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Unique Continuation for Schrodinger Operators with Unbounded Potentials*

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We consider unique continuation theorems for solution of inequalities $|\Delta u(x)| \leqslant W(x)\,|u(x)|$ with W allowed to be unbounded. We obtain two kinds of results. One allows $W \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ with $p \geqslant n-2$ for $n > 5$, $p > \frac{1}{3}(2n-1)$ for $n \leqslant 5$. The other requires fW^2 to be $-\Delta$-form bounded for all $f \in C_0^\infty$.

1. INTRODUCTION

In this paper, we want to consider unique continuation theorems in the following sense:

DEFINITION. We say that a function W on Ω, a connected open subset of $\mathbb{R}^n$, has the unique continuation property if and only if every function, u, obeying

$$|\Delta u(x)| \leqslant W(x)\,|u(x)| \tag{1.1}$$

which is equal to zero on some open set is identically zero on Ω.

The classical theorems going back to Carleman [2] and Müller [7] require that $W \in L^\infty_{\mathrm{loc}}$. While there is an extensive literature on replacing Δ in (1.1) by more general differential operators (see Hörmander [5]), there appears to have been no previous attempts at allowing W's with local singularities. This is a situation first emphasized by Lavine.

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In Section 2, we will use a method of Protter [8] and show that W has the unique continuation property if for any $f \in C_0^\infty$, there is a, b with

$$\langle \eta, |fW^2| \eta \rangle \leqslant a\langle \nabla\eta, \nabla\eta \rangle + b\langle \eta, \eta \rangle. \tag{1.2}$$

In Sections 3–7 we will exploit some ideas of Heinz [4] to prove the unique continuation property when $W \in L^p_{\text{loc}}(\mathbb{R}^n)$ $p \geqslant n-2$ for $n > 5$, $p > \frac{1}{3}(2n-1)$ for $n \leqslant 5$. We note that while these results strengthen those of Section 2, so far as L^p_{loc} conditions are concerned, it can happen that (1.2) holds with $W \notin L^p_{\text{loc}}$, p given by the above. In particular, if $n = 3N$, $r = (r_1, ..., r_N)$, $r_i \in R^3$ and $W = \sum_{i<j} V_{ij}(r_i - r_j) - E$, then (1.2) only requires that $V_{ij} \in L^3_{\text{weak,loc}}$ (independent of N), while for $N \geqslant 2$, L^p conditions require $V_{ij} \in L^p_{\text{loc}}$ with $p = 2N - 2/3$.

Our first result is a consequence of the following theorem which may be of interest in its own right. Let B^n be the unit ball in $\mathbb{R}^n$ and put

$$\|f\|_p = \left(\int_{B^n} |f(x)|^p \, dx \right)^{1/p}, \qquad p \geqslant 1.$$

Then we have

THEOREM 1.1. *Let p, q satisfy $1 < q \leqslant 2 \leqslant p$ and*

$$(n-2)(1/q - 1/p) \leqslant 1, \quad (n-1)/q < 1 + (n/p), \quad 1/p + 1/q \leqslant 1 \tag{1.3}$$

Then there is a constant C such that

$$\|r^k f\|_p \leqslant C\, \|r^k \Delta f\|_q \tag{1.4}$$

holds for all integers k and all $f \in C_0^\infty(R^n \backslash \{0\})$.

Since unique continuation is a local property, obviously one need only require that W be L^p_{loc} on $\Omega \backslash C$, with C a closed set of measure zero so that $\Omega \backslash C$ is connected. Thus the Müller theorem includes the important case of atomic potentials ($V_{ij}(r) = \alpha_{ij} |r|^{-1}$).

Let us emphasize the unsatisfactory nature of our results here. We are reasonable sure that any $W \in L^p_{\text{loc}} \mathbb{R}^n)$ with $p > n/2$ has the unique continuation property but we are unable to prove this.

Finally, we should mention some recent related results of Amrein and Berthier [1] who prove tht for certain potentials, V, with local singularities, $-\Delta + V$ has no eigenfunctions of compact support (this is one of the main applications of the unique continuation property, see Kato [6] and Section 7). Since we have not seen the most detailed results of Amrein and Berthier, we cannot make a comparison.

2. Protter's Inequalities

We assume that (1.1) holds with W^2 being $-\Delta$-form bounded in a ball B of radius $r_0 < 1$. By this we mean that

$$\| Wv \|^2 \leqslant C(\| \nabla v \|^2 + \| v \|^2), \qquad v \in C_0^\infty(B), \tag{2.1}$$

holds for some constant C, where the norm is that of $L^2(B)$. A sufficient condition for (7.1) to hold is that $W \in N_2^{\text{loc}}$, i.e., that

$$\int_{B \cap |x-y|<1} | W(x)|^2 \, |x-y|^{2-n} \, dx \leqslant C_0, \qquad y \in \mathbb{R}^n$$

(cf. [10]). In proving unique continuation under this hypothesis we shall make use of two inequalities due to Protter [8]. Take the center of B as the origin and put $w = w_\beta(r) = \exp\{r^{-\beta}\}$. Then there are constants β_0 and C_0 and a function $\sigma(\beta) \to 0$ as $\beta \to \infty$ such that if $\beta > \beta_0$

$$\beta^4 \int r^{-2\beta-2} \, |wv|^2 \, dx \leqslant C_0 \int r^{\beta+2} \, |w\Delta v|^2 \, dx \tag{2.2}$$

and

$$\int |w\nabla v|^2 \, dx \leqslant \sigma(\beta) \int r^{\beta+2} \, |w\Delta v|^2 \, dx, \tag{2.3}$$

where v vanishes outside B $wv \to 0$ as $r \to 0$ for every $\beta > 0$. Now suppose u satisfies (1.1) and vanishes near the origin. Let a be any number satisfying $0 < a < r_0$, and let φ be any function in $C_0^\infty(B)$ such that $\varphi \equiv 1$ for $|x| < a$. Put $v = \varphi u$. Then we have

$$\begin{aligned} \int_{r<a} r^{\beta+2} \, |w\Delta u|^2 \, dx &\leqslant \int r^{\beta+2} \, |wWv|^2 \, dx \\ &\leqslant C(\| \nabla(wv)\|^2 + \| wv \|^2). \end{aligned} \tag{2.4}$$

Since

$$\nabla(wv) = -\beta r^{-\beta-2} \vec{r} wv + w\nabla v,$$

we see by (2.2) and (2.3) that the left-hand side of (2.4) is bounded by

$$\sigma_1(\beta) \int r^{\beta+2} \, |w\Delta v|^2 \, dx,$$

where

$$\sigma_1(\beta) = 3C_0 \beta^{-2} + 2\sigma(\beta).$$

Take β so large that $\sigma_1(\beta) < \frac{1}{2}$. Then

$$\int_{r<a} r^{\beta+2} |w\Delta u|^2 \, dx \leqslant 2\sigma_1(\beta) \int_{r>a} r^{\beta+2} |w\Delta v|^2 \, dx. \tag{2.5}$$

Combining this with (2.2) we obtain

$$\int r^{-2\beta-2} |wv|^2 \leqslant \sigma_2(\beta) \int_{r>a} |w\Delta v|^2 \, dx, \tag{2.6}$$

where

$$\sigma_2(\beta) = C_0(1 + 2\sigma_1(\beta))\beta^{-4}.$$

But (2.6) implies

$$\int_{r<a} |u|^2 \, dx \leqslant a^{2\beta+2} \sigma_2(\beta) \int_{r>a} |\Delta v|^2 \, dx \to 0 \qquad \text{as} \quad \beta \to \infty.$$

This shows that $u \equiv 0$ for $|x| < a$. Since a was any value $< r_0$, we see that $u \equiv 0$ in B. The argument in an arbitrary domain is standard.

We have therefore proved:

THEOREM 2.1. *If W obeys (2.1), u obeys (1.1) and it vanishes in a small ball, then $u = 0$.*

3. ESTIMATE IN ONE DIMENSION

In order to prove the inequality that we shall use for our unique continuation theorem, we shall make use of a one-dimensional estimate applied to each partial wave. The estimate, which extends estimates of Heinz [4], is given by

THEOREM 3.1. *For s real, let*

$$L_s u = u'' - s(s+1)x^{-2}u. \tag{3.1}$$

Then

$$|(2s+1)x^\alpha f(x)|^{p'} \leqslant (|p(s+\alpha+1)|^{-1/p} + |p(s-\alpha)|^{-1/p})^{p'} \int_0^1 |y^{\alpha+1+1/p} L_s f(y)|^{p'} \, dy \tag{3.2}$$

for all real α, s, $1 < p < \infty$, $f \in C_0^\infty[(0, 1)]$, where $p' = p/(p-1)$.

Proof. If $g = L_s f$, it is easily checked that $(2s+1)f = u + v$, where

$$u(x) = \int_0^x x^{s+1} y^{-s} g(y)\, dy, \qquad v(x) = \int_x^1 x^{-s} y^{s+1} g(y)\, dy. \tag{3.3}$$

Since $L_s y^{-s} = L_s y^{s+1} = 0$ and f vanishes near 0 and 1, integration by parts yields

$$\int_0^1 y^{-s} g(y)\, dy = \int_0^1 y^{s+1} g(y)\, dy = 0. \tag{3.4}$$

If $s + \alpha + 1 = 0$ or $s = \alpha$, the right-hand side of (3.2) is infinite, so there is nothing to prove. If $\beta = -p(\alpha + s + 1) > 0$, we have

$$|x^\alpha u(x)| \leqslant x^{s+\alpha+1} \left(\int_0^x y^{\beta-1}\, dy\right)^{1/p} \left(\int_0^x y^{-p'((\beta-1)/p+s)} \,|\, g(y)^{p'} |\, dy\right)^{1/p'}.$$

Thus

$$|x^\alpha u(x)|^{p'} \leqslant \beta^{-p'/p} \int_0^1 |\, y^{\alpha+1+1/p} g(y)|^{p'}\, dy.$$

If $\beta < 0$, we have by (2.4)

$$|x^\alpha u(x)| \leqslant x^{s+\alpha+1} \left(\int_x^1 y^{\beta-1}\, dy\right)^{1/p} \left(\int_x^1 y^{-p'((\beta-1/p)+s)} \,|\, g(y)|^{p'} dy\right)^{1/p'}.$$

Thus

$$|x^\alpha u(x)|^{p'} \leqslant |\beta|^{-p'/p} \int_0^1 |\, y^{\alpha+1+1/p} g(y)|^{p'}\, dy.$$

Similarly, if $\sigma = p(s - \alpha) > 0$, we have

$$|x^\alpha v(x)| \leqslant x^{\alpha-s} \left(\int_0^1 y^{\sigma-1}\, dy\right)^{1/p} \left(\int_0^x y^{p'(s+1-(\sigma-1)/p)} \,|\, g(y)|^{p'}\, dy\right)^{1/p'}$$

which gives

$$|x^\alpha v(x)|^{p'} \leqslant \sigma^{-p'/p} \int_0^1 |\, y^{\alpha+1+1/p} g(y)|^{p'} dy.$$

If $\sigma < 0$, we get

$$|x^\alpha v(x)| \leqslant x^{\alpha-s} \left(\int_x^1 y^{\sigma-1}\, dy\right)^{1/p} \left(\int_x^1 y^{p'(\alpha+1-(\sigma-1)/p)} \,|\, g(y)|^{p'}\, dy\right)^{1/p'}$$

which gives

$$|x^\alpha v(x)|^{p'} \leqslant |\sigma|^{-p'/p} \int_0^1 |y^{\alpha+1+1/p} g(y)|^{p'} \, dy.$$

These inequalities give the desired result. ∎

4. Spherical Harmonics

In dealing with functions on $\mathbb{R}^n$, $n > 2$, we shall use partial wave expansions. If $r = |x|$ and $f(x)$ is a function in $L^2(|x| < 1)$, we can expand it in the form

$$r^\gamma f(r\xi) = \sum f_{lm}(r) \, Y_{lm}(\xi), \quad \gamma = (n-1)/2, \tag{4.1}$$

where $\xi = x/r$ and the Y_{lm} are surface harmonics (cf. [3]). For each integer $l \geqslant 0$, there are

$$h(l) = (2l + n - 2) \frac{(l + n - 3)!}{(n-2)! \, l!} \tag{4.2}$$

such polynomials. The $\{Y_{l,m}\}$ form a complete orthonormal sequence in $L^2(\Omega)$, where Ω is the unit sphere $|x| = 1$ in $\mathbb{R}^n$. The "coefficients" $f_{l,m}(r)$ are functions of r alone and are given by

$$f_{l,m}(r) = r^\gamma \int_\Omega f(r\xi) \, Y_{l,m}(\xi)^* \, d\xi. \tag{4.3}$$

If $v = r^\gamma h(r)$, then

$$v'' = r^\gamma (h'' + (n-1) r^{-1} h' + \tfrac{1}{4}(n-1)(n-3) r^{-2} h)$$

and

$$\Delta(h Y_{l,m}) = (h'' + (n-1) r^{-1} h' - l(l + n - 2) r^{-2} h) \, Y_{l,m}.$$

From this it follows that

$$\Delta f(r\xi) = r^{-\gamma} \sum L_s f_{l,m}(r) \, Y_{l,m}(\xi), \tag{4.4}$$

where

$$s(s+1) = l(l + n - 2) + \tfrac{1}{4}(n-1)(n-3).$$

This will be satisfied if we take

$$s = \tfrac{1}{2}(2l + n - 3). \tag{4.5}$$

An important property of the $Y_{l,m}$ is

$$\sum_{m=1}^{h(l)} |Y_{l,m}(\xi)|^2 = \frac{h(l)}{\omega}, \tag{4.6}$$

where ω is the surface area of Ω (cf. [3]).

5. L^p Inequality on Ω

Let $a(\xi)$ be a function in $L^\infty(\Omega)$. We can expand it in terms of surface harmonics. Thus

$$a(\xi) = \sum_{l,m} a_{l,m} Y_{l,m}(\xi) \tag{5.1}$$

where

$$a_{l,m} = \int_\Omega a(\xi) Y_{l,m}(\xi)^* \, d\xi. \tag{5.2}$$

Let $Y_l(\xi)$ be the $h(l)$-dimensional vector function $Y_l(\xi) = \{Y_{l,1}(\xi),..., Y_{lh}(\xi)\}$ and let a_l be the vector $\{a_{l,1},..., a_{l,h}\}$. Then (5.1) becomes

$$a(\xi) = \sum_l a_l \cdot Y_l(\xi). \tag{5.3}$$

Since the $Y_{l,m}$ are orthonormal, we have

$$\|a\|^2_{L^2(\Omega)} = \sum_l |a_l|^2. \tag{5.4}$$

Now (4.6) says that $|Y_l|^2 = h(l)/\omega$. Thus by the Schwarz inequality

$$\|a\|_{L^\infty(\Omega)} \leqslant \omega^{-1/2} \sum_l h(l)^{1/2} |a_l|. \tag{5.5}$$

If we now apply interpolation to these inequalities, we obtain

$$\|a\|^{p'}_{L^p(\Omega)} \leqslant C \sum_l h(l)^{1-p'/2} |a_l|^{p'}, \qquad 2 \leqslant p \leqslant \infty. \tag{5.6}$$

Moreover, a simple duality argument then gives

$$\sum_l h(l)^{1-q'/2} |a_l|^{q'} \leqslant C \|a\|^{q'}_{L^q(\Omega)}, \qquad 1 < q \leqslant 2. \tag{5.7}$$

6. Proof of Theorem 1.1

If we apply (5.6), we have

$$\|r^{\alpha} f(r\xi)\|_{L^p(\Omega)}^{p'} \leqslant C \sum_l h(l)^{1-p'/2} |r^{\alpha-\gamma} f_l(r)|^{p'} \tag{6.1}$$

by (4.1), where $f_l(r) = \{f_{l,1},..., f_{l,h}\}$. If we apply Theorem 3.1, we have

$$|(2l+n-2)\, r^{\alpha-\gamma} f_l(r)|^{p'}$$
$$\leqslant (|p(l+\alpha)|^{-1/p} + |p(l+n-\alpha-2)|^{-1/p})^{p'} \int_0^1 y^{\alpha-\gamma+1+1/p} |L_s f_l(y)|^{p'}\, dy \tag{6.2}$$

since s is given by (3.5). Note that

$$C^{-1}(l+1)^{n-2} \leqslant h(l) \leqslant C(l+1)^{n-2}. \tag{6.3}$$

Thus, if we put $g_l = L_s f_l$ and

$$m(l) = |p(l+\alpha)|^{-1/p} + |p(l+n-\alpha-2)|^{-1/p},$$

we get

$$\|r^{\alpha} f(r\xi)\|_{L^p(\Omega)}^{p'} \leqslant C \sum (l+1)^{(n-2)(1-p'/2)-p'} m(l)^{p'} \int_0^1 |y^{\alpha-\gamma+1+1/p} g_l(y)|^{p'}\, dy.$$

If k is an integer and $\delta = (n-1)/p$, we get

$$\|r^{k+\delta} f(r\xi)\|_{L^p(\Omega)}^{p'}$$
$$\leqslant C \sum (l+1)^{(n-2)(1-p'/2)-p'} m_k(l)^{p'} \int_0^1 |y^{k+\delta-\gamma+1+1/p} g_l(y)/^{p'}\, dy, \tag{6.4}$$

where

$$m_k(l) = |p(l+k+\delta)|^{-1/p} + |p(l+n-k-\delta-2)|^{-1/p}.$$

We estimate the right-hand side of (6.4) by

$$C\left(\sum (l+1)^{-\mu\rho}\right)^{1/\rho} \left(\sum m_k(l)^{p'\sigma}\right)^{1/\sigma}$$
$$\times \int_0^1 \left(\sum_1 (l+1)^{(n-2)(1-q'/2)} |y^{k+\delta-\gamma+1+1/p} g_k(y)|^{p'\tau}\right)^{1/\tau} dy, \tag{6.5}$$

where

$$\rho^{-1} + \sigma^{-1} + \tau^{-1} = 1, \qquad \mu\rho > 1,\ p'\sigma > p,\ p'\tau = q' \tag{6.6}$$

and

$$\mu = p'\left[1 + (n-2)\left(\frac{1}{p} - \frac{1}{q}\right)\right]. \tag{6.7}$$

(If $\mu = 0$, we take $\rho = \infty$.)

We shall show that under the hypotheses of Theorem 1.1 one can find μ, p, σ, and τ satisfying (6.6) and (6.7). Assuming this for the moment, we note that the first factor in (6.5) is finite. The second is bounded independently of k provided δ is not an integer. In fact we have

$$\sum_{l=0}^{\infty} |p(l+k+\delta)|^{-\nu} \leqslant |p|^{-\nu} \sum_{j=-\infty}^{\infty} |j+\delta|^{-\nu}, \tag{6.8}$$

where $\nu = p'\sigma/p > 1$. By (4.4) and (5.7), the last factor in (6.5) is bounded by

$$\begin{aligned} &C\int_0^1 \|r^{k+\delta+1+1/p}\,\Delta f(r\xi)\|_{L^q(\Omega)}^{p'}\,dr\Big) \\ &\quad\leqslant C\left(\int_0^1 \|r^{k+\delta+1+1/p}\,\Delta f(r\xi)\|_{L^q(\Omega)}^{q}\,dr\right)^{p'/q} \\ &\quad= C\,\|r^{k+\delta-\beta+1+1/p}\,\Delta f\|_q^{p'}, \end{aligned} \tag{6.9}$$

where $\beta = (n-1)/q$. Since

$$\delta - \beta + 1 + \frac{1}{p} = (n-1)\left(\frac{1}{p} - \frac{1}{q}\right) + 1 + \frac{1}{p} > 0, \tag{6.10}$$

this gives

$$\sup_{r\leqslant 1} \|r^{k+\delta} f(r\xi)\|_{L^p(\Omega)} \leqslant C\,\|r^k \Delta f\|_q. \tag{6.11}$$

Since

$$\|r^k f\|_p^p = \int_0^1 \|r^{k+\delta} f(r\xi)\|_{L^p(\Omega)}^p\,dr,$$

we obtain the desired inequality. It remains to show that one can find constants μ, σ, and τ satisfying (6.6) and (6.7) under the hypotheses of Theorem 1.1. Put $x = 1/p$, $y = 1/q$. Then (6.6) and (6.7) are implied by

$$\begin{aligned} &\mu > 0, \qquad (n-1)(y-x) < 1 + x \\ \text{or} \\ &\mu = 0, \qquad \rho = \infty,\ y < 2x. \end{aligned} \tag{6.12}$$

As a corollary of Theorem 1.1 we have

THEOREM 6.1. *For any $\varepsilon > 0$ there exists p, q such that the conclusion of Theorem 1.1 holds and*

$$1/q - 1/p = \frac{1}{n-2} \qquad \text{if } n > 5$$

$$> \frac{3}{2n-1} - \varepsilon \qquad \text{if } n \leqslant 5.$$

Proof. For $n > 5$, take $x = (n-3)/2(n-2)$, $y = (n-1)/2(n-2)$. For $n \leqslant 5$, take $x = (n-2)/(2n-1)$, $y = (n+1)/(2n-1) - \varepsilon$. Both (6.10) and (6.12) are satisfied. ∎

7. UNIQUE CONTINUATION THEOREM

THEOREM 7.1. *Let u obey (1.1) and $W \in L^r_{\mathrm{loc}}(\mathbb{R}^n)$ with*

$$r = n - 2 \qquad \text{if } n > 5$$

$$> \tfrac{1}{3}(2n-1) \qquad \text{if } n \leqslant 5.$$

Then if $u = 0$ in a small ball, then $u = 0$ everywhere.

Proof. By a standard connectedness argument, it suffices to show that there are some fixed R_0 depending only on local L^r norms of W so that $u(x) = 0$ for x near zero implies $u(x) = 0$ for $|x| < R_0$. Choose R_0 so small that $R_0 < 1$ and $(\int_{|x| < R_0} |W(x)|^r \, dx)^{1/r} \leqslant \frac{1}{2}$. Let χ be a C^∞ function supported in the unit ball which is identically one on the ball of radius R_0. Then, for k a negative integer, we let $p = (\frac{1}{2} - 1/2r)^{-1}$; $q = (\frac{1}{2} + 1/2r)^{-1}$ and use Theorem 6.1:

$$\begin{aligned}
\left(\int_{|x| \leqslant R_0} |r^k u|^p \, dx \right)^{1/p} &\leqslant \| r^k f \|_p \\
&\leqslant \| r^k \Delta f \|_q \\
&\leqslant \left(\int |r^k \Delta u|^q \, dx \right)^{1/q} + C R_0^k \\
&\leqslant \frac{1}{2} \left(\int |r^k u|^p \, dx \right)^{1/p},
\end{aligned}$$

where we use (1.1) and Holder's inequality in the last step. Taking $k \to -\infty$, we conclude that $u \equiv 0$ on the ball of radius R_0.

8. Application

Applications of unique continuation are often to eliminate the possibility of positive eigenvalues; here is a typical example; see Section XIII.13 of [9] for more complicated examples.

THEOREM 8.1. *Let W have compact support and lie in L^r with r given in Theorem 7.1. Then $-\Delta + W$ (the form sum) has no positive eigenvalues.*

Proof. Let W have support inside the ball of radius R_0. Suppose that $-\Delta u + Wu = Eu$ with $E > 0$. Expand u in spherical harmonies and use the fact that for $x > R_0$, the components u_{lm} obey a second-order equation whose solutions are Bessel functions which are easily seen to be non-square integrable. Thus $u(x) = 0$ if $|x| > R_0$. It follows that $u \equiv 0$ by Theorem 7.1.

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