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Brownian Motion, L^p Properties of Schrödinger Operators and the Localization of Binding*

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We use Brownian motion ideas to study Schrödinger operators $H = -\frac{1}{2}\Delta + V$ on $L^p(R^\nu)$. In particular: (a) We prove that $\lim_{t\to\infty} t^{-1} \ln \| e^{-tH} \|_{p,p}$ is p-independent for a very large class of V's where $\| A \|_{p,p}$ = norm of A as an operator from L^p to L^p. (b) For $\nu \geqslant 3$ and $V \in L^{\nu/2-\epsilon} \cap L^{\nu/2+\epsilon}$, we show that $\sup \| e^{-tH} \|_{\infty,\infty} < \infty$ if and only if H has no negative eigenvalues or zero energy resonances. (c) We relate the "localization of binding" recently noted by Sigal to Brownian hitting probabilities.

1. INTRODUCTION

Our goal here is to use Brownian motion ideas to study Schrödinger operators

$$H = -\tfrac{1}{2}\Delta + V. \tag{1.1}$$

As a starting point, we rely on certain estimates of Simon [13] obtained using Brownian motion. We emphasize that Carmona [3] independently obtained somewhat weaker results using similar methods and that earlier Herbst and Sloan [7] had proven some results with vaguely related methods. Moreover, both Carmona and Simon rely on beautiful estimates of Portenko [9] which were brought to their attention due to their rediscovery by Berthier and Gaveau [2].

To describe the results, we need to define two special classes of potentials, $\mathscr{V}_1$ and $\mathscr{V}_2$.

DEFINITION. $L_u^p(\mathbb{R}^\mu)$ is the family of uniformly L^p functions, i.e., $f \in L_u^p$ if and only if

$$\|f\|_{p,u} = \sup_x \left(\int_\Delta |f(x+y)|\, dy\right)^{1/p} < \infty,$$

where Δ is the unit cube centered at zero.

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DEFINITION. A function V on R is said to lie in $\mathscr{V}_1$ if and only if V is a finite sum of functions, each one *either* of the form

(1) g is bounded from below and $V \in L^1_{\text{loc}}$ or

(2) $g(x) = f(\pi x)$ where π is a linear function from R^ν *onto* R^μ and $f \in L_u{}^p(R^\mu)$ for some $p > \mu/2$.

DEFINITION. Let $\nu \geqslant 3$. We say V, a function on R^ν, is in $\mathscr{V}_2$ if and only if for some $p < \nu/2$ and some $q > \nu/2$:

$$V \in L^p \cap L^q.$$

Remarks. 1. The rather complicated form of $\mathscr{V}_1$ is made to accommodate two classes of V's of physical interest: (i) periodic V's; hence, we only require $L_u{}^p$ conditions rather than L^p conditions; (ii) multiparticle potentials which are sums of functions of fewer variables; if one only deals with $L^p(R^\nu)$ conditions, p must get larger as ν does but with the $g(x) = f(\pi x)$ condition, we can have ν get larger with μ fixed and have p fixed. Virtually *all* potentials of physical interest which lead to an H which is bounded from below are included in $\mathscr{V}_1$.

2. The $\nu \geqslant 3$ condition for $\mathscr{V}_2$ is due to the fact that for *any* V short range and negative, $-\frac{1}{2}\Delta + V$ has negative eigenvalues for $\nu = 1, 2$ but for $\nu \geqslant 3$, this is not true for V "small enough." This, in turn, is connected with recurrence properties of Brownian motion; see [13].

We can now state two basic results from [13]:

THEOREM 1.1. *Let $V \in \mathscr{V}_1$. Then H defined as an L^2 quadratic form on $C_0{}^\infty$ is closable, yielding a self-adjoint operator H which is bounded from below. The operators e^{-tH} $(t > 0)$ defined on L^2 extend from $L^p \cap L^2$ to bounded operators on L^p $(p < \infty$; for $p = \infty$, e^{-tH} is defined as the dual of the L^1 operator). For $p < \infty$, e^{-tH} is a strongly continuous semigroup obeying*

$$\| e^{-tH} \|_{p,p} \leqslant Ce^{At} \tag{1.2}$$

for suitable C, A independent of p (but dependent on V). Moreover, for any $t > 0$, $p \geqslant q$,

$$\| e^{-tH} f \|_p \leqslant C(p, q, t; V) \| f \|_q . \tag{1.3}$$

THEOREM 1.2. *If $V \in \mathscr{V}_2$, then for all sufficiently small real λ, $H(\lambda) = -\frac{1}{2}\Delta + \lambda V$ obeys*

$$\| \exp(-tH(\lambda)) \|_{p,p} \leqslant C \tag{1.4}$$

for all $t \geqslant 0$ and p.

Remarks. 1. We use extensively the fact that for $V \in \mathscr{V}_1$, the above-mentioned H obeys the Feynman–Kac formula,

$$(e^{-tH}f)(x) = E\left(\exp\left(-\int_0^t V(x+b(s))\,ds\right) f(x+b(t))\right), \tag{1.5}$$

where b is ν-dimensional Brownian motion and E is expectation with respect to the Brownian motion. Indeed, Theorem 1.1 is proven by establishing (1.5) for the L^2 semigroup and then using (1.5) to make the extension.

2. Under very weak additional conditions, $e^{-tH}(L^2)$ consists of continuous functions [13], so e^{-tH} cannot be strongly continuous on L^∞. (1.2) holds for the $p = \infty$ operator.

3. The small t behavior of (1.3) is discussed in detail in [13], yielding Sobolev estimates.

4. For alternative approaches to defining H on the L^p spaces, see [11, 14, 15].

With these preliminaries out of the way, we can describe the problems that concern us in this paper.

DEFINITION. For $V \in \mathscr{V}_1$ and $1 \leqslant p \leqslant \infty$,

$$\alpha_p(V) = \lim_{t\to\infty} t^{-1} \ln \| e^{-tH} \|_{p,p}\,. \tag{1.6}$$

The limit in (1.6) exists by a standard argument (essentially the one that says that $\mathrm{spr}(e^{-H}$ as an op on $L^p) \equiv e^{\alpha_p(V)}$ is given by the spectral radius formula) and the convexity of $\ln \| e^{-tH} \|_{p,p}$.

In Section 2, we prove that

THEOREM 1.3. *For any $V \in \mathscr{V}_1$, $\alpha_p(V)$ is independent of p.*

Remarks. 1. By duality and interpolation

$$\alpha_p = \alpha_{p'}\,; \qquad p' = (1 - p^{-1})^{-1} \tag{1.7a}$$

$$\alpha_p \leqslant \alpha_q \qquad \text{for} \quad 2 \leqslant p \leqslant q \leqslant \infty. \tag{1.7b}$$

Thus, the theorem follows from the inequality

$$\alpha_\infty \leqslant \alpha_2\,. \tag{1.8}$$

We actually prove that

$$\| e^{-tH} \|_{\infty,\infty} \leqslant C(t+1)^{\nu/2} e^{t\alpha_2} \tag{1.9}$$

from which (1.8) follows. We note that by duality and interpolation and (1.9)

$$\| e^{-tH} \|_{p,p} \leqslant C(t+1)^{\nu a/2} e^{t\alpha_2},$$

where $a = | 1 - 2p^{-1} |$.

2. If $\sigma_p = \{-\ln \lambda \mid \lambda \in \operatorname{spec}(e^{-H} \text{ as an op on } L^p)\}$, then the theorem asserts that $\inf(\sigma_p)$ is p-independent. Is it true that

$$\sigma_p \text{ is } p\text{-independent?} \tag{1.10}$$

This result, which would clearly imply Theorem 1.1, does not seem amenable to proof by the methods of this paper. For a much restricted class of V's, Weder [15] has proven some cases of (1.10); see also Section 5. Note that even were (1.10) known, (1.9) would be of independent interest.

The above is the only result that we prove for the big class $\mathscr{V}_1$. In Sections 3 and 4 deal with the smaller class $\mathscr{V}_2$. We begin by asking when

$$\beta_\infty = \sup_{t>0} \| e^{-tH} \|_{\infty,\infty} \tag{1.11}$$

is finite. One of the answers we get is the following:

DEFINITION. $V \in \mathscr{V}_2$ is called supercritical (resp. critical) [resp. subcritical] if and only if $\alpha_2(V) > 0$ (resp. $\alpha_2(V) = 0$ but $\alpha_2(\lambda V) > 0$ for all $\lambda > 1$) [resp. $\alpha_2(\lambda V) = 0$ for all $\lambda \in [0, \Lambda]$ with $\Lambda > 1$].

Remark. Since $V \in \mathscr{V}_2$ implies that $\sigma_2(H) \supset [0, \infty)$, $\alpha_2(\lambda V) \geqslant 0$ for all λ. Since $\alpha_2(\lambda V)$ is convex in λ and $\alpha_2(0V) = 0$, $\alpha_2(\lambda V)$ is monotone-nondecreasing in λ on $[0, \infty)$.

THEOREM 1.4. *Let $V \in \mathscr{V}_2$. Then $\beta_\infty < \infty$ if and only if V is subcritical.*

We prove Theorem 1.2 in Section 3 after some preliminaries of some independent interest involving the relation between β_∞ and the solution η of $H\eta = 0$ with $\eta \in L^\infty$.

In Section 4, we examine the following problem raised by Sigal and Ouchinnokov [12]: Suppose that V and W are two subcritical potentials; prove that $-\frac{1}{2}\Delta + V + W(\cdot - R)$ is subcritical for R large enough. For spherical symmetric V, W, this was proven by Sigal in his beautiful analysis of the Effimov effect [12]. An "elementary" proof of this fact "in general" was found by Klaus and Simon [8], who also showed that if V and W are both critical and $\nu = 3$, then

$$\alpha_2(V + W(\cdot - R)) \sim c/2R^2, \tag{1.12}$$

where $c = d^2$ and d is the unique solution of $x = e^{-x}$. Our goal in Section 4 is to prove the result for V, W subcritical where an attractive intuition is the following: when the wells V and W are far apart, the fact that in three or more dimensions hitting probabilities of a distant sphere go to zero should take over. The key to this argument is to relate subcriticality to Brownian motion and this is done by Theorem 1.4 since as we shall see

$$\beta_\infty = \sup_{x,t} E\left(\exp \int_0^t -V(x + b(s))\, ds\right). \tag{1.13}$$

Indeed, it was our interest in Sigal's problem that motivated the considerations in Sections 2 and 3 initially!

2. α_p Is Independent of p

Our goal here is to prove Theorem 1.3. Our method of proof is motivated, in part, by ideas in Carmona [4]. As indicated in Section 1, we prove (1.9), from which the theorem follows. By Theorem 1.1, we need only prove (1.9) for all sufficiently large t. As a preliminary, we note that since H is self-adjoint on L^2,

$$\| e^{-tH} f \|_2 \leqslant e^{\alpha_2 t} \| f \|_2 .$$

Thus by (1.3) for $t \geqslant 1$,

$$\begin{aligned} \| e^{-tH} f \|_\infty &\leqslant e^{\alpha_2 (t-1)}\, C(\infty, 2, 1;\, V) \| f \|_2 \\ &= D e^{\alpha_2 t} \| f \|_2 . \end{aligned}$$

In particular, let f be the characteristic function of *any* ball of radius R. Then

$$\| e^{-tH} f \|_\infty \leqslant D' R^{\nu/2} e^{\alpha_2 t}.$$

In particular, using (1.5),

$$E\left(\exp\left(-\int_0^t V(x + b((s))\, ds\right),\, | b(t) | \leqslant R\right) \leqslant D' R^{\nu/2} e^{\alpha_2 t}, \tag{2.1}$$

where $E(f, B) = \int_B f\, Db$.

By the Schwarz inequality

$$\begin{aligned} A &\equiv E\left(\exp\left(-\int_0^t V(x + b((s))\, ds\right),\, | b(t) | > R\right) \\ &\leqslant \| \exp[-t(-\tfrac{1}{2}\Delta + 2V)]1 \|_\infty^{1/2} E(| b(t) | > R)^{1/2} \\ &\leqslant \exp(\tfrac{1}{2} t \alpha_\infty(2V))\, f(R^2/t) \end{aligned}$$

by the scaling property Brownian motion. As $x \to \infty$, $f(x) \leqslant \exp(-cx)$, so choosing $R = at$ with a large enough, we can be sure that $A \to 0$ at $t \to \infty$. Thus, (2.1) implies that

$$\sup_x E\left(\exp\left(-\int_0^t V(x+b)(s)\right) ds)\right) \leqslant D'' t^{\nu/2} e^{\alpha_2 t}. \tag{2.2}$$

The proof is completed if we note that the left side of (2.2) is $\| e^{-tH} 1 \|_\infty$ since $(e^{-tH}1) \geqslant 0$ and that

$$\| e^{-tH} \|_{\infty,\infty} = \| e^{-tH} 1 \|_\infty . \tag{2.3}$$

(2.3), which we will use again, follows from the inequality $|f(x)| \leqslant \|f\|_\infty 1$ which implies $| e^{-tH} f | \leqslant \|f\|_\infty (e^{-tH}1)$ since e^{-tH} is positivity preserving. ∎

3. β_∞, THE GROUND STATE AND SUBCRITICALITY

Throughout this section we suppose that $\nu \geqslant 3$ and $V \in \mathscr{V}_2$. We use $H\eta = 0$, $\eta \in L^\infty$ as shorthand for

$$e^{-tH}\eta = \eta, \text{ all } t; \qquad \eta \in L^\infty. \tag{3.1}$$

Given any real-valued function $\eta \in L^\infty$ we define

$$\eta_+ = \operatorname{ess\,sup}_x \eta(x), \qquad \eta_- = \operatorname{ess\,inf}_x \eta(x), \qquad \eta_\infty = \operatorname{ess\,lim}_{x\to\infty} \eta(x),$$

if the latter limit exists.

PROPOSITION 3.1. *If η obeys* (3.1), *then η is a continuous function, η_∞ exists, and*

$$\eta(x) = \eta_\infty - \int c_\nu \,| x - y |^{-(\nu-2)}\, V(y)\, \eta(y)\, dy, \tag{3.2}$$

where c_ν is defined so that $c_\nu \,| x - y |^{-(\nu-2)}$ is the integral kernel of $(-\Delta)^{-1}$. Conversely, any $\eta \in L^\infty$ obeying (3.2) *obeys* (3.1).

Proof. Let $h = -(-\Delta)^{-1}(V\eta)$. By the Hölder and Young inequalities, $h \in L^\infty$. Moreover, both h and η obey $\Delta f = V\eta$ in the distributional sense, so $h - \eta$ is a function in L^∞ which is harmonic in the distributional sense. Thus $h - \eta$ is a constant $-\eta_\infty$, so (3.2) holds. But h is a continuous function going to zero at infinity, so the stated results on η hold. To prove the converse, note that if (3.2) holds, then $(H\phi, \eta) = 0$ for all $\phi \in C_0^\infty$, from which (3.2) follows.

LEMMA 3.2. *If η obeys* (3.1) *and η is nonnegative, then η* (*as a continuous function*) *is everywhere strictly positive.*

Proof. See Carmona [3] and Simon [13]. These authors discuss the case $\eta \in L^2$, but the proof is really only "local."

LEMMA 3.3. *Let η obey* (3.1) *with $\eta_\infty = 0$. Then $\phi = |V|^{1/2}\eta \in L^2$ and*

$$[|V|^{1/2}(-\Delta)^{-1}(\operatorname{sgn} V)|V|^{1/2}]\phi = -\phi. \tag{3.3}$$

Let $V_\pm = \max(\pm V, 0)$. Then $\psi = V_-^{1/2}\eta \in L^2$ and

$$[V_-^{1/2}(-\Delta + V_+)^{-1} V_-^{1/2}]\psi = \psi. \tag{3.4}$$

Proof. (3.3) is just a rewriting of the form that (3.2) takes when $\eta_\infty = 0$. Since $|V|^{1/2} \in L^{\nu+\epsilon} \cap L^{\nu-\epsilon}$, $\phi \in L^{\nu-\epsilon}$ and by the Young and Hölder inequality the operator $Q = |V|^{1/2}(-\Delta)^{-1}|V|^{1/2}$ is bounded from L^p to L^q for all $p \in [\nu/\nu - 1, \nu]$ and all $q \in [p - \delta, p + \delta]$ for suitably small δ (depending on ϵ). Thus, repeatedly using (3.3), we find that $\phi \in L^2$.

Since $|V_-| \leqslant |V|$, $\psi \in L^2$. Moreover, since $|e^{-t(-\Delta+V_+)}f| \leqslant e^{-t\Delta}|f|$, $(-\Delta + V_+)^{-1}$ has an integral kernel pointwise dominated by that of $(-\Delta)^{-1}$, $K = V_-^{1/2}(-\Delta + V_+)^{-1} V_-^{1/2}$ is a bounded operator from L^p to L^p, for $p \in [\nu/\nu - 1, \nu]$. As operators from L^p to suitable L^q:

$$(-\Delta)^{-1} = (-\Delta + V_+)^{-1} + (-\Delta + V_+)^{-1} V_+(-\Delta)^{-1}. \tag{3.5}$$

Thus, using (3.2) twice and (3.5)

$$\begin{aligned}
\eta &= -(\Delta)^{-1} V\eta, \\
&= -(-\Delta + V_+)^{-1} V\eta - (-\Delta + V_+)^{-1} V_+((-\Delta)^{-1} V\eta, \\
&= -(-\Delta + V_+)^{-1}(V - V_+)\eta, \\
&= (-\Delta + V_+)^{-1} V_-\eta.
\end{aligned}$$

Multiplying by $V_-^{1/2}$, (3.4) results. ∎

THEOREM 3.4. *There is always an $\eta \in L^\infty$ obeying* (3.1). *If there is a nonnegative η obeying* (3.1), *then all functions obeying* (3.1) *are multiples of this nonnegative function.*

Proof. Let A be the operator $(-\Delta)^{-1} V$ *on* L^∞. Since $|(Af)(x)| \leqslant [((-\Delta)^{-1}|V|)(x)]\,\|f\|_\infty$ and $|(Af)(x) - (Af)(y)| \leqslant c_\nu \int ||x - z|^{-(\nu-2)} - |y - z|^{-(\nu-2)}|\,|V(z)|\,dz$, the set $\{Af \mid \|f\|_\infty \leqslant 1\}$ is a family of uniformly equicontinuous functions, uniformly bounded and going uniformly to zero at infinity, and thus A is compact. We seek a solution of

$$\eta = \eta_\infty - A\eta. \tag{3.2$'$}$$

If $-1 \notin \sigma_\infty(A)$, then we let g be the function, $g = 1$ and $\eta = (1 + A)^{-1} g$. Then η obeys (3.2′) with $\eta_\infty = 1$. If $-1 \in \sigma_\infty(A)$, then by the Fredholm theory, the homogeneous equation (3.2′) with $\eta_\infty = 0$ has a solution η which thus also solves (3.2′). This implies the existence statement.

To prove uniqueness, suppose that η obeys (3.1) and is strictly positive. Suppose first that $\eta_\infty > 0$. For any other nonzero η' obeying (3.1), with $(\eta')_\infty = 0$, suppose that $g = \max(0, \eta')$ is not identically zero (by replacing η' by $-\eta'$, this is no loss). Since $(\eta')_\infty = 0$, the extended real-valued function η/g goes to infinity at infinity so, if $\lambda = \min_x (\eta(x)/g(x))$, then $\eta(x_0)/g(x_0) = \lambda$ for some finite χ_0. Let $\tilde{\eta} = \eta - \lambda\eta'$. Then $\tilde{\eta} \geqslant 0$ but $\tilde{\eta}(x_0) = 0$ and $(\tilde{\eta})_\infty \neq 0$. By Lemma 3.2, this is impossible.

If η' obeys (3.1), then $\tilde{\eta} = \eta' - \eta'_\infty \eta_\infty^{-1}$ will obey (3.1) with $(\tilde{\eta})_\infty = 0$. By the the above, $\tilde{\eta} = 0$, i.e., η' is a multiple of η. Thus we have uniqueness when the nonnegative solution of (3.1) has $\eta_\infty \neq 0$.

Now suppose η is a nonnegative solution with $\eta_\infty = 0$. Let η' be another solution. If $\eta'_\infty \neq 0$, we can suppose that $\eta'_\infty > 0$. It is then easy to see that for c large $\eta'_\infty + c\eta_\infty$ is strictly positive so we are back in the case already treated. We are thus reduced to the case $\eta'_\infty = 0$, $\eta_\infty = 0$. In that case, by Lemma 3.3, $\psi = V_-^{1/2}\eta$ and $\psi' = V_-^{1/2}\eta'$ both are in L^2 and obey $B\psi = \psi$, $B\psi' = \psi'$ with $B = V_-^{1/2}(-\Delta + V_+)^{-1} V_-^{1/2}$. Since B is a self-adjoint operator on L^2 (ess sup V_-) with strictly positive integral kernel, and $\psi \geqslant 0$, ψ' must be a multiple of ψ by the general theory of Perron–Frobenius (see, e.g., [10]). ∎

With these preliminaries, we can now turn to the connection between η and β_∞.

Proposition 3.5. *If* (3.1) *has a solution with* $\eta_- > 0$ (*so in particular*, η *is nonnegative*), *then* $\beta_\infty < \infty$ *and*

$$\beta_\infty \leqslant \eta_+/\eta_- . \tag{3.6}$$

Proof. Note that $1 \leqslant \eta_-^{-1}\eta$, so by (2.3) and the positivity of e^{-tH}:

$$\beta_\infty = \sup_t \| e^{-tH} 1 \|_\infty \leqslant \sup_t \eta_-^{-1} \| e^{-tH} \eta \|_\infty$$

$$= \eta_-^{-1} \| \eta \|_\infty = \eta_+\eta_-^{-1}. \quad ∎$$

Proposition 3.7. *If* $\beta_\infty < \infty$, *then* (3.1) *has a solution*, η, *with* $\eta_- > 0$ *and*

$$\eta_+/\eta_\infty \leqslant \beta_\infty . \tag{3.7}$$

Proof. Write $V = V_+ - V_-$ with $V_\pm = \max(\pm V, 0)$. Note first that

$$E\left(\int_0^\infty V_+(x + b(s))\, ds\right) = 2[(-\Delta)^{-1}V_+](x) \tag{3.8}$$

since $E(\int_0^\infty V_+(x + b(s))\, ds) = \int_0^\infty (e^{+(1/2)\Delta \cdot} V_+)(x)\, ds$. Thus,

$$\alpha \equiv \sup_x \left[E\left(\int_0^\infty V_+(x + b(s))\, ds \right) \right] < \infty$$

by Young's inequality and $V \in \mathscr{V}_1$. It follows by Jensen's inequality that

$$f(x) \equiv E\left(\exp\left(-\int_0^\infty V_+(x + b(s))\, ds \right) \right)$$

obeys

$$e^{-\alpha} \leqslant f(x) < 1. \tag{3.9}$$

We take

$$\eta(x) = \lim_{t \to \infty} (e^{-tH} f)(x). \tag{3.10}$$

To see that the limit exists, we note that the right side of (3.10) is just

$$E\left(\exp\left(-\int_0^t V(x + b(s))\, ds - \int_t^\infty V_+(x + b(s))\, ds \right) \right),$$

which is monotone-increasing in t. Since

$$(e^{-tH} f)(x) \leqslant \beta_\infty \| f \|_\infty = \beta_\infty$$

the limit in (3.10) exists and

$$\eta_+ \leqslant \beta_\infty . \tag{3.11}$$

Clearly, by the monotonicity of $(e^{-tH} f)(x)$ in t, $\eta(x) \geqslant f(x)$ so

$$\eta_- \geqslant e^{-\alpha}.$$

Next, we note that by the monotonicity,

$$(g, \eta) = \lim_{t \to \infty} (g, e^{-tH} f)$$

for any $g \in L^1$. It follows that $(e^{-sH} g, \eta) = (g, \eta)$ for any s, so by duality η obeys (3.1).

To complete the proof, we need only show that

$$\eta_\infty \geqslant 1 \tag{3.12}$$

so that (3.11) implies (3.7). But $\eta(x) \geqslant f(x)$ and, by Jensen's inequality and (3.8)

$$f(x) \geqslant \exp(-2[(-\Delta)^{-1} V_+](x)). \tag{3.13}$$

The right side of (3.13) goes to one at $x \to \infty$. ∎

Remark. Actually, in the above, $\eta_\infty = 1$. To see this, let $Q(x, b) = \int_0^\infty V(x + b)(s)\, ds$. Then $\eta(x) = E(e^{Q(x,\cdot)})$ and $Q(x, \cdot)$ obeys: (i) $\sup_x \times E(e^{(1+\epsilon)Q(x,\cdot)}) < \infty$ for some $\epsilon > 0$; (ii) $\sup_x E(e^{\delta|Q(x,\cdot)|}) < \infty$ for some $\delta > 0$; (iii) $E(|Q(x, \cdot)|) \to 0$ as $x \to \infty$. (ii) and (iii) imply that (iv) $E(|Q(x, \cdot)|^q) \to 0$ for any $q < \infty$. Now use the inequality $|e^y - 1| \leqslant y(e^y + 1)$ and Hölder's inequality to see that $E((e^Q - 1)) \leqslant E((e^Q + 1)^{1+\epsilon})\, E(|Q|^{1+\epsilon^{-1}})$. (i) and (iv) show that $\eta(x) \to 1$ at infinity.

We symmarize the last two results in a theorem:

THEOREM 3.8. *There exists an* η *solving* (3.1) *with* $\eta_- > 0$ *if and only if* $\beta_\infty < \infty$ *and*

$$\eta_+/\eta_\infty \leqslant \beta_\infty \leqslant \eta_+/\eta_- . \tag{3.14}$$

If $V \leqslant 0$, *then* $\eta_\infty = \eta_-$ *and equality holds in* (3.14).

Proof. All that remains is the equality when $V \leqslant 0$. In that case our above proof shows that for the η constructed there, $1 \leqslant \eta_- \leqslant \eta_\infty = 1$, where we use the remark above. ∎

To complete the proof of Theorem 1.4, we need to examine the connection between η's obeying (3.1) and criticality. Recall that $\sigma_p(A) =$ spectrum of A as an operator on L^p. $\sigma_p' = \sigma_p \backslash \{0\}$.

LEMMA 3.9. $1 \in \sigma_\infty((-\Delta + V_+)^{-1} V_-)$ *if and only if* $-1 \in \sigma_\infty((-\Delta)^{-1} V)$.

Proof. One direction is the argument at the end of Lemma 3.3 and the other direction follows by a similar argument reversing the roles of $-\Delta$ and $-\Delta + V_+$. ∎

LEMMA 3.10.(compare with [5]). $\sigma_2'((\text{sgn } V) \mid V \mid^{1/2}(-\Delta)^{-1} \mid V \mid^{1/2}) = \sigma_\infty'((-\Delta)^{-1} V)$ *and* $\sigma_2'(V^{1/2}(-\Delta + V_+)^{-1} V^{1/2}) = \sigma_\infty'((-\Delta + V_+)^{-1} V_-)$.

Proof. The two cases are virtually identical, so consider the first. Note that $A = (\text{sgn } V) \mid V \mid^{1/2}(-\Delta)^{-1} \mid V \mid^{1/2}$ is compact on L^2 and $B = (-\Delta)^{-1} V$ is compact on L^∞. If $B\eta = \lambda\eta$ with $\eta \in L^\infty$, then the argument in Lemma 3.3 shows that $(\text{sgn } V) \mid V \mid^{1/2} \eta = \phi \in L^2$ and $A\phi = \lambda\phi$. Conversely, let $A\phi = \lambda\phi$. Using the Hölder and Young inequalities, we saw in Lemma 3.2 that A maps L^p to L^q for $p \in [\nu/\nu^{-1}, \nu]$ and $q \in [p - \delta, p + \delta]$. Thus $\phi \in L^2 \cap L^\nu$ so that $\eta \equiv (-\Delta)^{-1} \mid V \mid^{1/2} \phi$ lies in L^∞. Clearly $B\eta = \lambda\eta$. ∎

Proof of Theorem 1.4. By Lemma 3.10, $\sigma_\infty((-\Delta + \lambda V_+)^{-1} \lambda V_-) \subset [0, \infty)$. Thus $1 \in \sigma_\infty((-\Delta + \lambda V_+)^{-1} \lambda V_-)$ for some $\lambda \leqslant 1$ if $L^\infty - \text{spr}((-\Delta + V_+)^{-1} V_-) \geqslant 1$. It follows using Lemmas 3.9 and 3.10 that if V is subcritical, then $L^\infty - \text{spr}((-\Delta + V_+)^{-1} V_-) < 1$ so that

$$\eta = (1 - (-\Delta + V_+)^{-1} V_-)^{-1} 1$$

is given by a Neumann series, each term of which is positive. Thus $\eta \geqslant 1$, so (3.1) has a solution with $\eta_- > 0$ and so $\beta_\infty < \infty$. If V is critical, then, by Lemmas 3.9 and 3.10, (3.1) has a solution $\tilde{\eta}$ with $\tilde{\eta}_\infty = 0$. By the uniqueness result, Theorem 3.4, there cannot also be a solution with $\eta_- > 0$; hence, $\beta_\infty = \infty$. If V is supercritical, then $\alpha_\infty = \alpha_2 > 0$ so β_∞ is surely infinite. ∎

4. Localization of Binding

Throughout this section we fix two potentials $V, W \in \mathscr{V}_1$ *both subcritical* and both supported in a sphere of radius r. Fix a unit vector $\hat{e}$ and let

$$H(R) = -\tfrac{1}{2}\Delta + V(x) + W(x - R\hat{e}).$$

We show that H is subcritical for R sufficiently large. We begin with consideration of the case $V, W \leqslant 0$, which is somewhat simpler:

Theorem 4.1. *Let V, W be as above and let $b = \max(\beta_\infty(V), \beta_\infty(W))$. Then for $(4r/R)\, b < 1$, $V + W(\cdot - R\hat{e}) \equiv q_R$ is subcritical and*

$$\beta_\infty(V + W(\cdot - R\hat{e})) \leqslant b(1 - 4R^{-1}r\, b)^{-1}.$$

Proof. Fix x_0 and R. Let $T = \{x \mid |x| = r \text{ or } |x - R\hat{e}| = r\}$ and let $S = \{x \mid |x| = R/2 \text{ or } |x - R\hat{e}| = R/2\}$. Define stopping times $\sigma_1, \tau_1, \sigma_2, \ldots$, inductively by:

$$\sigma_1 = \inf\{t \mid b(t) + x_0 \in S\}$$
$$\tau_i = \inf\{t > \sigma_i \mid b(t) + x_0 \in T\}$$
$$\sigma_i = \inf\{t > \tau_{i-1} \mid b(t) + x_0 \in S\}$$

with the convention that inf (empty set) $= +\infty$. Let

$$A_i = \{b \mid \tau_i(b) = \infty, \tau_{i-1}(b) < \infty\}.$$

Then

$$\bigcup_i A_i = \text{all paths} \tag{4.1}$$

since any path eventually leaves the space bounded by S and is Hölder continuous up to that time. If $b \in A_i$, then for $t > \sigma_i$, $V(x + b(s)) = W(x + b(s)) = 0$ so for any T,

$$-\int_0^T q(x + b(s)) \leqslant -\int_0^\infty q(x + b(s)) = \int_0^{\sigma_i} -q(x + b(s)).$$

Thus,

$$E\left(\chi_{A_i} \exp\left(-\int_0^T q(x+b(s))\right)\right) \leqslant E\left(\exp\left(-\int_0^{\sigma_i} q(x+b(s))\right) \chi_{\sigma_i<\infty}\right)$$

$$= E\left(\exp\left(-\int_0^{\tau_{i-1}} q(x+b)(s)\right) \chi_{\tau_i<\infty} E\left(\exp\left(-\int_{\tau_{i-1}}^{\sigma_i} q(x+b)(s)\right)\middle| b(s);\, s \leqslant \tau_{i-1}\right)\right)$$

$$\leqslant bE\left(\exp\left(-\int_0^{\tau_i} q\right)\chi_{\tau_i<\infty}\right) \leqslant b\left(\frac{4r}{R}\right) E(\cdots),$$

where we use the definition of β_∞ and the Dynkin–Hunt theorem (Brownian motion starts afresh at a stopping time) to get the b factor and then $E(\chi_{\tau_i<\infty} \mid b(s);\, s < \sigma_i) \leqslant 2$ (hitting probability for sphere of radius r at a distance R) $= 4r/R$. Thus, $E(\chi_{A_i} e^{-\int_0^T q}) \leqslant b^{i+1}(4r/R)^i$. (4.1) yields the theorem. ∎

Next, we consider the case where V can take both signs. This is more difficult since we cannot use $e^{-\int_0^T q} \leqslant e^{-\int_0^\infty q}$ and we are indebted to M. Aizenman for several useful suggestions.

THEOREM 4.2. *Let V, W be as in Theorem* 4.1 *but no longer required to be negative. Let* $\eta^{(V)}$, $\eta^{(W)}$ be the corresponding η's and let $b = \max(\eta_+^{(V)}/\eta_-^{(V)};\, \eta_+^{(W)}/\eta_-^{(W)})$. Then $-\Delta + q$ is subcritical for $4br/R < 1$,

$$\beta_\infty(q) \leqslant b(1 - 4br/R)^{-1}.$$

Proof. The proof follows the pattern above except that we must estimate

$$\sup_x E\left(\exp - \int_{\tau_{i-1}}^{\min(T,\sigma_i)} q(x+b(s))\, ds \mid b(s);\, s \leqslant \tau_{i-1}\right) \leqslant \max(\alpha(V), \alpha(W))$$

with

$$\alpha(V) = \sup_{|x|=r} E\left(\exp\left(-\int_0^\tau V(x+b)(s)\, ds\right)\right)$$

by the Dynkin–Hunt theorem. Here τ is a suitable stopping time

$$E\left(\exp\left(-\int_0^\tau V\right)\right)$$

$$\leqslant [\eta_-^{(V)}]^{-1} E\left(\exp\left(-\int_0^\tau V\right) \eta(x+b(\tau))\right)$$

$$= \eta_-^{(V)-1} E\left(\exp\left(-\int_0^\tau V(x+b(s))\right) E\left(\exp\left(-\int_\tau^\infty V(x+b(s))\, ds\right)\middle| b(s);\, s \leqslant \tau\right)\right)$$

$$= [\eta_-^{(V)}]^{-1}\, \eta(x) \leqslant b,$$

where we use the Dynkin–Hunt to rewrite $\eta(x) = E(\exp(-\int_0^\infty V(x + b(s))\, ds))$ as the stated conditional expectation. ∎

5. Some Extensions, Remarks, and Conjectures

In this final section we want to discuss a number of directions for further research on the questions treated herein.

A. *p-independence of* σ_p

We emphasize the question (1.10) in Section 1. For $V \in \mathscr{V}_2$ we have

THEOREM 5.1. *If* $V \in \mathscr{V}_2$, *then* $\sigma_p(-\frac{1}{2}\Delta + V)$ *is independent of p.*

Proof. Write $H = H_0 + V$, $H_0 = -\frac{1}{2}\Delta$. Let $Cf = \bar{f}$. Then $C[(H - z)^{-1}]^* C = (H - z)^{-1}$, so by duality and interpolation for $2 \leqslant p \leqslant \infty$, $\sigma_2(H) \subset \sigma_p(H) \subset \sigma_\infty(H)$, so it suffices to prove that $\sigma_2(H) = \sigma_\infty(H)$. Since $\sigma_2(H_0) = \sigma_\infty(H_0)$ and $V^{1/2}(H_0 - z)^{-1} | V |^{1/2}$ is L^2-compact and $(H_0 - z)^{-1} V$ is L^∞-compact, the essential spectra are equal. As in the proof of Lemma 3.10, $\sigma_2'(| V |^{1/2} \times (H_0 - z)^{-1} V^{1/2}) = \sigma_\infty'(H_0 - z)^{-1} V)$, so using the facts that when $z \notin [0, \infty)$, $z \in \sigma_\infty(H)$ if and only if $-1 \in \sigma_\infty(H_0 - z)^{-1} V$ and $z \in \sigma_2(H)$ if and only if $-1 \in \sigma_2(| V |^{1/2}(H_0 - z)^{-1} V^{1/2})$, we see that the result is proven. ∎

Remarks. 1. The above proof extends to $V \in L^{\nu/2+\epsilon} + (L^\infty)_\epsilon$.

2. The basic point is that L^∞ eigenvalues for $z \notin [0, \infty)$ automatically fall off exponentially so that they lie in L^2. Moreover, by Theorem 1.1, L^2 eigenfunctions are automatically in L^∞.

B. *Existence of* L^∞ *Positive Eigenfunctions*

There is some interest in finding $\eta > 0$ so that $H\eta = E\eta$ since $H - E$ is then automatically a Dirichlet form on $\eta^2\, dx$ [1, 5, 6]. If one combines Theorems 3.4 and 3.8 and the fact that a supercritical V always has a strictly positive L^2 ground state [10], then

THEOREM 5.2. *Let* $V \in \mathscr{V}_2$. *Then, there exists an E and an* η *in* L^∞ *with* $(H - E)\eta = 0$ *and* $\eta > 0$.

Question. Does this result extend to $\mathscr{V}_1$ and are the E and η then unique? For $V \in \mathscr{V}_1$, can one at least find E, η with $(H - E)\eta = 0$ and $\eta > 0$ without requiring that $\eta \in L^\infty$?

C. *Extending Theorem* 1.4 *to supercritical potentials, I*

One way of extending Theorem 1.4 to supercritical potentials is to ask about $\sup_t \| \exp[-t(H - \alpha_p(V))] \|_{p,p}$. Here we note that

THEOREM 5.3. *Let $V \in \mathscr{V}_2$ be supercritical. Then for $p \neq 1, \infty$*

$$\sup_t \| \exp[-t(H - \alpha_p(V))] \|_{p,p} < \infty.$$

Proof. Let P be the L^2 projection onto the lowest eigenvalue for H. Let $U_t = \exp[-t(H - \alpha_p(V))]$. Then

$$U_t P = P \tag{5.1}$$

$$\| U_t(1 - P) \|_{\infty,\infty} \leqslant c(1 + t)^{\nu/2} \tag{5.2}$$

$$\| U_t(1 - P) \|_{2,2} \leqslant (e^{-ta}) \tag{5.3}$$

where $a = \inf \sigma((H - \alpha_2(V)) \upharpoonright (1 - P) L^2) > 0$. The result now follows by interpolation. ∎

Remark. It should be easy to extend the result to $p = \infty$.

Question. What can be said for general $V \in \mathscr{V}_1$ about $\| \exp(-t \times [H - \alpha_p(V)]) \|_{p,p}$?

D. *Extending Theorem* 1.4 *to Supercritical Potentials, II*

For $V \in \mathscr{V}_2$, let Q be the L^2-projection onto the negative eigenvalues for H. A natural extension of Theorem 1.4 would be

Conjecture. $\sup_t \| e^{-tH}(1 - Q) \|_{\infty,\infty} < \infty$ if H has no zero energy eigenvalues or resonances.

The situation when there are zero eigenvalues is apt to be somewhat complicated since the corresponding eigenvector may or may not be L^1. One result we can mention (presumably of note only when $\nu = 3, 4$):

THEOREM 5.5. *If $V \in \mathscr{V}_2$ and if $H\eta = 0$ has no solution with $\eta \in L^2$ but has a solution with $\eta \in L^p$ for some $p < \infty$, then*

$$\sup_t \| e^{-tH}(1 - Q) \|_{\infty,\infty} = \infty.$$

Proof. We first note that as a map from L^p to L^p, $Q\eta = 0$ since $H\eta_i = E\eta_i$ with $E < 0$, implies that $\eta_i \in L^q$ (q = dual index to p) and $(\eta_i, \eta) = 0$, whence $Q\eta = 0$. Suppose $\sup_t \| e^{-tH}(1 - Q) \|_{\infty,\infty} < \infty$. Interpolating, with $\text{s-lim}_{t \to \infty} e^{-tH}(1 - Q) = 0$ in L^2, we see that $\text{s-lim}_{t \to \infty}(1 - Q) = 0$ in L^p, which is impossible since $e^{-tH}(1 - Q)\eta = \eta$. Thus, the sup is infinite. ∎

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Note added in proof. Results related to the problem mentioned following Theorem 5.2, i.e., non-L^∞ positive solutions of $(H - E)\eta = 0$ can be found in W. Most and J. Piepenbrink, *Pac. J. Math.* **75** (1978), 219–226 (and references therein).

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