, $\phi_n(x, k)$, $-\infty < x < \infty$, $-\pi \leqslant k \leqslant \pi$, and $\epsilon_n(k)$, with the following properties;

(a) $\phi_n(x, k) = e^{ikx}u_n(x, k)$ with $u_n(x+1, k) = u_n(x, k)$,

(b) $\epsilon_n(k)$ and $f(x)u_n(x, k)$ (for $f \in C_0^\infty$) are real analytic in k for $k \in (0, \pi)$ and $(-\pi, 0)$ (fu_n analytic as an L^2-valued function); they are continuous up to the boundary points $0, \pm\pi$,

(c) $\epsilon_1(k) \leqslant \epsilon_2(k) \leqslant \cdots \leqslant \epsilon_n(k) \leqslant \to \infty$, with strict inequality for $k \neq 0, \pm\pi$. $\epsilon_n(k) = \epsilon_n(-k)$.

(d) for $g \in L^1 \cap L^2$, $g_n^\dagger(k) = \int \overline{\phi_n(x, k)} g(x)dk$ obeys $\sum_n \int_{-\pi}^{\pi} |g_n^\dagger(k)|^2 dk = \|g\|_2^2$

(e) For $a_1, a_2, \ldots, a_n \in C_0^\infty[(-\pi, 0) \cup (0, \pi)]$,

$$a^b(i) = \sum_n \int \phi_n(x, k) a_n(k) dk$$

obeys $\|a^b\|_2^2 = \sum_n \int_{-\pi}^{\pi} |g_n(k)|^2 dk$.

(f) ^{b} and † extend to unitary operators between $L^2(-\infty, \infty)$ and $\bigoplus_1^\infty L^2(-\pi, \pi)$ and are inverse to one another

(g) For $g \in D(H_0)$:

$$(H_0 g)_n^\dagger(k) = \epsilon_n(k) g_n^\dagger(k).$$

We define the *critical values* of H_0 to be the values $\epsilon_n(0)$, $\epsilon_n(\pm\pi)$ and the value $\epsilon_n(k_0)$ for any n, k_0 with $(d\epsilon_n/dk)(k_0) = 0$. Since the ϵ_n are analytic and $\epsilon_n \to \infty$ as $n \to \infty$, the critical values are discrete in $(-\infty, \infty)$, i.e., their only possible limit point is ∞. (We note that generically, the ϵ_n are all analytic on $[-\pi, \pi]$ with end points associated; then by symmetry $(\partial\epsilon_n/\partial k)(k_0) = 0$ for $k_0 = 0, \pm\pi$ so that the $\epsilon_n(0)$, $\epsilon_n(\pm\pi)$ are critical values in the usual sense. However, it can happen that ϵ_n continues smoothly through 0 and continues into ϵ_{n+1} in such a way that $(\partial\epsilon_n/\partial k)(0) \neq 0$. In this case one can, by giving some additional arguments, remove this $\epsilon_n(0)$ from the critical values.)

Now fix n and an interval $[a, b]$ in $(0, \pi)$ or $(-\pi, 0)$ containing no critical points for ϵ_n. Let $\mathcal{O}$ be an open set containing the set of velocities $\{\partial\epsilon_n/\partial k \mid k \in [a, b]\}$. Then for $\eta \in C_0^\infty(a, b)$ and $\psi(x) = \int_a^b \eta(k)\phi_n(x, k)dk$ and $x/t \notin \mathcal{O}$:

$$\left(\int_{|x-y| \leqslant 1} |(e^{-itH_0}\psi)(y)|^2\right)^{1/2} \leqslant C_l(1 + |x| + |t|)^{-l} \left\| \left(1 - \frac{d^2}{dk^2}\right)^l \eta \right\|_2 \tag{5.1}$$

(5.1) follows by writing

$$(e^{-itH_0}\psi)(x) = \int_a^b \eta(k) u_n(x, k) e^{-i\epsilon_n(k)t + ikx} dk$$

using $e^{-i\epsilon_n(k)t + ikx} = \{[ix - it(\partial\epsilon_n/\partial k)]\partial/\partial k\}^l e^{-i\epsilon_n(k)t + ikx}$ and integrating by parts. Since $\partial^m u/\partial k^m$ is periodic in x, it has no growth in x. The fact that the local L^2-norm occurs on the left of (5.1) follows from the fact that we only stated analyticity of $u(x, k)$ in local L^2-sense. By a little more work one can prove for $x/t \notin \mathcal{O}$

$$|(e^{-itH_0}\psi)(x)| \leqslant \text{RHS of (5.1)} \tag{5.1$'$}$$

for $(\partial/\partial x)u(x, k)$ is also locally in L^2 and periodic which yields L^∞ bounds on $u(x, k)$. This solves problem (I).

To solve problem (II), we first note that it isn't necessary to define x-space localization operators for all ψ's at once. We need only consider ψ of the above form $\int_a^b \eta(k)\phi_n(x, k)dk$. Let j be a function on the circle $(-\pi, \pi)$ which is positive definite as a function on the group, of very small support, and with a normalization condition given below. Let $j_m(k) = e^{imk}j(k)$ $(m \in Z)$ which is also positive definite. Define

$$A_m\psi = \int_{-\pi}^{\pi} (j_m * \eta)(k)\phi_n(x, k)dk.$$

A_m is a positive operator since j_m is positive definite and $(\psi, A_m\psi) = \int_{-\pi}^{\pi}\overline{\eta(k)}(j_m * \eta)(k)dk$. Moreover, using (5.1′) and translation covariance, we see that for $x - m/t \notin \mathcal{O}$:

$$\left|\left[e^{-itH_0}(A_m\psi)\right](x)\right| \leqslant C_l(1 + |x - m| + |t|)^{-l}\|\psi\|_2 \tag{5.2}$$

so long as supp η + supp $j \subset [a, b]$ (sum mod 2π). In addition, if j is normalized so that

$$\sum_m \left[\int e^{imk}j(k)dk\right] = 1$$

then $\sum A_m = 1$ (to obtain this result expand η in a Fourier series). The A_m's solve Problem (II).

We can now prove:

THEOREM 5.1. *Let W, V be functions on $(-\infty, \infty)$ with $W(x + 1) = W(x)$, $\int_0^1 |W(x)|dx < \infty$ and $\sup_n \int_n^{n+1}[(1 + |x|)^{1+\epsilon}|V(x)|]^2dx < \infty$. Let $H = -d^2/dx^2 + V + W$ and $H_0 = -d^2/dx^2 + W$. Then the conclusions of Theorem 2.1 hold.*

Proof. Existence of $\Omega^\pm$ follows from (5.1) and the argument following lemma 1 in Section 2. The other conclusions follow if we can prove an Enss decomposition principle since $F(|x| \leqslant n)(H + i)^{-1}$ is compact on account of $Q(H) = Q(-d^2/dx^2)$. If $[a, b]$ contains no critical values, we can decompose any $\psi \in E_{[a, b]}(H_0)$ as a finite sum of terms of the form $\int_{\alpha_n}^{\beta_n} \eta_n(k)\phi_n(x, k)dk$. Using (5.1), (5.2) and the A_m's we can now mimic the proof of theorem 2.1. □

Example 5.2. (Quasi-1 dimensional periodic systems). Let W be periodic on R^ν with independent periods $a_1, \ldots, a_\nu$. Here we want to consider $-\Delta + W + V$ where V is periodic under translation by $a_1, a_2, \ldots, a_{\nu-1}$ with falloff in the a_ν direction. This is somewhat artificial but the analysis will be useful in the next section to analyze "half-solids" and dislocations in solids. The point is that one can make direct integral decompositions $H_0 \equiv -\Delta + W = \int_{[0,2\pi)^{\nu-1}}^{\oplus} h_0(\theta)d\theta$, $H \equiv H_0 + V = \int_{[0,2\pi)^{\nu-1}}^{\oplus} h(\theta)d\theta$ where $h_0(\theta)$ is $-\Delta + W$ on a one dimensional tube with θ-dependent boundary conditions on the boundary of the tube; see [10]. The analysis of Example 5.1 goes over without any real change to study $h_0(\theta)$ and $h(\theta)$ for fixed θ. $h_0(\theta)$ still has a nice eigenfunction expansion. We single out as *singular points* those points where some eigenvalue $\epsilon_n(k, \theta)$ is degenerate and the degeneracy is removed by varying k. Again the singular and critical values are discrete. The only real change is that (5.1′) fails in general; rather L^2 can be replaced by a suitable L^p. The result is:

THEOREM 5.2. *Let V, W be functions which are uniformly locally L^p where $p = 2$ ($\nu \leqslant 3$), $p > 2$ ($\nu = 4$) or $p = \nu/2$. Suppose that $a_1, \ldots, a_\nu$ is a basis for $\mathbb{R}^\nu$ and that both V and W are periodic under translation by $a_1, \ldots, a_{\nu-1}$. Suppose moreover that W is periodic under a_ν and $(1 + |x \cdot a_\nu|)^{1+\epsilon} V$ is uniformly locally L^p. Let $h_0(\theta)$, $h(\theta)$ be the fibers of the direct integral decomposition of $H_0 = -\Delta + W$ and $H = -\Delta + V + W$ induced by the translation symmetry under $a_1, \ldots, a_{\nu-1}$. Then $h_0(\theta)$, $h(\theta)$ obey the conclusions of Theorem* 2.1 *for each fixed θ.*

Example 5.3. (ν-dimensional solids) Let W be periodic on $\mathbb{R}^\nu$ with independent periods $a_1, \ldots, a_\nu$. We want to discuss $H = -\Delta + W + V$ where V has falloff in all directions. We take $H_0 = -\Delta + W$. By a standard analysis [29], H_0 has an eigenfunction expansion similar to that in the one dimensional case; namely, for $k \in B$ (the Brillouin zone, a suitable set in $\mathbb{R}^\nu$), there are eigenvalues $\epsilon_n(k)$ and functions $\phi_n(x, k)$ in $L^2_{\rm loc}$ so that

(a) $\phi_n(x, k) = e^{ikx}u_n(x, k)$ with $u_n(x + a_j, k) = u(x, k)$; $j = 1, \ldots, \nu$.

(b) $\epsilon_n(k)$ and $f(x)u_n(x, k)$ (for $f \in C_0^\infty$) are real analytic in the neighborhood of any k_0 in B for which $\epsilon_n(k_0) \neq \epsilon_j(k_0)$ ($j \neq n$).

(c) $\epsilon_1(k) \leqslant \cdots \leqslant \epsilon_j(k) \leqslant \cdots \to \infty$.

(d) For $g \in \mathcal{S}$, $g_n^\dagger(k) = \int \overline{\phi_n(x, k)} g(x)dx$ obeys $\sum_n \int_B |g_n^\dagger(k)|^2 d^\nu k = \|g\|_2^2$.

(e) For $\alpha_1, \ldots, \alpha_n \in C_0^\infty(B)$, $\alpha^b(x) = \sum_n \int_B \phi_n(x, k)\alpha_n dk$ obeys $\|\alpha^b\|_2^2 = \sum_n \int_B |\alpha_n(k)|^2 d^\nu k$.

(f) ^{b} and † extend to unitaries between $L^2(\mathbb{R}^\nu)$ and $\bigoplus_1^\infty L^2(B)$ and are inverse to one another

(g) For $g \in D(H_0)$:

$$(H_0 g)_n^\dagger(k) = \epsilon_n(k) g_n^\dagger(k).$$

We define a *singular value* of H_0 to be the value $\epsilon_n(k_0)$ at a point k_0 with a $\epsilon_n(k_0)$ degenerate eigenvalue i.e., $\epsilon_n(k_0) = \epsilon_j(k_0)$ and a *critical value* is the value $\epsilon_n(k_0)$ at a point with $\nabla \epsilon_n(k_0) = 0$. As in Remark 6 following theorem 2.1, the only possible limit points for C_ν lie in S_ν so that $\overline{C_\nu \cup S_\nu}$ being countable is equivalent to $\overline{S}_\nu$ being countable. By following the arguments in Example 5.1, one immediately sees that

THEOREM 5.3. *Let W be a function which is uniformly local L^p where $p = 2$ ($\nu \leqslant 3$), $p > 2$ ($\nu = 4$) or $p = \nu/2$ and where W is periodic in n independent directions. Suppose that $(1 + |x|)^{1+\epsilon} V \in L^p + L^\infty$ and moreover that S_ν the singular values for $-\Delta + W$ has countable closure. Then the conclusions of Theorem 2.1 hold for $H = -\Delta + V + W$, $H_0 = -\Delta + W$.*

The key difference from the one dimensional case is that we do not know that $\overline{S_\nu}$ is countable in case $\nu \neq 1$. Indeed, we expect that "usually" S_ν contains whole intervals if $\nu \geqslant 4$ while if $\nu \leqslant 3$, S_ν will "usually" be discrete. For, Wigner and von Neumann [42] have proven that the variety of $n \times n$ self-adjoint matrices with a degenerate eigenvalue has codimension 3. Thus, when $\nu \leqslant 3$, the singular points are to be expected to be discrete but for $\nu \geqslant 4$, whole "curves" will occur on which ϵ_n need not be constant. One can ask whether the degeneracies that occur at singular points are really serious or whether we might accommodate them with more effort. Typically, the form of band function that worries us is of the form:

$$\epsilon(k_1, k_2, k_3) = k_1 + k_2 \pm \left((k_1 - k_2)^2 + k_3^2\right)^{1/2} \tag{5.3}$$

The occurrence of the two dimensional square root means that the behavior of $\int e^{i\epsilon(k)t - ikx}\eta(k)dk \equiv u(x, t)$ will be similar to that of the two dimensional wave equation rather than what we have when ϵ is smooth; in that case, $u(0, t)$ only falls off like a power of t (namely t^{-1}) so that any Cook-type method is likely to fail when a band function like (5.3) is present. Of course, a trace class method can still work (and will if $V = O(|x|^{-\nu-\epsilon})$). Notice that while (5.3) is three-dimensional, the Wigner-von Neumann result says that such functions are very unlikely as band-functions until $\nu \geqslant 4$.

Example 5.4. (magnons) The framework for magnon scattering is given in [39, 28]. The basic operators are now finite difference operators on $l^2(\mathbf{Z}^\nu)$. Using the appropriate complex exponentials, it is easy to make a "stationary phase analysis" and prove an analog of lemma 1; indeed, this is done in [28] among other places. For the two magnon sector, it is easy to obtain an analog of Theorem 2.1, indeed, one can add longer range potentials than the natural one that arises from the usual nearest neighbor coupling. Since most of these conclusions (except for absence of singular spectrum) are known [28], we will give no details except to note the potentiality for treating n-magnon scattering once the Schrödinger case is completed.

§6. The Kupsch-Sandhas method: local singularities, the half solid. In this section, we wish to discuss situations that occur when some bounded region of configuration space is badly behaved due to obstacles or severe local singularities of the potential. Cook's method in this case was discussed by Kupsch-Sandhas [24] whose method we combine with that of Enss. In the discussion below, J is typically multiplication by a function which is smooth, identically one near infinity and zero on the bad set. Suppose that H_0 is a constant coefficient pseudo-differential operator and that for some self-adjoint H

$$JF(k \in K) \subset D(H) \tag{6.1}$$

for any compact set K in k-space. Then one can define

$$Q = HJ - JH_0 \tag{6.2}$$

as an operator from $\cup_K \operatorname{Ran} F(k \in K)$ to L^2. While one could presumably (with only a little more effort) handle cases where Q is a form or where only $Q(k^{2N} + 1)^{-1}$ is bounded, the discussion is simpler if we assume relative boundedness:

THEOREM 6.1. *Let $H_0 = P(k)$ with P vaguely elliptic. Let H be a self-adjoint operator and J a bounded operator so that*

(a) *$JF(|x| \geqslant R) = F(|x| \geqslant R)$ for all large R.*

(b) *$F(|x| \leqslant R)(H + i)^{-1}$ is compact for each $R < \infty$.*

(c) (6.1) *holds.*

(d) *For each $z \notin \sigma(H_0)$: the function $h(R) \equiv \|Q(H_0 - z)^{-1}F(|x| \geqslant R)\|$ with Q given by* (6.2) *obeys*

$$h(0) < \infty \tag{6.3}$$

$$\int_0^\infty h(R)\,dR < \infty. \tag{6.4}$$

Then the conclusions of Theorem 2.1 *hold.*

Proof. We will indicate only the changes needed in the scheme of section 2. By Cook's argument and lemma 1 (of section 2), one easily sees that

$$\underset{t \to \mp\infty}{s\text{-lim}}\, e^{itH} J e^{-itH_0} = \tilde{\Omega}^{\pm}$$

exists in just the way we proved Theorem 2.1(a). Writing

$$(1 - J)(H_0 + i)^{-1} = (1 - J)F(|x| \leqslant R)(H_0 + i)^{-1} + (1 - J)F(|x| \geqslant R)(H_0 + i)^{-1}$$

and using the fact that, by hypothesis (a), the second term is zero for R large, we see that $(1 - J)(H_0 + 1)^{-1}$ is compact. By a standard approximation argument [28],

$$\underset{t \to \pm\infty}{s\text{-lim}}\, e^{itH}(1 - J)e^{-itH_0} = 0$$

so $\Omega^{\pm}$ exist and equal $\tilde{\Omega}^{\pm}$.

The proof is completed if we describe the changes necessary in the proof of the Enss decomposition principle since once this holds, all three conclusions of the theorem follow easily. The main change is in Lemmas 3–5 of section 2. We first claim that for Φ as in lemma 4:

$$a(R) \equiv \|[\Phi(H)J - J\Phi(H_0)]F(|x| \geqslant R)\| \to 0 \tag{6.5}$$

as $R \to \infty$. As in the proof of that lemma, one need only consider the case $\Phi(x) = (x - z)^{-1}$ with z near $\pm i$. But in that case

$$a(R) \leqslant \|(H - z)^{-1}\| \, \|Q(H_0 - z)^{-1}F(|x| \geqslant R)\|$$

goes to zero by (6.4).

Now let ϕ_n obey the hypotheses of Theorem 2.2. Let Φ be the function used in the proof of Theorem 2.2. Then, by (6.5) and hypothesis (a),

$$\|[\Phi(H) - J\Phi(H_0)]F(|x| \geqslant n)\phi_n\| \to 0$$

Moreover, since $\Phi(H_0)$ is convolution with a fixed function in $\mathcal{S}$:

$$\left\| F\left(|x| \leqslant \frac{n}{2}\right)\Phi(H_0)F(|x| \geqslant n)\phi_n \right\| \to 0.$$

Since, by hypothesis (a), $(1 - J)F(|x| \geqslant n/2) \to 0$, we see that

$$\|(1 - J)\Phi(H_0)F(|x| \geqslant n)\phi_n\| \to 0.$$

Using these estimates and $\|F(|x| \leqslant n)\phi_n\| \to 0$, we see that

$$\|[\Phi(H) - \Phi(H_0)]\phi_n\| \to 0.$$

Given this, the proof follows that in Theorem 2.2 except that in estimating $(\Omega^{\pm} - 1)\phi^{\text{in}}_{\text{out}}$, one uses $\Omega^{\pm} = \tilde{\Omega}^{\pm}$ and Cook's argument for $e^{itH}Je^{-itH_0}$. □

In the example below, we obtain a variety of results on the absence of singular spectrum. The only method of proof previously available was the "twisting trick" of Davies-Simon [10].

Example 6.1. (Severe local singularities) We can recover the results of a variety of authors [7, 12, 25, 34] on completeness and absence of singular spectrum [10] for $-\Delta + V$ where V is allowed to have severe local singularities. We suppose that $V = V_+ - V_-$ where V_+ is locally L^1 away from some closed set K of measure zero and positive and where V_- is $-\Delta$-form bounded with relative bound strictly less than one. Moreover, we suppose that $F(|x| \geqslant R_0)V$ obeys the hypothesis in Enss' original paper for some R_0 i.e., $W = F(|x| \geqslant R_0)V$ obeys (2.1) with $N = 1$. Taking J to be multiplication by a function which is smooth and equal to 0 (resp. 1) for $|x| \leqslant (3/2)R_0$ (resp. $\geqslant 2R_0$), it is easy to verify the hypotheses for Theorem 6.1 for H, the form sum of $-\Delta$ and V.

Example 6.2. (Schrödinger obstacle scattering) Let Ω be a closed bounded set and let H_1 be some self-adjoint extension of $-\Delta$ on $\mathbb{R}^\nu\backslash\Omega$. Pick any basis $\phi_1, \ldots, \phi_n, \ldots$ for $L^2(\Omega)$ and let $H = H_1 \oplus \sum n(\phi_n, \cdot)\phi_n$ on $L^2(\mathbb{R}^\nu) = L^2(\Omega) \oplus L^2(\mathbb{R}^\nu\backslash\Omega)$. Pick J to be a multiplication operator obeying $1 - J \in C_0^\infty$ and $J = 0$ on a neighborhood of Ω. All the hypotheses of Theorem 6.1 hold trivially with one exception: hypotheses (b) will hold or not depending on the boundary condition in H_1. For Dirichlet boundary conditions, it is automatic; for Neumann boundary conditions, it will hold if, for example, Ω obeys the segment condition [11]. In any event, if it holds, then H_1 as an operator on $L^2(\mathbb{R}^\nu\backslash\Omega)$ will have empty singular spectrum and only zero as a possible accumulation point of eigenvalues (in this generality, this is a result of [10]). There is also a completeness result.

Example 6.3. (Accoustical scattering from obstacles; Example 1.2) Given Kato's analysis [23], the results of Example 6.2 immediately recover completeness results for acoustical scattering, i.e., for the equation $u_{tt} = -H_1 u$. One also obviously gets the absence of singular spectrum for the generator. These results can also be obtained directly by combining the methods of Example 2.4 and Theorem 6.1.

Example 6.4. (Half-Solid = Example 1.5). For one dimensional and quasi-one dimensional problems, one can use the Kupsch-Sandhas "J-trick" with J's that are zero on a half-line. Consider, for example a bounded function V periodic on $(-\infty, \infty)$ and let W be equal to V (resp. 0) for $x > 0$ (resp. < 0). Let $H = -d^2/dx^2 + W$, $H_0^{(l)} = -d^2/dx^2$ and $H_0^{(r)} = -d^2/dx^2 + V$. Then one can show (recovering results of [10]) that (b), (d), (e) of Theorem 2.1 hold and

$$\Omega_{l,r}^{\pm} = \underset{t\to\mp\infty}{s\text{-lim}}\, e^{itH} J_{l,r} e^{-itH_0^{(l,r)}} \tag{a$'$}$$

exist, where J_l is multiplication by a smooth function which is 0 (resp. 1) for $x > 1$ (resp. $x < -1$) and $J_r = 1 - J_l$.

$$\begin{aligned}\operatorname{Ran}\Omega_l^+ \oplus \operatorname{Ran}\Omega_r^+ &= \operatorname{Ran}\Omega_r^+ \oplus \operatorname{Ran}\Omega_r^- \\ &= \text{absolutely continuous space for } H. \qquad \text{(c$'$)}\end{aligned}$$

To do this, one need only follow the proof of Theorem 2.1 with the following changes: (i) Rather than deal with $\Phi(H) - \Phi(H_0)$ we deal with $\Phi(H) - (\Phi(H_0^{(l)}) J_l + \Phi(H_0^{(r)})J_r)$. (ii) When analyzing $\phi_{n,\,\substack{\text{in}\\ \text{out}}}$ make a further breakup into left and right. (iii) Use Example 5.1 to analyze $H_0^{(r)}$. With no additional effort, one can allow V to have local singularities and add a "short range," i.e., $|x|^{-1-\epsilon}$ and locally singular perturbation to W. By using Example 5.2 one can handle the fibers of a potential which is periodic in $\nu - 1$ directions and "half-periodic" in the last direction, e.g., a potential which is a periodic function times the characteristic function of a half-space $\{x \mid x \cdot q \geqslant 0\}$ (so long as q has "rational indices," i.e., $\{x \mid x \cdot q = 0\} \cap \{\text{periods for } V\}$ is a $\nu - 1$ dimensional lattice). Or the potential which is two different periodic functions on the left and right of the plane $x \cdot q = 0$ can be treated. Such a potential can be used to describe scattering from lattice dislocations in solids.

§7. **Magnetic fields.** In this section, we will describe scattering in a constant magnetic field, a subject recently analyzed by Avron et al. [3], using both trace class methods and the Agmon-Kuroda method. Roughly speaking, we will recover the results of [3] with two differences:

(a) In their trace class theory, [3] allows form perturbation V's; we could do this, but for simplicity, do not.

(b) In the azimuthally symmetric case, we will allow considerable growth in the direction perpendicular to the field; in [3] the trace class theory but not the Agmon-Kuroda theory is done in this generality so that our results on absence of singular spectrum are new in this situation.

Throughout this section, we fix a number $B > 0$ and let

$$H_0 = \left(\frac{1}{i}\frac{\partial}{\partial x} + \frac{B}{2}y\right)^2 + \left(\frac{1}{i}\frac{\partial}{\partial y} - \frac{B}{2}x\right)^2 + \left(\frac{1}{i}\frac{\partial}{\partial z}\right)^2 \tag{7.1}$$

on $L^2(\mathbb{R}^3)$, which is the Hamiltonian of a particle of charge 1 in a magnetic field $(0, 0, B)$. Notice that in terms of the operators $L_z = (1/i)(y(\partial/\partial x) - x(\partial/\partial y))$ and $H_{\text{osc}} = -(\partial^2/\partial x^2) - (\partial^2/\partial y^2) + (B^2/4)(x^2 + y^2)$, we can write

$$H_0 = -\frac{\partial^2}{\partial z^2} + H_{\text{osc}} - BL_z\,. \tag{7.2}$$

Viewed as operators on $L^2(\mathbb{R}^2)$, H_{osc} and L_z can be simultaneously diagonalized: There is a complete set of functions $f_{n,m}(x, y)$ such that $(m = 0, \pm 1, \ldots, \; n = 0, 1, \ldots)$ $L_z f_{n,m} = m f_{n,m}$; $H_{\text{osc}} f_{n,m} = B(2n + |m| + 1) f_{n,m}$. Moreover, $f_{n,m}(x, y)$ is a polynomial in x and y times $\exp[-(B/4)(x^2 + y^2)]$. These results are proven in §3 of [3]. Thus, we can define an eigenfunction transform:

$$\hat{\phi}_{n,m}(k) = (2\pi)^{-1/2} \int e^{-ikz}\, \overline{f_{n,m}(x, y)}\, \phi(\vec{r})\, dr \tag{7.3}$$

so that

$$\int |\phi(\vec{r})|^2 d^3r = \sum_{n,m} \int_{-\infty}^{\infty} dk\, |\hat{\phi}_{n,m}(k)|^2 \tag{7.4}$$

and

$$(e^{-itH_0}\phi)(\vec{r}) = \sum_{n,m} f_{n,m}(x, y) \int_{-\infty}^{\infty} \exp\left(-i\left\{(k^2 + E_{n,m})t - kz\right\}\right) \hat{\phi}_{n,m}(k)\, dk \tag{7.5}$$

where

$$E_{n,m} = B(2n + (|m| - m) + 1). \tag{7.6}$$

Our goal is to compare $H_0 + V$ and H_0. Our results are much stronger in case V is *azimuthally symmetric*, i.e., commutes with rotations in the x, y plane; for, in that case, $F(|z| < a)(H + i)^{-1}F(L_z = m)$ is compact, so the RAGE theorem implies that a wave function in $\mathcal{H}_{\text{cont}}$ must make large excursions in z.

THEOREM 7.1. *Let H_0 be given by* (7.1). *Let V be a symmetric operator obeying*:

(1) *V commutes with rotations about the z axis.*

(2) *$Ve^{-cH_0}F(L_z = m)$ is bounded for some $c > 0$ and each m.*

(3) *$H_0 + V \upharpoonright F(L_z = m)$ is bounded from below on* $\text{Ran}(e^{-cH_0})$ *for each fixed m and the corresponding quadratic form closure defines an operator H with*

$$Q(H \upharpoonright F(L_z = m)) \subset Q(H_0 \upharpoonright F(L_z = m)). \tag{7.7}$$

(4) *Let $h(R) = \| Ve^{-cH_0}F(|z| > R)\|$. Then*

$$\int_0^\infty h(R)dR < \infty. \tag{7.8}$$

Then the conclusions of Theorem 2.1 *hold with two changes*: *in* (d) $\overline{C \cup S}$ *must be replaced by* $\{E_{n,m}\}_{n=0}^\infty$ *and "limit points of the point spectrum of H" by "limit points of the point spectrum of $H \upharpoonright F(L_z = m)$" and* (c) *reads*

$$\sigma_{\text{ess}}(H \upharpoonright F(L_z = m)) = \sigma_{\text{ess}}(H_0 \upharpoonright F(L_z = m)) = [(-m + |m| + 1)B, \infty].$$

Proof. In this proof, fix m and use H, H_0 to stand for $H \upharpoonright F(L_z = m)$, $H_0 \upharpoonright F(L_z = m)$. As in the proof of Theorem 2.1, we need only prove: (1) the Enss decomposition principle where $F(|x| \leqslant n)$ is replaced by $F(|z| \leqslant n)$ (i.e., vector x replaced by its component parallel to B) and $\overline{S_v \cup C_v}$ by $\{E_{n,m}\}_{n=0}^\infty$. (2) Compactness of $F(L_z = m)F(|z| \leqslant n)(H + i)^{-1}$. The second fact follows from (7.7) and the compactness $F(L_z = m)F(|z| \leqslant n)(H_0 + 1)^{-1/2}$ (compactness in the x, y directions comes from the discreteness of the spectrum of H_{osc}).

In the proof of the Enss decomposition principle, we must replace lemma 1 of §2 by lemma 1 below. Otherwise the proof is directly patterned on that in section 2, except that only the z-direction is singled out: the decomposition into cubes is replaced by a decomposition into slabs: $\alpha < |z| < \alpha + 1$. We use the fact that $H_0 - (-d^2/dz^2)$ is quantized so that if $E_\Omega(H_0)\phi = \phi$ and $\text{dist}(\Omega, \{E_{n,m}\}_{n=0}^\infty) = d$, then supp $\hat{\phi}_{n,m}(k) \subset \{k \mid k^2 \geqslant d\}$ for all n. Lemma 2 is not needed since only two directions are involved; in Lemma 3, we deal with $[f(H_0 \upharpoonright (L_z = m)), g_R(z)]$ and $[f(k), g_R(z)]$. (k is the k of (7.3), i.e., $(1/i)(\partial/\partial z)$.) Moreover, $|k|^{2N} + 1$ is replaced by e^{-cH_0} in lemma 4. Finally, in Lemma 5 where hypothesis (iii) of "regular perturbation" is used, we use (7.7) which implies that $H_0^{1/2}E_{(a,b)}(H)$ is bounded (and $(H_0 + 1) \leqslant C_\epsilon e^{-cH_0} + \epsilon$). □

LEMMA 1. *Let ϕ be a function with $L_z\phi = m\phi$ and* supp $\hat{\phi}_{n,m}(k) \subset [(1/2)\alpha, (1/2)\beta]$ *for all n. Then, for* $z \notin [z_0 + (\alpha - \delta)t, z_0 + (\beta + \delta)t]$

$$\left[\int |(e^{-itH_0}\phi)(x, y, z)|^2 dx\, dy\right]^{1/2} \leqslant C_{M,\delta}(1 + |z - z_0| + t)^{-M} \| \, |1 + (z - z_0)|^M \phi\|. \tag{7.9}$$

Proof. By (7.5):

$$\text{LHS of (7.9)} = \left[\sum_n \left(\int \exp(-i(k^2 t - kz))\hat{\phi}_{n,m}(k)\right)^2\right]^{1/2}$$

and by (7.3), (7.4):

$$\|(1 + z^2)^{M/2}\phi\|^2 = \sum_n \left\| \left(1 - \frac{d^2}{dk^2}\right)^{M/2} \hat{\phi}_{n,m} \right\|^2_{L^2(dk)}$$

so that (7.9) follows from (2.5). □

Example 7.1. (Local potentials) Using the explicit formula for the kernel of the operation e^{-tH_0}, one easily shows that for c large enough, $(1 - d^2/dz^2) e^{(1-\epsilon)B\rho^2/4} e^{-cH_0} \upharpoonright (L_z = m)$ is bounded. Thus, for local potentials, $h(R) \leqslant \sup_{|\alpha| > R} \omega_\alpha$ where

$$\omega_\alpha = \left(\int_{\alpha < z < a+1} e^{(1-\epsilon)B\rho^2/2} |V(\rho, z)|^2 dx\, dy\, dz\right)^{1/2}.$$

For a local potential (7.7) will hold if $V = V_1 + V_2$ with $V_1 \geqslant -c\rho^2 - d$ where $c < B^2/4$ and V_2 in $L^2 + L^\infty$. Thus, V can obey the hypotheses of Theorem 7.1 and still have considerable growth in the ρ direction, e.g., if $0 \leqslant V(\rho, z) \leqslant C(1 + |z|)^{-1-\epsilon}\exp((1 - \epsilon)B\rho^2/4)$.

In order to treat the non-azimuthally symmetric case we need weak information on the changes of the x, y support of $e^{-itH_0}\phi$. Since H_0 is quadratic in the velocities, one might hope that some kind of classical allowed trajectories lemma like Lemma 1 extends to the x, y directions. The problem is that the components of the velocity (namely $v_x = i[H_0, x]$ and $v_y = i[H_0, y]$) do *not* commute so one cannot talk about the "values" of v_x, v_y. However, one can talk about $v_x^2 + v_y^2$. We have:

LEMMA 2. *Fix $\rho_0 \in \mathrm{R}^2$ and let $\rho(x, y, z) = (x, y)$. Fix $D > 0$ and $M = 2^k$. Let ϕ be any vector with $\langle \phi, H_0^M \phi \rangle \leqslant D\langle \phi, \phi \rangle$. Then, for all t and R*

$$\int_{|\rho - \rho_0| \geqslant R} |e^{-itH_0}\phi|^2 dx\, dy\, dz \leqslant C_{M,D}(1 + R)^{-2M} \|(1 + (\rho - \rho_0)^M)\phi\|^2 \tag{7.10}$$

Remark. At first sight, the lack of t dependence seems surprising in (7.10). But the projections of the classical orbits in x, y are periodic in time and so is the LHS of (7.10).

Proof. Write $H_0 = \tilde{H}_0 - d^2/dz^2$. Then since d^2/dz^2 commutes with $\tilde{H}_0$ and $F(|\rho - \rho_0| \geqslant R)$, H_0 in the (LHS) of (7.10) can be replaced by $\tilde{H}_0$. Since the spectrum of $\tilde{H}_0$ lies in $\{Bn\}$, we see that the left side of (7.10) is periodic in t. We therefore need only prove (7.10) for $0 \leqslant t \leqslant 2\pi/B$. Moreover (see [4]), there are operators $U(a)$ (*not* the usual translations) commuting with $\tilde{H}_0$ so that $U(a)\rho U(a)^{-1} = \rho + a$ so that we can take $\rho_0 = 0$ in (7.10). Next notice that

$$(1+R)^{2M}\int_{|\rho|\geqslant R}|\psi|^2 \leqslant \int (1+|\rho|)^{2M}|\psi|^2.$$

Thus, (7.10) is reduced to showing that, for $0 \leqslant t \leqslant 2\pi/B$ and for $\langle \phi, H_0^M \phi\rangle \leqslant D\langle \phi, \phi\rangle$, we have that

$$\left\langle \phi, \left[1 + \left(\rho^2(t)\right)^M\right]\phi\right\rangle \leqslant C_{M,D}\left\langle \phi, \left(1+\rho^2\right)^M \phi\right\rangle \tag{7.11}$$

where $\rho(t) = e^{+itH_0}\rho e^{-itH_0}$.

We prove (7.11) using ideas of Radin-Simon [26]. Clearly, it suffices to show that

$$\left.\left\langle \phi, \frac{d}{dt}\left(\rho^2(t)\right)^M \phi\right\rangle\right|_{t=0} \leqslant C'\left\langle \phi, \left(1+\rho^2\right)^M \phi\right\rangle \tag{7.12}$$

for then if the left side of (7.11) is $f(t)$, we have $df/dt \leqslant c'f$ whence $f(t) \leqslant f(0)\exp(c't)$. To prove (7.12), we use: (i) $[\dot{\rho}_j, \rho_i] = -2i\delta_{ij}$. (ii) $\dot{\rho}^2$ commutes with H_0 and is less than $4H_0$ so that $\langle \phi, (\dot{\rho}^2)^M \phi\rangle \leqslant 4^M D\,\|\phi\|^2$. (iii) The Schwartz inequality repeatedly. For example, if $M = 2$:

$$\begin{aligned}
\left\langle (\rho^4)^{\cdot} \right\rangle &= \langle \dot{\rho}\cdot\rho\rho^2\rangle + \langle \rho\cdot\dot{\rho}\rho^2\rangle + 2 \text{ others}\\
&= 4\langle \dot{\rho}\cdot\rho\rho^2\rangle + b\langle \rho^2\rangle \qquad \text{(using (i) above)}\\
&\leqslant 4\left\langle (\dot{\rho}\cdot\rho)^2\right\rangle^{1/2}\left\langle \rho^4\right\rangle^{1/2} + b\langle \rho^2\rangle \qquad \text{(using (iii))}\\
&\leqslant 2\left\langle (\dot{\rho}\cdot\rho)^2\right\rangle + 2\langle \rho^4\rangle + b\langle \rho^2\rangle\\
&\leqslant 2\langle \dot{\rho}\dot{\rho}\rho\rho\rangle + 2\langle \rho^4\rangle + b'\langle \rho^2\rangle \qquad \text{(using (i))}\\
&\leqslant \langle \dot{\rho}^4\rangle + 3\langle \rho^4\rangle + b'\langle \rho^2\rangle\\
&\leqslant C'\left\langle \phi, \left(1+\rho^2\right)^2 \phi\right\rangle \qquad \text{(using (ii))}. \quad \square
\end{aligned}$$

Remark. Using the Frohlich-Lieb method [15] of proving chessboard estimates for lattices of length $M \neq 2^n$, one can probably extend this result to $M \neq 2^n$.

We can now prove a result for non-azimuthally symmetric V.

THEOREM 7.2. *Let H_0 be given by* (7.1). *Let V be a symmetric operator obeying*

(1) $\|V\phi\| \leqslant a\|H_0\phi\| + b\|\phi\|$

(2) $h(R) \equiv \|V(H_0+i)^{-1}F(|z|>R)\|$ *is in* $L^1(0,\infty)$.

(3) $l(R) \equiv \|V(H_0+i)^{-1}F(\rho>R)\|$ *goes to zero as* $R\to\infty$.

Then all the conclusions of Theorem 2.1 *hold for* $H = H_0 + V$ *except that* $\overline{C \cup S}$ *is replaced by* $\{(2k+1)B\}_{k=0}^{\infty}$ *in* (d) *and* (e) *reads* $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0) = [B,\infty)$.

Proof. Once again, we need only follow the proof of Theorem 2.1. Now $F(|z| \leqslant n)F(|\rho| \leqslant n)(H_0+i)^{-1}$ is compact, so in the Enss decomposition principle (a) becomes $\|F(|z|\leqslant n)F(|\rho|\leqslant n)\phi_n\| \to 0$ and in (b), $\overline{S_v \cup C_v}$ is replaced by $\{(2k+1)B\}$. We still decompose into slabs in z in forming $\phi_{n,\,\substack{\mathrm{in}\\ \mathrm{out}}}$. The only new feature is in the proof that

$$\int_0^\infty \|Ve^{-itH_0}\phi_{n,\,\mathrm{out}}\|\,dt \to 0 \tag{7.13}$$

as $n\to\infty$ (and the analog for $\phi_{n,\,\mathrm{in}}$). Let us denote the integrand in (7.13) as $G_n(t)$. We have that

$$\begin{aligned} G_n(t) \leqslant{}& \|V(H_0+i)^{-1}F(z \geqslant \alpha t - \beta)\|\;\|(H_0+i)\phi_{n,\,\mathrm{out}}\| \\ &+ (a+b)\|F(z \leqslant \alpha t - \beta)e^{-itH_0}(H_0+i)\phi_{n,\,\mathrm{out}}\|. \end{aligned}$$

By using lemma 1 of this section and choosing α suitably, we obtain an upper bound uniform in n, on $G_n(t)$ which is in $L^1(dt)$ so (7.13) holds if we can show that

$$G_n(t) \to 0 \quad \text{as} \quad n\to\infty, \quad t \text{ fixed}. \tag{7.14}$$

To prove this, we write

$$Ve^{-itH_0}\phi_{n,\,\mathrm{out}} = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}$$

where $\phi_{n,\,\alpha,\,\mathrm{out}}$ is the contribution from the slab $\alpha < z < \alpha + 1$ in the construction of $\phi_{n,\,\mathrm{out}}$.

$$\mathrm{I} = V(H_0+i)^{-1}F\left(|z| > \frac{n}{2}\right)e^{-itH_0}(H_0+i)\phi_{n,\,\mathrm{out}}$$

$$\mathrm{II} = V(H_0+i)^{-1}F\left(|z| < \frac{n}{2}\right)F\left(|\rho| > \frac{n}{2}\right)e^{-itH_0}(H_0+i)\phi_{n,\,\mathrm{out}}$$

$$\mathrm{III} = V(H_0+i)^{-1}F\left(|z| < \frac{n}{2}\right)F\left(|\rho| < \frac{n}{2}\right)e^{-itH_0}(H_0+i)\sum_{|\alpha|\geqslant n}\phi_{n,\,\alpha,\,\mathrm{out}}$$

$$\mathrm{IV} = V(H_0+i)^{-1}F\left(|z| < \frac{n}{2}\right)F\left(|\rho| < \frac{n}{2}\right)e^{-itH_0}(H_0+i)\sum_{|\alpha|\leqslant n}\phi_{n,\,\alpha,\,\mathrm{out}}.$$

$\|\mathrm{I}\| \to 0$ since it can be bounded by const. $h(n/2)$, $\|\mathrm{II}\| \to 0$ since it can be bounded by const. $l(n/2)$. $\|\mathrm{III}\|$ goes to zero in the usual way by controlling each term using lemma 1 and getting a $1/(|\alpha|+1)^m$ falloff for t fixed. To show that $\|\mathrm{IV}\|$ goes to zero, we introduce a smooth partition of unity $\sum_\beta \chi_\beta(\rho)$ in the x, y plane, with $\chi_\beta(\rho)$ centered in the square close β in $\mathbf{Z}^2$. Using lemma 2 of this section, the β contribution is bounded for $|\beta| > (3/4)n$ by const.$(1 + |\beta|)^{-M}$ and the sum of the terms for $|\beta| < (3/4)n$ is controlled by the hypothesis $F(|z| < n)F(|\rho| < n)\phi_n \to 0$. □

§8. **Electric fields.** In this section, we want to consider scattering for the pair (H, H_0) with $H = H_0 + V$ and

$$H_0 = -\Delta - x_1 \tag{8.1}$$

with x_1 the first component of x in R^ν. H_0 is the energy of a particle of charge 1 in a constant electric field $E = (1, 0, \ldots, 0)$. Existence of $\Omega^\pm(H, H_0)$ was considered by Avron-Herbst [2] and Veselic-Weidmann [41] exploting Cook's method. Using the Agmon-Kuroda method and special properties of Airy functions, Herbst [17] proved completeness and the absence of singular spectrum. We will recover these results here with the Enss method.

We begin by recalling two results from Avron-Herbst:

LEMMA 1. ([2]) *H_0 is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^\nu)$ and*

$$e^{-itH_0} = e^{-it^3/3}e^{itx_1}e^{-itp_\perp^2}e^{-i(p^2t + t^2p)} \tag{8.2}$$

where $p = i^{-1}\partial/\partial x_1$ and $p_\perp^2 = -\Delta - p^2$.

Proof. ([2]) One easily checks that on $\mathcal{S}$:

$$H_0 = e^{-ip^3/3}(p_\perp^2 - x_1)e^{ip^3/3}$$

(since $x_1 = i\partial/\partial p$). The self-adjointness follows, and

$$\begin{aligned} e^{-itH_0} &= e^{-ip_\perp^2 t}e^{-ip^3/3}e^{ix_1 t}e^{ip^3/3} \\ &= e^{-ip_\perp^2 t}e^{ix_1 t}e^{-i(p+t)^3/3}e^{ip^3/3} = (8.2). \quad \square \end{aligned}$$

(8.2) is the basic formula used by [2] in their analysis of $\|Ve^{-itH_0}\phi\|$ and it will also be used by us in a similar way. It says that except for phase factors $(e^{-itH_0}\phi)(x)$ is obtained by first translating by t^2 units to the right and then applying $e^{+i\Delta t}$.

LEMMA 2. (*essentially in* [2]) *Suppose that $V = V_1 + V_2$ with V_2 bounded, V_1, $-\Delta$-bounded with relative bound a and V_1x_1 $-\Delta$-bounded. Then V is H_0-bounded with relative bound a.*

Proof. ([2]) Write (with $T = -\Delta$):

$$\begin{aligned} V_1(H_0 + ib)^{-1} &= V_1(T + ib)^{-1} + V_1(T + ib)^{-1}x_1(H_0 + ib)^{-1} \\ &= V_1(T + ib)^{-1} + V_1x_1(T + ib)^{-1}(H_0 + ib)^{-1} \\ &\quad + 2iV_1(T + ib)^{-1}p(T + ib)^{-1}(H_0 + ib)^{-1}. \end{aligned}$$

Using the fact that A-bound of $B = \lim_{b\to\infty}\|B(A + ib)^{-1}\|$, the result follows. □

We will also need the following result of Herbst [17]:

LEMMA 3. ([17]) *Let* $\rho = (1 + x_1^2)^{1/2}$. *Then for any* δ *real,* $\rho^{-\delta} R \rho^{\delta} = R\Gamma_\delta$ *with* Γ *bounded.* $(R = (H_0 + i)^{-1})$. *In particular, if* $V\rho^\delta$ *is* H_0*-bounded, then* $V(H_0 + i)^{-1}\rho^\delta$ $(= V\rho^\delta R \Gamma_\delta)$ *is bounded.*

Proof. See Lemma 2.2 of [17]. □

Lemmas 2 and 3 are mainly needed to identify which potentials obey the hypotheses of the theorems below. We begin with the one dimensional case which is somewhat simpler than the general case.

THEOREM 8.1. *Let* $H_0 = (-d^2/dx^2) - x$ *on* $L^2(-\infty, \infty)$. *Let* V *be symmetric and* H_0*-bounded with relative bound less than* 1 *and let* $H = H_0 + V$. *Let*

$$h(R) = \| V(H_0 + i)^{-1} F(x > R) \| \tag{8.3}$$

Suppose that

$$\int_0^\infty h(R^2)\, dR < \infty. \tag{8.4}$$

Then (H, H_0) *obey all the conclusions of Theorem* 2.1 *with* $\overline{S_v \cup C_v}$ *replaced by* ϕ *in* (*d*) *and* $\sigma_{\text{ess}}(H_0) = (-\infty, \infty)$.

Remarks. Note $x > R$, not $|x| > R$ in (8.3) and $h(R^2)$ not $h(R)$ in (8.4). This is because e^{-itH_0} pushes one to $+\infty$ (never $-\infty$) and as t^2, not t.

Example 8.1. By lemmas 2 and 3, if V is a function and $(1 + x^2)^{1/4+\epsilon} V$ is uniformly in L^2_{loc}, then the hypotheses of the theorem hold. Actually, as noted by Herbst, $(1 + x^2)^{1/4+\epsilon} V$ need only lie in L^2_{loc} on $(0, \infty)$ and one only needs that $(1 + x^2)^{-1/4+\epsilon} V$ lie in L^2_{loc} on $(-\infty, \infty)$.

Example 8.2. By combining the ideas in the proof of Theorem 8.1 and the ideas of Section 6, one can analyze $H = -d^2/dx^2 - |x| + V$. There are now wave operators $\Omega^{\pm}_{l,r}(H, H_0^{l,r})$ with $H_0^r = -d^2/dx^2 - x$ and $H_0^l = -d^2/dx^2 + x$. Ω_l^+ and Ω_r^+ have orthogonal ranges and span the orthogonal complement of the eigenvectors for H.

In trying to follow the scheme of Section 2, one difficulty immediately arises. Suppose that a state has more or less definite energy: $H \sim E_0$. As $t \to \infty$, x goes to ∞, so $p^2 = H + x$ gets large. Thus, lemma 1 in §2 will not suffice since it deals with functions supported on a compact set in p-space. We thus prove:

LEMMA 4. (1-*dimension*) *Suppose that* $\phi \in L^2$ *and* $\hat{\phi}$ *has support in* $(-1, \infty)$ (*resp.* $(-\infty, 1)$). *Then for all* M, *and all* $t \geqslant 0$ (*resp.* $t \leqslant 0$):

$$|(e^{-itH_0}\phi)(x)| \leqslant C_M (1 + |x - a - x_{\text{crit}}(t)| + |t|)^{-M} \times \left\| \left(1 - \frac{d^2}{dx^2}\right)\left(1 + (x - a)^2\right)^{M/2} \phi \right\|_2$$

in the region $x < a + x_{\text{crit}}(t) \equiv a + t^2 - 3|t|$.

Proof. Suppose that $a = 0$. Let $\phi(x, t) = e^{it^3/3} e^{-ixt}(e^{-itH_0}\phi)(x)$. Then, by (8.2):

$$\phi(x, t) = (2\pi)^{-1/2} \int \exp(-if(k, x, t))\hat{\phi}(k) \tag{8.5}$$

where $f(k, x, t) = k^2 t + k(t^2 - x)$. In the region $x < x_{\text{crit}}$, $k > -1$, $t > 0$ (resp. $x < x_{\text{crit}}$, $k < 1$, $t < 0$), $f' \equiv \partial f/\partial k$ is non-vanishing; indeed in that region:

$$f' = 2kt + t^2 - x > -2|t| + t^2 - x = |x_{\text{crit}} - x| + |t|.$$

Thus, by (8.5)

$$\phi(x, t) = (2\pi)^{-1/2} \int \left[\left(\frac{i}{f'} \frac{\partial}{\partial k} \right)^m \exp(-if(k, t)) \right] \hat{\phi}\, dk.$$

When we integrate by parts, some derivatives of f' enter. Using

$$\frac{\partial^m}{\partial k^m} \frac{1}{f'} = \frac{(2t)^n n!}{(f')^{n+1}} (-1)^n$$

we see that in the region in question where $f' > |t|$, $|\partial^n/\partial k^m(1/f')| \leqslant C_m |f'|^{-1}$. Thus $\psi = (1 + \cdots)^M \phi$ obeys

$$\begin{aligned} |\psi(x, t)| &\leqslant C \sum_{\alpha=0}^{n} \left\| \left(\frac{d}{dk} \right)^\alpha \hat{\phi} \right\|_1 \\ &\leqslant C' \sum_{\alpha=0}^{n} \left\| (1 + k^2) \left(\frac{d}{dk} \right)^\alpha \hat{\phi} \right\|_2 \qquad \text{(by Schwarz)} \\ &\leqslant C'' \left\| (1 + k^2) \left(1 - \frac{d^2}{dk^2} \right)^{m/2} \hat{\phi} \right\|_2 . \end{aligned}$$

This proves the lemma for $a = 0$. If U_a is the conventional translation by a units $U_a H U_a^{-1} = H + a$ so that $U_a e^{-itH} U_a^{-1} = e^{-ita} e^{-itH}$. Thus, the result for $a = 0$ implies the result for $a \neq 0$. □

The next lemma is related to results of Herbst-Simon [18] and uses some of their ideas in its proof:

LEMMA 5. (*ν-dimensions*) (a) *Suppose that g is* C^∞ *with all derivatives in* L^∞ *and suppose that* $\operatorname{supp} g \subset \{x \mid x_1 < 0\}$. *Then* $p_j g(H_0 + i)^{-1}$ *is a bounded operator.*

(b) *If g is as defined in* (a), *then* $|x_1|^{1/2} g(H_0 + i)^{-1}$ *is bounded.*

(c) *In one dimension,* $F(x_1 < a)(H_0 + i)^{-1}$ *is compact for any a. In ν dimensions,* $F(x_1 < a)F(|x_\perp| < b)(H_0 + i)^{-1}$ *is compact where* $x_\perp = x - (x_1, 0, \ldots, 0)$.

(d) *If* $g \subset C_0^\infty$ *has support in* $\{x \mid x_1 < 0\}$, *then* $p^2 g(H_0 + i)^{-1}$ *is bounded.*

Proof. (a) Let $\phi \in (H_0 + i)\mathcal{S}$ which is dense. We need only show, for such ϕ, that $\|p_j g(H_0 + i)^{-1}\phi\| \leqslant D\|\phi\|$. Since $-gx_1 g \geqslant 0$, we have that

$$\sum_j \|p_j g(H_0 + i)^{-1}\phi\|^2 \leqslant \langle (H_0 + i)^{-1}\phi, gH_0 g(H_0 + i)^{-1}\phi\rangle \tag{8.6}$$

Using $[H_0, g] = -2\sum_j p_j(i\partial_j g) + \Delta g$, we can write the right side of (8.6) as a sum of three terms. Two of them, viz $\langle (H_0 + i)^{-1}\phi, g^2 H_0(H_0 + i)^{-1}\phi\rangle$ and $\langle (H_0 + i)^{-1}\phi, g(\Delta g)(H_0 + i)^{-1}\phi\rangle$ are bounded by $c\|\phi\|^2$. The third, viz; $-2i\langle p_j g(H_0 + i)^{-1}\phi, \partial_j g(H_0 + i)^{-1}\phi\rangle$ is clearly bounded by $c\|\phi\|(\sum_j \|p_j g(H_0 + i)^{-1}\phi\|)$, so using $ab \leqslant (1/2)\epsilon a^2 + (1/2)\epsilon^{-1}b^2$:

$$\langle (H_0 + i)^{-1}\phi, (H_0 + i)(H_0 + i)^{-1}\phi\rangle \leqslant \frac{1}{2}\sum_j \|p_j g(H_0 + i)^{-1}\phi\|^2 + d\|\phi\|^2 \tag{8.7}$$

(8.6) and (8.7) imply the desired result.

(b) By (a), (8.7), and $-gx_1 g \leqslant gH_0 g$, we see that

$$-\langle (H_0 + i)^{-1}\phi, (gx_1 g)(H_0 + i)^{-1}\phi\rangle \leqslant c\|\phi\|^2.$$

(c) As in the proof of lemma 2, $F(-n < x_1 < a)F(|x_1| < b)(H_0 + i)^{-1}$ is compact, so we need only show that $\|F(x_1 < n)(H_0 + i)^{-1}\| \to 0$ as $n \to +\infty$. Pick g obeying the hypotheses of (a) so that $g = 1$ for $x_1 < -1$. Then, for $n > 1$:

$$\|F(x_1 < -n)(H_0 + i)^{-1}\| \leqslant \|F(x_1 < -n)|x|_1^{-1/2}\| \, \||x_1|^{1/2}g(H_0 + i)^{-1}\| \leqslant cn^{-1/2}$$

goes to zero.

(d) $p^2 g(H_0 + i)^{-1} = [p^2, g](H_0 + i)^{-1} + gH_0(H_0 + i)^{-1} + gx(H_0 + i)^{-1}$. The first term is bounded by part (a); the second and third are trivially bounded. □

The next lemma will allow us to control the right side of the estimate in Lemma 4.

LEMMA 6. *Consider H_0 in 1-dimension. Let $f \in C_0^\infty(R)$ be positive with $\int f(y)dy = 1$ and let $f_\alpha(x) = \int_\alpha^{\alpha+1} f(x - y)dy$. Let $\eta_\pm(k)$ be two C^∞ functions with $\eta_+(k) = \eta_-(-k)$, $0 \leqslant \eta_\pm \leqslant 1$, $\eta_+ + \eta_- = 0$ and* supp $\eta_+ \subset [-1, \infty)$. *Let $\Phi \in C_0^\infty(R)$. Then for $\alpha > 0$ and m an even integer*:

$$\|(1 + k^2)(1 + (x - \alpha)^m)\eta_+(k)f_\alpha(x)\Phi(H_0)\| \leqslant C_m(1 + \alpha)$$

where C is independent of α.

Proof. Since $\|(H_0 + \alpha + i)\Phi(H_0)\| \leqslant C(1 + \alpha)$, it suffices to prove that

$$\|(1 + p^2)(1 + (x - \alpha)^m)\eta_+(p)f_\alpha(x)(H_0 + i + \alpha)^{-1}\| \leqslant C_m \tag{8.8}$$

But the left side of (8.8) is independent of α since $H_0 \to H_0 - \alpha$ under

$x \to x + \alpha$. Thus, we can take α_0 so negative that $\operatorname{supp} f_{\alpha_0} \subset (-\infty, 0)$. Since $(H_0 + i + \alpha_0)(H_0 + i)^{-1}$ is bounded we are reduced to showing that

$$\|(1 + p^2)(1 + x^m)\eta_+(p)g(x)(H_0 + i)^{-1}\| < \infty \tag{8.9}$$

where g is in C_0^∞ with support in $(-\infty, 0)$. (8.9) holds with the $(1 + p^2)$ dropped since the x^m can be commuted through the $\eta_+(p)$ and $x^a g(x)$ is bounded. Next replace $(1 + p^2)$ by p_j. Then p_j can be commuted through to give $p_j|x|^a g(x)(H_0 + i)^{-1}$ which is bounded by lemma 5(a) and terms of the type already treated. Repeating this argument using lemma 5(d), we are done. □

The final lemma is needed to handle the following problem: in proving the Enss decomposition principle in §2, we needed the fact that $\Phi(H_0)$ does not greatly destroy the localization of states in x-space; specifically, we used the fact that $\|F(|x| \leqslant (1/2)n)\Phi(H_0)F(|x| \geqslant n)\| \to 0$ as $n \to \infty$. Since $\Phi(H_0)$ was fairly explicitly known as a convolution operator, this was quite easy. In the present situation, H_0 is more complicated so that it will be easier to use a piece of soft analysis:

LEMMA 7. *Let $\{A_m\}_{m=1}^\infty$ be a family of compact operators and let $\{B_n\}_{n=1}^\infty$ be a family of bounded operators with s-lim $B_n = 0$. Then, one can find $M(n) \to \infty$ as $n \to \infty$ so that*

$$\|B_n A_m\| \leqslant \frac{1}{m} \tag{8.10}$$

for $m \leqslant M(n)$.

Remarks. 1. The application we have in mind, $A_m = \Phi(H_0)F(x < m)$ and $B_n = F(x > n)$.

2. Since $\Phi(H_0) = e^{-ip^3/3}\Phi(-x)e^{ip^3/3}$ for $H_0 = -d^2/dx^2 - x$, one can probably prove directly that (8.10) holds for the case of interest by exploiting properties of Airy functions—one would then obtain information on $M(n)$. As in the case $H_0 = -\Delta$, we suppose that if Φ in C^∞ all that is needed is $n - M(n) \to +\infty$.

Proof. Fix m. Then $\|B_n A_m\| \to 0$ as $n \to \infty$ so that we can find $N(m)$ so that $N(m + 1) \geqslant 1 + N(m)$ and $\|B_n A_m\| \leqslant 1/m$ for $n \geqslant N(m)$. Now let $M(n) = \sup\{m \mid n \geqslant N(m)\}$. □

Proof of Theorem 8.1. Given the compactness result Lemma 5(c), it suffices to prove the Enss decomposition principle under the hypotheses: $\|F(x < n)\phi_n\| \to 0$ as $n \to \infty$ and $E_{(a,b)}(H_0)\phi_n = \phi_n$. Suppose such a ϕ_n is given. As usual, $\|[\Phi(H) - \Phi(H_0)]\phi_n\| \to 0$ as $n \to \infty$, so we pick $\Phi \equiv 1$ in (a, b) with support in $(a - 1, b + 1)$ and let $\phi_{n,w}^{(1)} = \phi_n - \Phi(H_0)\phi_n$ which goes to zero as $n \to \infty$. Moreover by lemma 7 and lemma 5(c) of this section, we can find $m(n) \to \infty$ so that

$$\|F(x < m(n))\Phi(H_0)\phi_n\| \to 0. \tag{8.11}$$

(Since $(H_0+i)\Phi(H_0)$ is bounded, $F(x<m)\Phi(H_0)$ is compact.) Now pick $f\in C_0^\infty(R)$ with $\int f(y)=1, f\geqslant 0$ and $\operatorname{supp} f\subset(-1,1)$. Let $f_\alpha(x)=\int_\alpha^{\alpha+1} f(x-y)$ and let η_+, η_- be as given in lemma 6. Let

$$\phi_{n,w}^{(2)}=\sum_{\alpha\leqslant m(n)-2} f_\alpha(x)\Phi(H_0)\phi_n$$

$$\phi_{n,\substack{\text{in}\\\text{out}}}=\sum_{\alpha\geqslant m(n)-1}\eta_\mp(p)f_\alpha(x)\phi_n=\sum_{\alpha\geqslant m(n)-1}\phi_{n,\mp,\alpha}$$

By (8.11), $\|\phi_{n,w}^{(2)}\|\to 0$. By lemmas 4 and 6, for $2x<m(n)-1+t^2-3|t|$, $\alpha\geqslant m(n)-1$ and $\pm t>0$,

$$|e^{-itH_0}\phi_{n,\pm,\alpha}(x)|\leqslant\frac{1}{(1+t^2+\alpha+|x|)^M}(1+|\alpha|)$$

so that, in the usual way,

$$\|F\big(x<\tfrac{1}{2}\big(m(n)-1+t^2-3|t|\big)\big)e^{-itH_0}\phi_{n,\substack{\text{out}\\\text{in}}}\|\leqslant C(1+|t|)^{-M}$$

for $\pm t>0$. A similar result (with slight extension of lemma 6) holds for $\|F(..)e^{-itH_0}(H_0+1)\phi_{n,\substack{\text{out}\\\text{in}}}\|$. □

Remark. There is an interesting distinction between the proof of Theorem 8.1 and that of Theorem 2.1: in §2, the f_α are chosen so that $\hat{f}_\alpha$ has compact support; in the above, we take f_α itself to have compact support. Since there are no bad regions of momentum space, the restriction on $\hat{f}_\alpha$ is not needed. It turns out above to be somewhat simpler to take f_α to have compact support, so we do so. However, this is not essential: for the proof of lemma 5(a) only used $\operatorname{supp} g\subset\{x\mid x_1<0\}$ to assure that $-gx_1g\geqslant 0$. If g has falloff as $x_1\to\infty$, then $-gx_1g+c\geqslant 0$ for suitable c. With this observation, $f\in\mathcal{S}$ can be accommodated.

In the ν-dimensional case, our result is:

THEOREM 8.2. *Let $H_0=-\Delta-x_1$ on $L^2(\mathbb{R}^\nu)$. Let V be symmetric and H_0-bounded with relative bound less than* 1 *and let $H=H_0+V$. Let*

$$h(R)=\|V(H_0+i)^{-1}F(x_1>R)\|$$

and

$$k(R)=\|V(H_0+i)^{-1}F(|x_\perp|>R)\|.$$

Suppose that (8.4) *holds and that*

$$k(R)\to 0\qquad as\quad R\to\infty. \tag{8.12}$$

Then all the conclusions of Theorem 2.1 *hold with $\overline{S_v\cup C_v}$ replaced by ϕ in (d) and $\sigma_{\text{ess}}(H_0)=(-\infty,\infty)$.*

Example 8.3. Suppose that $(1 + x_1^2)^{+1/2+\epsilon}V(H_0 + i)^{-1}$ is compact and V is a multiplication operator. Then, as in Example 8.1, (8.4) holds by lemmas 2 and 3. Since $V(H_0 + i)^{-1}$ is compact and $F(|x_\perp| > R) \to 0$ strongly, (8.11) holds. Thus the "potentials of type A" of Herbst [17] obey the hypotheses of Theorem 8.2 so that this theorem includes Theorem 1.3 of Herbst [17] as a special case.

Proof of Theorem 8.2. It is no longer true by compactness alone that if we wait long enough the particle will have to enter the region $x_1 > n$; rather it will have to enter either that region or the region $x_1 < n$, $|x_\perp| > R_n$ where R_n is any sequence with $R_n \to \infty$. The part of the wave function in $x_1 > n$ will be treated as in Theorem 8.1. For the remainder, we will use a new idea: namely in the region $x_1 < -n$, $H_0 \gtrsim p^2 + n$ so conservation of energy makes it very unlikely that the particle enters that region. What remains is the piece $|x_1| < n$, $|x_\perp| > R_n$. If we wait a time $\sim n^{1/2}$, the particle will get to the region $x_1 > n$ where we can use the method of Theorem 8.1. Thus, we need only pick R_n so large that there is negligible interaction in the time period $t < n^{1/2}$. In the formal proof below, the reader should keep this heuristic sketch in mind. Set $F_n = F(x_1 < n)F(|x_\perp| < R_n)$ where $R_n \to \infty$ will be picked below. Since $F_n(H_0 + i)^{-1}$ is compact, we only need a version of the Enss decomposition principle supposing $\|F_n\phi_n\| \to 0$ and $E_{(a,b)}(H)\phi_n = \phi_n$. Since $1 - F_n = F(x_1 > n) + F(|x_\perp| > R_n)F(x_1 < n)$ (8.4) and (8.11) imply that $[\Phi(H) - \Phi(H_0)]\phi_n \to 0$ in the usual way. Moreover, by lemma 7, we can find $m(n) \to \infty$ so that $\|F_{m(n)}\Phi(H_0)\phi_n\| \to 0$. For α an integer, let $f_\alpha(x_1)$ be given as in the proof of Theorem 8.1. For $\alpha > m(n)$, we decompose $f_\alpha\Phi(H_0)\phi_n$ into an in and out piece, $\phi^{>}_{n,\text{in},\alpha}$ and $\phi^{>}_{n,\text{out},\alpha}$ and show that

$$\left\| F\big(x_1 < 1/2(m(n) - 1 + t^2 - 3|t|)\big)e^{-itH_0} \sum_{\alpha > m(n)} \phi^{>}_{n,\substack{\text{out}\\ \text{in}}} \right\| \leqslant C(1 + |t|)^{-M}$$

as in that proof by extending lemma 4 to replace $|(e^{-itH_0}\phi)(x)|$ by $[\int dx_\perp |(e^{-itH_0}\phi)(x_1, x_\perp)|^2]^{1/2}$. Let $\phi^{(0)}_{n,\text{in}}, \phi^{(0)}_{n,\text{out}}$ be similar sums for $|\alpha| \leqslant m(n)$. By using lemma 1 in the extended form and lemma 6 we see that for $x_1 < t^2 - 3|t| + m(n) \equiv y_c(n, t)$

$$\left[\int dx_\perp |e^{-itH_0}\phi^{(0)}_{n,\substack{\text{in}\\ \text{out}}}(x_1, x_\perp)|^2\right]^{1/2} \leqslant C_M(1 + |x_1 - y_c(n, t)| + |t|)^{-M}(m(n) + 1)^{M+1} \tag{8.13}$$

(Do *not* decompose into strips in x_1.) For $t > m(n)$ and $x_1 < m(n) + (1/2)t^2$ the left hand side of (8.13) is dominated by $C'_M(1 + m(n)^2 + |t| + |x_1|)^{-M}(m(n) + 1)^{M+1} \leqslant C'_M(1 + m(n)^2 + |t| + |x_1|)^{-(M-1)/2}$ and thus

$$\int_{\pm m(n)}^{\pm\infty} \left\| F\Big(x_1 < m(n) + \frac{1}{2}t^2\Big)e^{-itH_0}(H_0 + i)\phi^{(0)}_{n,\substack{\text{out}\\ \text{in}}} \right\| dt \to 0 \tag{8.14}$$

To control $\int_0^{\pm m(n)} \ldots$, we use the ideas of section 2. Use $\phi_n^{(0)}$ to denote $(H_0 + i)\phi_{n,\,\substack{\text{out}\\ \text{in}}}^{(0)}$. Since $p_\perp^2 < H_0 + x_1$, we have that

$$\left(\phi_n^{(0)}, p_\perp^2 \phi_n^{(0)}\right) \leqslant m(n) + c. \tag{8.15}$$

Let $G_n(p_\perp)$ be a smooth function in the $p_\perp$-space which is 1 if $p_\perp < m(n)^2$ and 0 if $p_\perp > 2m(n)^2$: Choose G_n so that $G_n(p_\perp) = g(p_\perp / m(n)^2)$ for a suitable $g \in C_0^\infty$. Let $\phi_n^{(1)} = G_n(p_\perp)\phi_n^{(0)}$, $\phi_n^{(2)} = [1 - G_n(p_\perp)]\phi_n^{(0)}$. By (8.15) $\|\phi_n^{(2)}\|^2 \leqslant m(n)^{-3}$ so that

$$\int_0^{m(n)} \| F(\ldots) e^{\pm itH_0} \phi_n^{(2)} \| \leqslant m(n) m(n)^{-3/2} \to 0.$$

Next notice that

$$\begin{aligned} &\| F(|x_\perp| < \tfrac{1}{2} R_{m(n)}) G_n(p) F(|x_\perp| > R_{m(n)}) \| \\ &\quad = \left\| F\left(|x_\perp| < \tfrac{1}{2} R_{m(n)}/m(n)^2\right) g(p_\perp) F(|x_\perp| > R_{m(n)}/m(n)) \right\| \end{aligned}$$

goes to zero in norm so long as $R_n / n^2 \to 0$. Thus

$$\| F(|x_\perp| < \tfrac{1}{2} R_{m(n)}) \phi_n^{(1)} \| \to 0. \tag{8.16}$$

Thus, as in section 2, so long as $m(n)^3 + m(n) < (1/4) R_{m(n)}$, we can decompose $\phi_n^{(1)} = F(|x_\perp| < (1/2) R_{m(n)})\phi_n + \tilde{\phi}_n^{(1)}$. [We throw the first piece into the waste basket, checking that the resulting change in $\phi_n^{(0)}$ does not invalidate (8.14) and that

$$\| F(|x_\perp| < \tfrac{1}{4} R_{m(n)}) e^{-itH_0} \tilde{\phi}_n^{(1)} \| \leqslant (1 + R_{m(n)} + |t|)^{-M}.$$

holds.] Thus

$$\begin{aligned} &\int_0^{m(n)} \| V e^{-itH_0} (H_0 + i)^{-1} \tilde{\phi}_n^{(1)} \| \, dt \\ &\qquad \leqslant 0\left((1 + R_{m(n)})^{-1}\right) + m(n) k\left(\tfrac{1}{4} R_{m(n)}\right). \end{aligned}$$

If we pick R_n so that $R_n / n^2 \to 0$, $R_n > 4(n^3 + n)$ and $nk((1/4)R_n) \to 0$, then the $\phi_n^{(0)}$ pieces can be controlled.

Finally, we must deal with $\phi_n^<$, the sum of $f_\alpha \Phi(H_0)\phi_n$ over $\alpha < m(n)$. Since $|x_1|^{1/2} F(x_1 < -1)(H_0 + i)^{-1}$ is bounded, $\| F(x_1 < -m(n) + 2)(H_0 + i)^{-1} \| \to 0$ as $n \to \infty$ and thus $\| F(x_1 < (-m(n) + 2)) \Phi(H_0) \| \to 0$ so $\|\phi_n^<\| \to 0$. Thus $\phi_n^<$ can be thrown into the wastebasket. □

§9. Schrödinger operators with absorption (optical models). In this last section we want to consider operators $H = H_0 + V$, $H_0 = -\Delta$, where V is no longer symmetric; rather $\operatorname{Im}(\phi, V\phi) \leqslant 0$ for all ϕ in a suitable dense domain.

Under suitable circumstances, $B_t = e^{-itH}$ for $t \geqslant 0$ will be a well-defined semigroup of contractions and we will analyze the large time behavior of $B_t\phi$. Such operators are of interest in part because they arise in the study of the "optical model" in nuclear physics. Typically in that case V is not a local potential even if H is obtained by making an "optical approximation" to a many-body Schrödinger operator with local potentials.

In a recent paper, Davies [9] has advocated the study of B_t of the above type and he has begun their study by developing a trace class theory. In particular, for V with $(H - i)^{-n}V(H_0 - i)^{-m}$ trace class for some n, m, he proved that

$$W\phi \equiv \lim_{t\to\infty} e^{itH_0}B_t\phi \tag{9.1}$$

exists for all ϕ in a certain subspace $\mathcal{H}_{ac}^H$. (Since $\mathcal{H}_{ac}^H \subset \mathcal{H}_b^\perp$, the space defined below, when V obeys (2.1) with $N = 1$, our conclusions below imply those of Davies.)

We will follow the suggestion of Davies [9] and continue the study of such operators by developing some abstract features (Theorems 9.1, 9.2 below) and then apply the Enss method. We note that one can also extend the Agmon-Kuroda theory to this situation if V is a local potential with $r^{-1-\epsilon}$ falloff. Indeed, if one follows the proof of Theorem XIII.33 in [29], the self-adjointness of V enters in only one place: namely in the proof of Lemma 8, on page 176. So long as $\operatorname{Im} V \leqslant 0$, that proof still goes through for the case $(H_0 - x - i0)^{-1}$ but not for $(H_0 - x + i0)^{-1}$. The net result is that if V is an Agmon potential in the sense of [29] with $\operatorname{Im} V = 0$ replaced by $\operatorname{Im} V \leqslant 0$, then away from the real eigenvalues, considered as a map from L^2_δ to $L^2_{-\delta}$ ($\delta > 1/2$), $(H + V - x - i\epsilon)^{-1}$ has a nice limit as $\epsilon \downarrow 0$. Presumably this can be used to provide an alternative proof to Theorem 9.3 below when V is local and has strict $r^{-1-\epsilon}$ falloff.

We begin with some general theory.

LEMMA 1. *Let iH be the generator of a contraction semigroup on a Hilbert space, $\mathcal{H}$ (we use the conventions of* [27] *so $B_t = e^{-itH}$ is the semigroup). Then $-iH^*$ is also the generator of a semigroup; indeed, if B_t (resp. C_t) is the semigroup generated by iH (resp. $-iH^*$), then $C_t = B_t^*$.*

Proof. Let $C_t = B_t^*$ which is clearly a contraction semigroup. Let $-iJ$ be its generator. Then

$$\left[(-iJ + 1)^{-1}\right]^* = (iH + 1)^{-1}$$

by the formula relating the resolvent of the generator to its semigroup. That $J = H^*$ is now just some graph chasing. □

THEOREM 9.1. *Let iH be the generator of a contraction semigroup on a Hilbert space $\mathcal{H}$. Suppose that $H\phi = E\phi$ with E real. Then $H^*\phi = E\phi$. In particular:*

(a) *If $H\phi = E\phi$, $H\psi = \lambda\psi$ with $E \neq \lambda$ and at least one real, then $(\phi, \psi) = 0$.*

(b) *If* $\mathcal{H}_b$ = *span of the eigenvectors of* H *with real eigenvalues then* $\mathcal{H}_b$ *and* $\mathcal{H}_b^\perp$ *are invariant spaces for* H.

Proof. If $H\phi = E\phi$, then $B_t\phi = e^{-iEt}\phi$ $(B_t = e^{-itH})$. Thus $(\phi, (B_t^* B_t)\phi) = (\phi, \phi)$. Since $B_t^* B_t$ is a self-adjoint operator of norm 1, this implies that $B_t^* B_t\phi = \phi$ so $e^{-itE}B_t^*\phi = \phi$ and thus, by the lemma $H^*\phi = E\phi$.

Under the circumstances of (a), suppose that E is real. Then $(E-\lambda)(\phi, \psi) = (H^*\phi, \psi) - (\phi, H\psi) = 0$, so $(\phi, \psi) = 0$. To prove (b) note that $\mathcal{H}_b$ is clearly left invariant by H and also H^* so $\mathcal{H}_b^\perp$ is left invariant by H^* and also H. ☐

Next we need to extend the result of Ruelle [31] to this situation. Since the spectral theorem is not available, we cannot use Wiener's theorem and fall back on the mean ergodic theorem:

THEOREM 9.2. *Let* H *and* $\mathcal{H}_b$ *be as in Theorem* 8.2. *Suppose that* C *is a bounded operator with* $C(H-i)^{-1}$ *compact. Then*

$$\frac{1}{T}\int_0^T \|Ce^{-itH}\phi\|^2 dt \to 0 \quad \text{as} \quad T\to\infty \quad \text{for any} \quad \phi\in\mathcal{H}_b^\perp .$$

Proof. Since H leaves $\mathcal{H}_b^\perp$ invariant, $D(H)\cap\mathcal{H}_b^\perp$ is dense so we can suppose that $\phi\in D(H)$. Writing

$$Ce^{-itH}\phi = C(H+i)^{-1}e^{-itH}[(H+i)\phi]$$

and thinking of $(H+i)\phi$ as a new ϕ, we are reduced to the case where C is compact. By an approximation argument, we can suppose that C is finite rank and hence rank 1. Let P_b be the orthogonal projection onto $\mathcal{H}_b^\perp$. Then for $\phi\in\mathcal{H}_b^\perp$, we can replace C by $(P_bC^*CP_b)^{1/2}$ without changing $\|Ce^{-itH}\phi\|$ so we are reduced to the case where C is a rank one projection and $\mathcal{H}_b = \{0\}$.

In that case, define a one parameter contraction semigroup $\mathcal{B}_t$ on $\mathcal{I}_2$, the Hilbert-Schmidt operators on $\mathcal{H}$, by

$$\mathcal{B}_t(A) = B_t^* A B_t$$

We claim that if $\mathcal{B}_t(A) = A$ for all t, then $A = 0$. For $\mathcal{B}_t(A^*) = \mathcal{B}_t(A)^*$ so $\mathcal{B}_t(A) = A$ implies that $\mathcal{B}_t(1/2(A+A^*)) = 1/2(A+A^*)$ and similarly for $(2i)^{-1}(A-A^*)$. Thus, without loss we need only prove the claim for $A = A^*$. Since A is Hilbert-Schmidt, and we can take $A\to -A$, we can suppose that either $\|A\| = 0$ or that $\|A\|$ is an eigenvalue of finite multiplicity. In the second case, if $A\phi = \|A\|\phi$, then $\langle B_t\phi, AB_t\phi\rangle = \|A\|\,\|\phi\|^2 \geqslant \|A\|\,\|B_t\phi\|^2$ so $AB_t\phi = \|A\|B_t\phi$ and $\|B_t\phi\| = \|\phi\|$. Thus B_t is a semigroup of isometries on the finite dimensional space $V = \{\eta \mid A\eta = \|A\|\eta\}$. Thus $V\subset\mathcal{H}_b$ which is assumed to be $\{0\}$. Thus $\|A\| = 0$, i.e., our claim is proven. Similarly, since $\mathcal{B}_t^*(A) = B_tAB_t^*$, we have that $\mathcal{B}_t^*(A) = A$, all t, has no solutions and thus $\cup_t \operatorname{Ran}(\mathcal{B}_t - I)$ is dense in $\mathcal{I}_2$.

Let $A = \mathcal{B}_t(C) - C$. Then:

$$\left\| \frac{1}{T} \int_0^T \mathcal{B}_s(A) ds \right\| \leq \frac{1}{T} \left[\int_0^t + \int_T^{T+t} \|\mathcal{B}_s(C)\| \right] \leq \frac{2t}{T} \|C\|_2$$

goes to zero. By the above density result,

$$\| \frac{1}{T} \int_0^T \mathcal{B}_s \, ds \| \to 0$$

for all A. Choosing $A = (\eta, \cdot)\eta$ and $Q = (\phi, \cdot)\phi$, we have that

$$\frac{1}{T} \int_0^T |(\eta, B_s\phi)|^2 ds = \mathrm{Tr}\left(Q^* \frac{1}{T} \int_0^T \mathcal{B}_s(A) ds \right) \to 0.$$

which is the required rank 1 result. □

Remarks. 1. The last paragraph of the proof is just a transcription of part of the proof of the mean ergodic theorm.

2. These last two theorems show that in many ways $\mathcal{H}_b$ is the natural candidate for the "bound states of H".

We need one last technical result of a general nature:

LEMMA 2. *Let H, $\mathcal{H}_b$ be as above. Then $\{(H - i)^{-2}H\phi \mid \phi \in D(H) \cap \mathcal{H}_b^\perp\}$ is dense in $\mathcal{H}_b^\perp$.*

Proof. Let $\eta \in \mathcal{H}_b^\perp$ be orthogonal to all such vectors. Since $(H - i\lambda)^{-2}H\phi = (H - i)^{-2}H[(H - i)(H - i\lambda)^{-1}]^2\phi$, $(H - i)(H - i\lambda)^{-1}$ is bounded and leaves $\mathcal{H}_b^\perp$ invariant and since $(-i\lambda)(H - i\lambda)^{-1} \to 1$ strongly as $\lambda \to \infty$, we conclude that $(\eta, H\phi) = 0$ for all $\phi \in \mathcal{H}_b^\perp$. Since $\eta \in \mathcal{H}_b^\perp$, this is also true for $\phi \in \mathcal{H}_b$. Thus $H^*\eta = 0$ so $\eta \in \mathcal{H}_b$. Thus $\eta = 0$. It follows that the set in question is total in $\mathcal{H}_b^\perp$. □

The main result of this section is

THEOREM 9.3. *Let $H_0 = -\Delta$. Let V be an operator on $L^2(\mathbb{R}^\nu)$ with*

(i) $\|V\phi\| \leq a\|H_0\phi\| + b\|\phi\|$, *some* $a < 1$.

(ii) $\mathrm{Im}(\phi, V\phi) \leq 0$ *for all* $\phi \in D(H_0)$

(iii) *The Enss condition* (2.1) *holds where*

$$h(R) = \|V(H_0 - i)^{-1}F(|x| > R)\|.$$

Let $H = H_0 + V$ and $B_t = e^{-itH}$. Then

(a) $\Omega^+ = s\text{-}\lim_{t \to -\infty} B_{-t}e^{-itH_0}$ *exists*

(b) *For all $\phi \in \mathcal{H}_b^\perp$, the limit $W\phi$ of* (9.1) *exists.*

(c) *The only possible limit point of real eigenvalues of H is* 0 *and any non-zero real eigenvalue has finite multiplicity.*

Remarks. 1. Under hypotheses (i) and (ii), $H = H_0 + V$ is automatically closed on $D(H_0)$ and $-iH$ is the generator of a contraction semigroup; see Theorem X.50 of [27].

2. No special role is played by $H_0 = -\Delta$ and one can presumably extend this theorem to the setup of Section 2.

3. There are a number of significant questions left open by this theorem. If $V(H_0 - i)^{-1}$ is compact, then $\sigma_{\text{ess}}(H) = [0, \infty)$ in the sense that away from $[0, \infty)$, $\sigma(H)$ consists only of eigenvalues of finite multiplicity (see Theorem XIII.14 of [29]) and $[0, \infty) \subset \sigma(H)$. This is presumably also true only under the hypotheses of the theorem. How can it be proven? Similarly, can a non-zero real be a limit point of complex eigenvalues?

4. Nothing is said about $\operatorname{Ran} \Omega^+$. By the intertwining relation $B_s\Omega^+ = \Omega^+ e^{-isH_0}$, proven in the usual manner, one deduces that for $H^*\phi = E\phi$ and $s > 0$:

$$|((\Omega^+)^*\phi, e^{-ish_0}(\Omega^+)^*\phi)| = e^{-s(\operatorname{Im} E)}\|(\Omega^+)^*\phi\|^2. \tag{9.2}$$

This shows that $(\Omega^+)^*\mathcal{H}_b = 0$ since $e^{-isH_0} \to 0$ weakly. Thus $\overline{\operatorname{Ran} \Omega^+} \subset \mathcal{H}_b^\perp$. However, it will not be all of $\mathcal{H}_b^\perp$; for (9.2), the Paley-Wiener theorem and the positivety of H_0 imply that $(\Omega^+)^*\phi = 0$ for an eigenvector of H^* with $\operatorname{Im} E > 0$. It is natural to conjecture that $\overline{\operatorname{Ran} \Omega^+}$ is the orthogonal complement of the eigenvectors of H^*, but is it true?

5. Nothing is said about $\overline{\operatorname{Ran} W}$. Is it all of L^2?

6. Under very special circumstances where V is "short range" and "small", Kato [22] has shown that H and H_0 are similar under two invertible maps that are defined in a "time-independent" way but which presumably equal W and $(\Omega^+)^{-1}$; that is W and Ω^+ should be invertible in those circumstances. Are there other circumstances where Ω^+ is invertible from L^2 to its range or where W is invertible, at least on some reasonable space like the Range of Ω^+?

7. We remark that the stated conclusions of the theorem do not exclude the possibility $W = \Omega^+ = 0$ so that $H = -iI$ obeys the conclusions of the theorem! However the methods of proof do show that $W \neq 0$, $\Omega^+ \neq 0$. Indeed with stationary phase methods, one easily finds lots of ϕ's obeying

$$\int_{-\infty}^{0} \|Ve^{-itH_0}\phi\|\,dt \equiv \gamma_+(\phi) < \|\phi\| \tag{9.2a}$$

and lots of ϕ's obeying

$$\int_{0}^{\infty} \|Ve^{-itH_0}\phi\|\,dt \equiv \gamma_-(\phi) < \|\phi\|. \tag{9.2b}$$

Letting $\Omega_t = B_{-t}e^{-itH_0}$ for $t < 0$ and using $\|d/dt(\Omega_t\phi)\| = \|Ve^{-itH_0}\phi\|$ one finds that for $t < s < 0$

$$\|(\Omega_t - \Omega_s)\phi\| \leqslant \int_t^s \|Ve^{-iuH_0}\phi\|\,du \tag{9.3}$$

so that under the condition (9.2a),

$$\|(\Omega^+ - 1)\phi\| < \|\phi\|.$$

Moreover writing

$$\left(B_t - e^{-itH_0}\right)\phi = -\int_0^t \frac{d}{du}\left(B_{t-u}e^{-iuH_0}\phi\right)du$$

one finds that for $t > 0$:

$$\|\left(B_t - e^{-itH_0}\right)\phi\| \leq \int_0^t \|Ve^{-isH_0}\phi\|\,ds. \tag{9.4}$$

Using the unitarity of e^{-itH_0} and (b) of the theorem, $w\text{-}\lim_{t\to\infty} e^{itH_0}B_t = WP$ where P is the projection onto $\mathcal{H}_b^\perp$. Thus, by (9.4) and (9.2b):

$$\|(WP - 1)\phi\| < \|\phi\|$$

Of course, $\|(A - 1)\phi\| < \|\phi\|$ implies that $A\phi \neq 0$ and that $P_{\text{Ran}\,A}\phi \neq 0$.

Proof of Theorem 9.3. As already noted by Davies [9], Cook's method works for Ω^+, essentially since (9.3) holds so (a) is easy.

Now let $\phi \in \mathcal{H}_b^\perp \cap D(H)$ and let $\eta = (H - i)^{-2}H\phi$. We will prove that $W\eta$ exists, so, by lemma 2, W exists on all of $\mathcal{H}_b^\perp$. As usual, by using Theorem 8.2, and the compactness of $F(|x| < n)(H_0 + i)^{-1}$ we can find $t_n \to \infty$ so that

$$\|F(|x| < n)\eta_n\| \to 0; \qquad \eta_n \equiv e^{-it_nH}\eta \tag{9.5a}$$

and

$$\|F(|x| < n)\phi_n\| \to 0; \qquad \phi_n = e^{-it_nH}\phi \tag{9.5b}$$

Suppose that we find a function $\epsilon(M) \to 0$ as $M \to \infty$ and for each M a decomposition

$$\eta_n = \eta_{n,M,w}^{(1)} + \eta_{n,M,w}^{(2)} + \eta_{n,M,\text{out}} + \eta_{n,m,\text{in}} \tag{9.6}$$

with (all limits as $n \to \infty$ with M fixed)

$$\|\eta_{n,M,w}^{(1)}\| \to 0, \qquad \|\eta_{n,M,w}^{(n)}\| \leq \epsilon(M)$$

$$\|\eta_{n,M,\text{in}}\| \to 0, \qquad \int_0^\infty \|Ve^{-itH_0}\eta_{n,M,\text{out}}\|\,dt \to 0.$$

Then letting

$$\alpha_n = \sup_{t>0} \|\left(B_t - e^{-itH_0}\right)\eta_n\|$$

and using (9.4) on the η_{out} piece we see that

$$\overline{\lim_{n\to\infty}}\, \alpha_n \leq 2\epsilon(M).$$

Taking $M \to \infty$, we see that

$$\lim_{n\to\infty} \alpha_n = 0.$$

But for $t, s > t_n$, we have that

$$\|e^{+itH_0}B_t\eta - e^{+itH_0}B_s\eta\|$$
$$\leqslant \|(B_{t-t_n} - e^{-i(t-t_n)H_0})\eta_n\| + s\text{-term} \leqslant 2\alpha_n$$

so the sequence $e^{+itH_0}B_t\eta$ is Cauchy. Thus, the existence of the decomposition (9.6) proves conclusion (b).

The main differences from the scheme of Section 2 are two-fold:

(1) We do not have the full functional calculus, so we will make do with a special Φ which does not yield a function with compact support in momentum space but rather one which cannot peak near $p = 0$ or ∞ as $n \to \infty$. The pieces near 0 and ∞ will be put in $\eta_{n,w}^{(2)}$.

(2) We will need a direct argument that $\|\eta_{n,M,\text{in}}\| \to 0$. This argument is due to Enss [14] and we are grateful for his permission to use it.

Now let $\Phi(x) = (x-i)^{-2}x = (x-i)^{-2} + i(x-i)^{-1}$. Let $\Psi_M(x)$ be a function which is C^∞ and which is 0 if $|x| < M^{-1}$ or $|x| > M$ and 1 if $|x| > 2M^{-1}$ and $|x| < (1/2)M$. Notice that if $\|\Psi_M\|_\infty \leqslant 1$, then

$$\epsilon_1(M) \equiv \|(1-\Psi_M)\Phi\|_\infty \to 0 \qquad \text{as} \quad M \to \infty \tag{9.7}$$

since Φ vanishes at 0 and ∞. Let $f_{\alpha,M}(x)$, $g_{\alpha,M}^{\text{in}}(k)$, $g_{\alpha,M}^{\text{out}}(k)$ be functions of the type used in section 2 for decomposing functions supported in $M^{-1} < k^2 < M$ with $P(k) = k^2$. Let (drop all the M's as subscript except on Ψ),

$$\eta_{n,w}^{(1)} = [\Phi(H) - \Phi(H_0)]\phi_n + \sum_{|\alpha| \leqslant (1/2)n} f_\alpha(x)\Psi_M(H_0)\Phi(H_0)\phi_n$$

$$\eta_{n,w}^{(2)} = (1-\Psi_M)(H_0)\Phi(H_0)\phi_n$$

$$\eta_{n,\text{ex}} = \sum_{|\alpha| > (1/2)n} g_\alpha^{\text{ex}}(k) f_\alpha(x)\Psi_M(H_0)\Phi(H_0)\phi_n$$

for ex = in or out. Clearly $\eta_w^{(1)} + \eta_w^{(2)} + \eta_{\text{in}} + \eta_{\text{out}} = \Phi(H)\phi_n = \eta_n$ as required. By the usual argument, *since* $\Phi(x)$ *is a polynomial in* $(x-i)^{-1}$, $\|[\Phi(H) - \Phi(H_0)] F(|x|n)\| \to 0$ and moreover, $\Psi_M\Phi_0$, as a fixed function in C_0^∞, doesn't delocalize much, so $\|\eta_{n,w}^{(1)}\| \to 0$. Next ($\epsilon_1$ given by (9.7)),

$$\|\eta_{n,w}^{(2)}\| \leqslant \epsilon_1(M)\|\phi_n\| \leqslant \epsilon_1(M)\|\phi\| \equiv \epsilon(M).$$

The standard argument shows that $\int_0^\infty \|Ve^{-itH_0}\eta_{n,\text{out}}\| dt \to 0$ so all that remains is the promised direct argument of Enss [14] that $\|\eta_{n,\text{in}}\| \to 0$.

Let A_n be the operator $\sum_{|\alpha|>(1/2)n} g_\alpha^{\text{in}}(k) f_\alpha(x) \Psi_M(H_0)\Phi(H_0)$. That $A_n\phi_n$ goes to zero will follow from the two facts:

$$s\text{-lim}\, A_n e^{-it_nH_0} = 0 \tag{9.8}$$

$$\lim\| A_n\big(e^{-it_nH_0} - B_{\tau_n}\big)\| = 0. \tag{9.9}$$

(9.9) is clearly equivalent to $\|(B_{\tau_n}^* - e^{+it_nH_0})A_n^*\| = 0$. By the B^* analog of (9.4) this follows from

$$\lim_{n\to\infty} \int_0^\infty \| V^* e^{+itH_0} A_n^* \| \, dt. \tag{9.10}$$

(9.10) follows from the calculations in section 2 if we notice that the order of factors in $g_\alpha f_\alpha \Phi$ is irrelevant to the arguments and that all estimates are uniform in the vector. Thus (9.9) is proven. To prove (9.8), we note that by the proof of (9.10),

$$\| F(|x| < R) e^{+it_nH_0} A_n^* \| \to 0$$

for each R, so that $A_n e^{-it_nH_0}\psi \to 0$ for any ψ with compact support. This completes the proof of conclusion (b).

Now suppose (c) is false. Then by Theorem 9.1, we can find an orthonormal sequence η_n with $H\eta_n = H^*\eta_n = E_n\eta_n$ and $E_n \to E \neq 0$. Letting $(E_n - i)^2 E_n^{-1}\eta_n = \phi_n$, we obtain a decomposition of the form (9.6) with all the properties of that decomposition except that $\|\eta_{n,\text{in}}\| \to 0$ is not directly proven. Rather:

$$\int_0^\infty \| V^* e^{+itH_0} \eta_{n,\text{in}} \| \, dt \to 0.$$

Given this decomposition, write

$$\langle \eta_n, \eta_n \rangle = a + b + c + d$$

where:

$a = \langle \eta_n, \eta_{n,w}^{(1)} \rangle$ goes to zero as $n \to \infty$, $b = \langle \eta_n, \eta_{n,w}^{(2)} \rangle$ is smaller than $\epsilon(M)$,

$$\begin{aligned} |c| &= |\langle B_t^* \eta_n, \eta_{n,\text{out}} \rangle| = |\langle \eta_n, B_t \eta_{n,\text{out}} \rangle| \\ &= \lim_{t\to\infty} \big|\big\langle \eta_n, \big(B_t - e^{-itH_0}\big)\eta_{n,\text{out}} \big\rangle\big| \\ &\leqslant \sup_{t>0} \big\| \big(B_t - e^{-itH_0}\big) \eta_{n,\text{out}} \big\| \end{aligned}$$

goes to zero as $n \to \infty$. Similarly, using $|d| = |\langle B_t \eta_n, \eta_{n,\text{in}} \rangle|$ we see that $d \to 0$. Thus

$$\lim_{n\to\infty} \langle \eta_n, \eta_n \rangle \leqslant \epsilon(M)$$

which violates the normalization condition if M is properly chosen. □

Appendix: An amusing inequality. Clearly, the development of lore about phase space decompositions should be useful in further developments of the Enss method. In a preliminary version of the proof of lemma 3 in Section 2, we found an inequality that may be useful in further developments. Let X, P denote the ν-tuples of operators on $L^2(\mathbb{R}^\nu)$, X_j = multiplication by x_j and $P_j = i^{-1}\nabla_j$. Let $\|\cdot\|_2$ = Hilbert-Schmidt norm. The inequality:

$$\|A\|_2^2 = (2\pi)^\nu \int |a(y,k)|^2 dy\, dk \tag{A.1}$$

$$A = \int a(y,k) e^{iyX} e^{ikP} dy\, dk \tag{A.2}$$

is well-known. One way of verifying (A.1) is to note that

$$(A\phi)(x) = \int b(x,p)\hat{\phi}(p)\,dp$$

where ^ is the Fourier transform and

$$b(x,p) = (2\pi)^{-\nu/2} \int a(y,k) e^{iyx} e^{ikp} e^{ipx} dy\, dk$$

so that $\|b\|_2^2 = (2\pi)^\nu \|a\|_2^2$ by the Plancherel theorem.

Our result is:

THEOREM A.1 $\|[f(X), g(P)]\|_2 \leqslant (2\pi)^\nu \|\nabla f\|_2 \|\nabla g\|_2$.

Proof. Clearly

$$[f(X), g(P)] = \int \hat{f}(y)\hat{g}(k)[e^{iyX}, e^{ikP}]$$

$$= \int \hat{f}(y)\hat{g}(k) e^{iyX} e^{ikP}(1 - e^{iyk})$$

so $[f(X), g(P)]$ has the form of (A.2) with

$$|a(y,k)| \leqslant |\hat{f}(y)|\,|\hat{g}(k)|\,|y|\,|k|.$$

Now use (A.1) and the Plancherel theorem. □

Note added in proof

E. Mourre has noted that Ex. 4.4 (treating $V = \nabla\cdot W$ with $W = O(r^{-1-\epsilon})$) can be extended to treat potentials V which are $-\Delta$ bounded and which are of the form $V = -\Delta W_1 + \nabla\cdot W_2 + W_3$ with $W_i(r^{-1-\epsilon})$ at infinity. For one can write $-\Delta W,= \sum_i [p_i, [p_i, W_1]]$. Noting that when $D(H) = D(H_0)$, and $(H_0+1)^{-1} - (H+1)^{-1}$ is compact, Theorem 4.1 can be extended to allow V's with $(H+i)^{-1} V (H_0+i)^{-1} F(x > R) = h(R) \in L^1(0,\infty)$, one finds a complete spectral analysis of such objects. An example of such V's would be $\sin(r^\alpha)/r^\beta$ with $\alpha > 1$ and $\beta + 2(\alpha - 1) > 1$. It is clear that if one knows that

$D(H^m) = D(H_0^m)$ for some m, then V can be the $2m$-th derivative of a function which is $O(r^{-1-\epsilon})$ at infinity. In this way one can treat potentials of the above type so long as $\beta > 1/2$ and $\alpha > 1$.

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