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A Remark on Dobrushin's Uniqueness Theorem

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Ten years ago, Dobrushin [1] proved a beautiful result showing that under suitable hypotheses, a statistical mechanical lattice system interaction has a unique equilibrium state. In particular, there is no long range order, etc.; see [6, 7] for related material, Israel [4] for analyticity results and Gross [3] for falloff of correlations.

There does not appear to have been systematic attempts to obtain very good estimates on precisely when Dobrushin's hypotheses hold, except for certain spin $\frac{1}{2}$ models [6, 4]. Our purpose here is to note that with one simple device one can obtain extremely good estimates which are fairly close to optimal.

Let Ω be a fixed compact space (single spin configuration space), $d\mu_0$ a probability measure on Ω and for each $\alpha \in Z^\nu$, Ω_α a copy of Ω. For X a finite subset of $\mathbb{Z}^\nu$, let $\Omega^X = \mathop{\times}_{\alpha \in X} \Omega_\alpha$. An interaction is an assignment of a continuous function, $\Phi(X)$, on Ω^X to each finite $X \subset \mathbb{Z}^\nu$. While it is not necessary for Dobrushin's theorem, it is convient notationally to suppose Φ translation covariant in the obvious sense.

Let $\mathscr{E} = \mathop{\times}_{\alpha \neq 0} \Omega_\alpha$ be the set of "external fields" to $\alpha = 0$. Given $s \in \Omega_0$, $t \in \mathscr{E}$, Φ with $\sum_{0 \in X} \|\Phi(X)\|_\infty < \infty$, we define $H(s|t)$ on Ω_0 by

$$H(s|t) = \sum_{0 \in X} \Phi(X)(s, t)$$

and for any t, the probability measure $\nu_t = e^{-H(\cdot|t)} d\mu_0(\cdot)/\mathbb{Z}_t$ with $\mathbb{Z}_t = \int e^{-H(s|t)} d\mu_0(s)$. Let

$$\varrho_i = \sup\{\tfrac{1}{2}\|\nu_t - \nu_{t'}\| \mid t_k = t'_k \text{ for } k \neq i\}, \tag{1}$$

where the norm on measures is the total variation norm:

$$\|\nu\| = \sup\{|\nu(f)| \mid f \in C(\Omega);\ \|f\|_\infty = 1\}.$$

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Dobrushin's theorem says that if

$$\sum_{i \neq 0} \varrho_i < 1 \tag{2}$$

then there is a unique equilibrium state for Φ. Our main result here is:

Theorem. *If* $\sum_{0 \in X} (|X|-1) \|\Phi(X)\|_\infty < 1$, *then* (2) *holds.*

Remarks. 1. There are long range models (see [5]) where the sum is $1+\varepsilon$ and there are multiple states.

2. For purely pair interactions, if $a = \sum_{i \neq 0} \|\Phi(\{i,0\})\|$ our condition is $a<1$. By comparison Gross [3], who investigated when (2) holds, required (Corollary 4.2 of [3]) $4ae^{4a}<1$, i.e. $a<a_0 \sim 0.142$.

Lemma. *Let* $d\mu_0$ *be a probability measure on* Ω *and let* $d\mu_h = e^h d\mu_0 / \int e^h d\mu_0$ *for any* $h \in C(\Omega)$. *Then* $\|\mu_h - \mu_g\| \leqq \|h-g\|_\infty$.

Proof. Let $\nu_\theta = \mu_{\theta h + (1-\theta) g}$. Let $q = h-g$ and let $f \in C(\Omega)$ with $\|f\|_\infty = 1$. Then, with $\langle q \rangle_\theta = \nu_\theta(q)$:

$$|\mu_h(f) - \mu_g(f)|$$

$$\leqq \int_0^1 \nu_\theta([q - \langle q \rangle_\theta] f) d\theta$$

$$\leqq \int_0^1 \nu_\theta(|q - \langle q \rangle_\theta|) d\theta \tag{3}$$

$$\leqq \int_0^1 \nu_\theta(|q - \langle q \rangle_\theta|^2)^{1/2} d\theta \tag{4}$$

$$\leqq \int_0^1 [\nu_\theta(q^2)]^{1/2} d\theta \leqq \|q\|_\infty,$$

where we used $\frac{d}{d\theta} \nu_\theta(f) = \nu_\theta(fq) - \nu_\theta(f)\nu_\theta(q)$ *in the first step, then the Schwarz inequality and finally that* $\nu_\theta((q - \langle q \rangle_\theta)^2) = \nu_\theta(q^2) - [\nu_\theta(q)]^2 \leqq \nu_\theta(q^2)$.

Proof of the Theorem. Clearly, if $t_k = t'_k$ for $k \neq i$:

$$\|H(\bullet | t) - H(\bullet | t')\|_\infty \leqq \sum_{\{0,i\} \subset X} \|\Phi(X)(\bullet, t) - \Phi(X)(\bullet, t')\|_\infty \tag{5}$$

$$\leqq 2 \sum_{\{0,i\} \subset X} \|\Phi(X)\|_\infty . \tag{6}$$

Thus, by the lemma

$$\sum_{i \neq 0} \varrho_i \leqq \sum_i \sum_{\{0,i\} \subset X} \|\Phi(X)\|_\infty = \sum_{0 \in X} (|X|-1) \|\Phi(X)\|_\infty . \quad \square$$

One can often do better by looking at the guts of the above proof. Let me give some examples in a number of remarks:

1. In going from (5) to (6) we can clearly replace $\|\Phi(X)\|_\infty$ by $\frac{1}{2}[\max(\Phi(X)) - \min(\Phi(X))]$ and thus we can also make this replacement in the theorem.

2. Since $\langle |q-\langle q\rangle|^2\rangle \leqq \langle |q-\alpha|^2\rangle$ for any constant α, we have that $\|\mu_h-\mu_g\| \leqq \|h-g-\alpha\|_\infty$ for any constant α.

3. By using (2), we can recover Lanford's proof [6] that for $\Omega=\{0,1\}$, $\Phi(X) = A_X\varrho^X\left(\varrho^X = \underset{\alpha\in X}{\pi}\varrho_\alpha;\ \varrho_\alpha=0 \text{ or } 1 \text{ on } \{0,1\}\right)$, (2) holds if $\sum_{0\in X}|A_x|(|X|-1)<4$, For in that case, if $t_i=1$, $t'_i=0$:

$$H(\bullet|t)-H(\bullet|t')=\sum_{\{0,i\}\subset X}\Phi(X)(\bullet|t)$$

so that

$$\|H(\bullet|t)-H(\bullet|t')-\tfrac{1}{2}\sum_{\{0,i\}\subset X}A_X\|\leqq\tfrac{1}{2}\sum_{\{0,i\}\subset X}|A_X|.$$

We have thus picked up two factors of 2.

4. If $\Omega=\{-1,1\}$, and $\omega_a(\pm 1)=e^{\pm a}/2\cosh a$, then by a direct computation

$$\|\omega_a-\omega_b\|=|\tanh b-\tanh a|\leqq 2\tanh\tfrac{1}{2}|b-a|.$$

If $\Phi(X)=-J_X\underset{\alpha\in X}{\pi}\sigma_\alpha$, then $|v_t-v_{t'}|\leqq 2\tanh\frac{1}{2}\left[2\sum_{\{0,i\}\subset X}|J_X|\right]\leqq\sum_{\{0,i\}\subset X}\tanh|J_X|$. This shows that if $\sum_{0\in X}(|X|-1)\tanh(|J_X|)<1$, there is no multiple phase and if $J_X=0$ for $|X|$ odd and $\mu_0(\pm 1)=\frac{1}{2}$; no spontaneous magnetization. (This is also noted by Israel [4]). This improves results of Griffith's [2] who considered only pair interactions and $J_X\geqq 0$, i.e. *Griffith's result follows from Dobrushin's theorem.*

5. Let $\Omega=[-1,1]$, $S^X=\underset{\alpha\in X}{\pi}S_\alpha$ and $\Phi(X)=-J_XS^X$. Let $d\mu_0=dx$ and $\omega_a=e^{ax}dx/\text{Normalization}$. Then $\omega_a((S-\langle S\rangle)^2)$ takes its maximum at $a=0$ by the GHS inequality so, by (4), $\|v_a-v_b\|_\infty\leqq\sqrt{1/3}\,|a-b|$. Thus, the 1 in $\Sigma\ (|X|-1)\ \|\Phi(X)\|_\infty$ can be replaced by $\sqrt{3}=1.73$ compared with the $\pi/2=1.57$ obtained by Israel [4] with different methods. If one can show $\omega_a(|s-\langle s\rangle|)$ has its maximum at $a=0$, $\sqrt{3}$ can be replaced by 2 using (3).

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