

The Classical Limit of Quantum Partition Functions

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Abstract. We extend Lieb’s limit theorem [which asserts that $SO(3)$ quantum spins approach S^2 classical spins as $L \rightarrow \infty$] to general compact Lie groups. We also discuss the classical limit for various continuum systems. To control the compact group case, we discuss coherent states built up from a maximal weight vector in an irreducible representation and we prove that every bounded operator is an integral of projections onto coherent vectors (i.e. every operator has “diagonal form”).

1. Introduction

This paper is motivated in the first place by a beautiful paper of Lieb [23] who considers the following situation. Let A be a finite set and let $H(S_\alpha)$ be a function of $R^{3|A|}$ variables $\{S_{\alpha,i}\}$, $\alpha \in A$, $i = 1, 2, 3$ which is multiaffine, i.e. a sum of monomials which are of degree zero or one in the variables at each site. Define

$$Z_{c\ell}(\gamma) = \int \prod_{\alpha \in A} [d\Omega(S_\alpha)/4\pi] \exp(-H(\gamma S_\alpha)) \tag{1.1}$$

where $d\Omega$ is the usual (unnormalized) measure on the unit sphere, S^2 , in R^3 . For each $\ell = 1/2, 1, 3/2, \dots$, let

$$Z_Q^\ell(\gamma) = (2\ell + 1)^{-|A|} \text{Tr}(\exp[-H(\gamma L_\alpha/\ell)]) \tag{1.2}$$

where $\{L_\alpha\}$ is a family of independent spin ℓ quantum spins, i.e. L_α acts on $(C^{2\ell+1})^{|A|}$ thought of as a tensor product with $L_\alpha = 1 \otimes \dots \otimes L \otimes \dots \otimes 1$ (not 1 only in the α th factor) and \tilde{L} the usual vector of angular momentum ℓ . Then Lieb [23] proves:

$$Z_{c\ell}(\gamma) \leq Z_Q^\ell(\gamma) \leq Z_{c\ell}(\gamma + \ell^{-1}\gamma). \tag{1.3}$$

This demonstrates convergence of Z_Q to $Z_{c\ell}$ as $\ell \rightarrow \infty$ in a sufficiently strong way that one can interchange the $\ell \rightarrow \infty$ and the $|A| \rightarrow \infty$ limit in the free energy per unit volume.

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Lieb's proof depends on developing some results on Bloch coherent vectors and on two inequalities which he proves about projections onto such vectors. Independently of Lieb and at about the same time Berezin [3, 4] proved some abstract inequalities about projections onto overdetermined vectors which include Lieb's inequalities as a special case. Conversely, Lieb's methods can be used to prove the abstract Berezin inequalities. These "Berezin-Lieb" inequalities are the basis of much of what we do; we discuss them in Sect. 2. Motivated by Lieb's paper, various other problems have been discussed. [5, 15, 40].

Our goal in this paper is to extend Lieb's analysis from the sequence of representations of $SO(3)$ to more general compact Lie groups. This is clearly a natural mathematical question but our interest comes from some work of Dunlop-Newman [8]. They prove a Lee-Yang theorem for S^2 -spins by using Lieb's result and Asano's theorem [2] on a Lee-Yang theorem for spin 1/2 quantum spins and its extension to spin ℓ spins by Griffith's trick [14]. While this is a rather indirect way of proving a Lee-Yang theorem for S^2 -spins, it is the only way we know! In order to extend the Dunlop-Newman result from S^2 to S^N (and thereby to $(\phi^4)_{N+1}$ field theories [8]), one must try to generalize Lieb's result and this is our reason for interest in the general problem. We succeed up to the point of reducing Lee-Yang for S^N spins to an analog of Asano's result for certain spinors; see Sect. 7 below.

The preceding discussion focuses attention on discovering what replaces S^2 as the classical limit space. There has been one other example computed; namely Fuller and Lenard [9] consider the sequence of spherical harmonic representations of $O(n)$ and discover, by ad hoc means, that the limit space is $G(n, 2)$, the Grassman manifold of oriented two planes in R^n . Our goal is to give a general procedure that computes the classical limit manifold in general. It will turn out that this involves isolating the "proper" set of coherent vectors on Lie groups. Such sets have been considered by Klauder [18], Perelomov [37], and Gilmore [10, 11]. The first two authors take general families of coherent vectors; only the last author emphasizes the virtue of taking coherent vectors based on maximal weight vectors; as we shall see, only these seem to be suitable for controlling the classical limit. Gilmore notes that his coherent states are parametrized by homogeneous spaces but when he discusses the classical limit [12], these spaces, which are the limit manifold, get lost!

The classical limit manifolds turn out to be coadjoint orbits; i.e. orbits under the natural action of the basic Lie group on the dual of its Lie algebra. We describe these things in more detail in Sect. 5 and give several examples in Sect. 6.

For the time being, we note that in general Lie groups there are distinct families of orbits and this is because there will be different classical limit spaces depending on which family of representations is used. For example, for $SO(4)$, the limit for the sequence of spherical harmonic representations is the four dimensional manifold $S^2 \times S^2$ while for the spinor representations, it is the two dimensional manifold $S^2 \cup S^2$.

We also note that the Kostant-Souriau [20, 38] method of "geometric" quantization focuses attention on coadjoint orbits in certain nilpotent Lie groups. From their point of view, our result here are most natural.

As we shall discuss in Sect. 5, coadjoint orbits have a natural group invariant symplectic structure, not an unreasonable thing for classical mechanical systems! Since only S^2 among the spheres have such a structure, $S^n (n \neq 2)$ is never a classical limit manifold. This appears to be serious for extending the Dunlop-Newman idea but, as we shall see in Sect. 7, the difficulties can be overcome.

Next we summarize the contents of this paper. In Sect. 2, we present the Berezin-Lieb inequalities. These relate the partition function associated to an operator A to the classical partition function of its lower symbol $a(x) = \text{Tr}(P(x)A)$ and its upper symbol g with $A = \int g(x)P(x)d\mu(x)$ (if such exists) where $P(x)$ is a family of rank one projections with $\int P(x)d\mu(x) = 1$. Further properties of upper and lower symbols in general are found in Appendix 1 and in the case where $P(x)$ is built from maximal weight vectors in Appendix 2. As a warmup to our main interest in spin systems, we discuss in Sect. 3 the classical limit of $\text{Tr}(e^{-H})$, $H = -\Delta + V(x)$ using Lieb's ideas. Such an approach has already been found by Thirring [40]. We include it here for several reasons: first, Thirring's proof is not yet widely available; second, we wish to note the simple extension to allow magnetic fields and finally, we wish to discuss some connections with other methods of controlling the classical limit in this case. In Sect. 4, we discuss the classical limit of the pressure of multiparticle systems in the thermodynamic limit. This section lacks the polish of much of our other discussion in the sense that a much more comprehensive result is both desirable and presumably possible. It also lacks the ideological purity of much of the rest of the paper eschewing the use of coherent state methods alone. So far as we know, these are the first results on this subject and we hope it will be brought to a higher level by others. In Sect. 5, we summarize various features of compact Lie group theory, especially Weyl's theory of representations; this is partly to establish notation and partly for the reader's convenience. In Sect. 6, in many ways the central section of the paper, we combine Sects. 2 and 5 with the machinery of Appendix 2 to extend Lieb's limit theorem to certain sequences of representations. There should be classical limit theorems for the sequence whose maximal weight is $L\lambda$ where $L = 1, 2, \dots$ and λ is any dominant weight (see Sect. 5 for definitions). Unfortunately, for technical reasons discussed in Appendix 2, we are limited to the case λ a fundamental weight (minimal dominant weight). Most interesting examples are included however; in particular, that of Fuller-Lenard. At the end of Sect. 6, we discuss various examples including that of $O(2n)$ spinors. These representations are further discussed in Appendix 3. Finally in Sect. 7, we describe the relevance of our work to an approach to proving the Lee-Yang theorem. In Appendix 2, we prove that for coherent vectors built on maximal weight vectors, every operator has a diagonal representation.

We end this introduction with the following remark: it seems to me that there has been in the literature entirely too much emphasis on quantization, (i.e. general methods of obtaining quantum mechanics from classical methods) as opposed to the converse problem of the classical limit of quantum mechanics. This is unfortunate since the latter is an important question for various areas of modern physics while the former is, in my opinion, a chimera.

A sketch of some of our results appeared in [37].

2. Coherent Projections and Berezin-Lieb Inequalities

Let \mathcal{H} be a Hilbert space, (X, Σ, μ) a measure space. A family of *coherent projections* is a weakly measurable map $x \mapsto P(x)$ from X to the orthogonal projections on \mathcal{H} so that

$$\begin{aligned} & \text{i) } \dim P(x) = 1 \text{ for all } x \in X \\ & \text{ii) } \int P(x) d\mu(x) = 1 \end{aligned} \tag{2.1}$$

in the sense that

$$\int (\varphi, P(x)\psi) d\mu(x) = (\varphi, \psi) \tag{2.1'}$$

for all $\varphi, \psi \in \mathcal{H}$. [In case $\int d\mu = \infty$, (2.1') supposes that the integral is absolutely convergent. By the Schwarz inequality on \mathcal{H} and then on $L^2(X, d\mu)$, this follows if $\int (\varphi, P(x)\varphi) d\mu(x) < \infty$ for all φ .]

Lemma 2.1. *If $d = \dim(\mathcal{H})$, then $\int d\mu(x) = d$.*

Proof. If $d < \infty$, just take traces of both sides of (2.1). If $d = \infty$, let $\{\varphi_i\}_{i=1}^\infty$ be an orthonormal basis and note that for any n

$$\int d\mu(x) \geq \int \sum_{i=1}^n (\varphi_i, P(x)\varphi_i) d\mu(x) = n. \quad \square$$

The next result shows that coherent projections always arise from coherent vectors. This result uses the underlying hypothesis always made by reasonable men that $\dim \mathcal{H}$ is countable.

Proposition 2.2. *If $P(x)$ is a family of coherent projections then, there exists a measurable family $\psi(x)$ of unit vectors so that $P(x) = (\psi(x), \cdot)\psi(x)$. Moreover, the $\psi(x)$ are total.*

Proof. This is a simple application of the von Neumann selection principle. Let $\{\varphi_i\}_{i=1}^N$ be an orthonormal basis for \mathcal{H} (N finite or countable). Let $A_n = \{x | (\varphi_i, P(x)\varphi_i) = 0, i = 1, \dots, n-1, (\varphi_n, P(x)\varphi_n) \neq 0\}$. Then $\bigcup_{n=1}^\infty A_n = X$ since the φ_i are a basis. On A_n , let

$$\psi(x) = P(x)\varphi_n / \|P(x)\varphi_n\|.$$

Then $P(x) = (\psi(x), \cdot)\psi(x)$ is obvious since $P(x)$ is rank one. If $(\psi(x), \eta) = 0$, then $(\eta, P(x)\eta) = 0$ for all x so $\|\eta\|^2 = 0$ by (2.1). \square

Definition. Let A be a bounded operator on \mathcal{H} and $P(x)$ a fixed family of coherent projections. Then $L(A)$ is the function on X given by:

$$L(A)(x) = \text{Tr}(AP(x)) \tag{2.2}$$

$L(A)$ is called the *lower symbol* of A .

Notice that

$$\|L(A)\|_\infty \leq \|A\|. \tag{2.3}$$

Next, let $f \in L^\infty(X, d\mu)$. Since $(\varphi, P(x)\psi) \in L^1$ for all φ, ψ with

$$\begin{aligned} \int |(\varphi, P(x)\psi)| d\mu(x) & \leq \int \|P(x)\varphi\| \|P(x)\psi\| d\mu(x) \\ & \leq [\int (\varphi, P(x)\varphi) d\mu(x)]^{1/2} [\int (\psi, P(x)\psi) d\mu(x)]^{1/2} \\ & = \|\varphi\| \|\psi\| \end{aligned}$$

we have that for any $f \in L^\infty$:

$$\int f(x)(\varphi, P(x)\psi)d\mu(x) \equiv a(\varphi, \psi)$$

is a convergent integral with

$$|a(\varphi, \psi)| \leq \|f\|_\infty \|\varphi\| \|\psi\|. \tag{2.4}$$

Thus, there is a bounded operator A_f with $(\varphi, A_f\psi) = a(\varphi, \psi)$.

Definition. The map $f \mapsto A$ from $L^\infty(X)$ to $\mathcal{L}(\mathcal{H})$ is denoted by U . If $A = U(f)$ for some f , f is called the *upper symbol* of A .

We write

$$U(f) = \int f(x)P(x)d\mu(x)$$

(2.4) says that

$$\|U(f)\| \leq \|f\|_\infty. \tag{2.4'}$$

What we call upper and lower symbols, Berezin [4] calls contravariant and covariant symbols respectively. We prefer to use upper and lower since the inequalities always go that norms of operators are bounded above by norms of upper symbols and below by norms of lower symbols. One disadvantage of dropping the Berezin names is that the names suggest a kind of duality; in fact, while Berezin does not note a duality, there is a strong duality; see Appendix 1. In that appendix, we discuss a number of aspects of upper and lower symbols; in particular, when the following holds :

Definition. We say a family of coherent projections is *complete* if and only if $\text{Ran}(U)$ is sequentially strongly dense in $\mathcal{L}(\mathcal{H})$, the bounded operators.

Completeness of the $P(x)$ should not be confused with the completeness of the coherent vectors $\psi(x)$ which is guaranteed by Proposition 2.2. In fact, as we see in Appendix 1, there exist both complete and incomplete families of coherent projections.

In this section we want to give proofs of the following pair of results.

Theorem 2.3. (First Berezin-Lieb Inequality). *Let $a(x) = L(A)(x)$. Then, if A is self-adjoint*

$$\int \exp(a(x))d\mu(x) \leq \text{Tr}(\exp(A)). \tag{2.5}$$

Theorem 2.4 (Second Berezin-Lieb Inequality). *Let $A = U(f)$. Then if f is real valued,*

$$\text{Tr}(\exp(A)) \leq \int \exp(f(x))d\mu(x). \tag{2.6}$$

Remarks. 1. We only give detailed proofs in case $\dim(\mathcal{H}) < \infty$. Fairly simple approximation arguments work for the case $\dim \mathcal{H} = \infty$ so long as the inequalities are suitably interpreted.

2. These inequalities were independently obtained by Berezin [3, 4] and Lieb [23] (Lieb had specific P 's but his proofs can be extended). Their published proofs of the second inequality are quite different. The basic proof we give below of (2.6) is an elegant unpublished proof of Lieb which is related to one step in the published proof of Berezin. We then give an instructive alternate proof of (2.6).

3. Since only Jensen’s inequality is used, the results extend if \exp is systematically replaced by any convex function, Φ . This was noted by Berezin [4]. With this remark, Wehrl’s inequality [42] relating quantum and “classical” entropy can be viewed as a special case of the Berezin-Lieb inequalities.

Proof of Theorems 2.3 and 2.4 (assuming $\dim \mathcal{H} < \infty$). Let $\{\varphi_i\}_{i=1}^N$ be a complete orthonormal set of eigenfunction for A with $A\varphi_i = \mu_i\varphi_i$.

Define

$$c_i(x) = (\varphi_i, P(x)\varphi_i).$$

Then (2.7) implies that, for each i

$$\int c_i(x) d\mu(x) = 1 \tag{2.7}$$

and the completeness of the φ_i imply that, for each x

$$\sum_i c_i(x) = 1. \tag{2.8}$$

The first bound will follow from Jensen’s inequality based on (2.8) and then (2.7) used to evaluate some integrals while the second bound will come from Jensen’s inequality based on (2.7) and then (2.8) to do some sums. Explicitly, if $a = L(A)$, then any x :

$$\begin{aligned} \exp(a(x)) &= \exp\left(\sum_i \mu_i c_i(x)\right) \\ &\leq \sum_i c_i(x) e^{\mu_i} \quad [\text{by Jensen and (2.8)}]. \end{aligned}$$

Integrating $d\mu(x)$ and using (2.7), we obtain the first inequality.

Now, let f be real valued and $A = U(f)$. Then

$$\begin{aligned} (\varphi_i, e^A \varphi_i) &= e^{(\varphi_i, A \varphi_i)} = \exp\left(\int f(x) c_i(x) d\mu(x)\right) \\ &\leq \int e^{f(x)} c_i(x) d\mu(x) \quad [\text{by Jensen and (2.7)}]. \end{aligned}$$

Summing over i and using (2.8), the second inequality results. \square

In [23], Lieb obtains the upper bound by a Golden-Thompson inequality. Here is a second proof which yields an alternate proof of the second inequality.

Alternate proof of Theorem 2.4 (Again, if $\dim \mathcal{H} < \infty$). By adding a constant to f we can suppose that $f \geq 0$. Thus, it suffices to show that if $f \geq 0$

$$\text{Tr}([\int f(x) P(x) d\mu(x)]^n) \leq \int f^n(x) d\mu(x). \tag{2.9}$$

By Proposition 2.2, $P(x) = (\varphi(x), \cdot) \varphi(x)$ for some φ . Let $K(x, y) = (\varphi(x), \varphi(y))$. Then

$$\text{Tr}(P(x_1) \dots P(x_n)) = K(x_1, x_2) K(x_2, x_3) \dots K(x_n, x_1).$$

Let B be the integral operator with kernel K [K is Hilbert-Schmidt since $\int (K(x, y))^2 d\mu(x) = \int (\varphi(y), P(x)\varphi(y)) d\mu(x) = 1$ and $\int (d\mu(y)) < \infty$]. Then the left side of (2.9) is

$$\text{Tr}((AB)^n)$$

where $A =$ multiplication of f . For general self-adjoint A, B one has the Golden-Thompson type inequality (see e.g. [35]).

$$\text{Tr}((AB)^n) \leq \text{Tr}(A^n B^n).$$

But, by (2.1), $\int K(x, y)K(y, z)d\mu(y) = K(x, z)$, i.e. $B^2 = B$. Thus

$$\text{Tr}(A^n B^n) = \text{Tr}(A^n B) = \int f^n(x)K(x, x)d\mu(x) = \int f^n(x)d\mu(x)$$

since $K(x, x) = 1$. [We brush over the fact that since K is only assumed measurable, one cannot compute the trace so cavalierly; this is easily handled using $\int K(x, y)K(y, x)d\mu(y) = 1$.]

This proves (2.9) and so Theorem 2.4. \square

3. Schrodinger Operators with Confining Potentials

In this section we wish to prove:

Theorem 3.1. *Let V, a be functions on R^v (respectively real valued and R^v -valued) so that:*

- (a) $|V(x)| \leq Ce^{D|x|^2}; |a(x)| \leq Ce^{D|x|^2}$
- for some C, D .
- (b) V, a are continuous
- (c) $|V(x)| \geq Ax^\delta$ some $A, \delta > 0$.
- (d) $\text{div} a = 0$ in distributional sense.

Let $p_\hbar = \hbar i^{-1} \nabla$ and $H_\hbar = (p_\hbar - a)^2 + V$. Let $Z_Q = \hbar^{v/2} \text{Tr}(e^{-H_\hbar})$ and

$$Z_{cl} = \int \frac{d^v k d^v y}{(2\pi)^v} \exp(-k^2 - V(y)). \tag{3.1}$$

Then

$$\lim_{\hbar \rightarrow 0} Z_Q = Z_{cl}. \tag{3.2}$$

Remarks. 1. There are three general approaches to this kind of result: Dirichlet-Neumann bracketing (see [27, 28, 34]), path integrals (see [36]) and coherent vectors as we use here.

2. This result is not new. A stronger result was proven by Combes et al. [7] using $D - N$ bracketing and another proof using Stochastic and Wiener integrals can be found in [36]. For the case $a = 0$, the precise proof we give was found originally by Thirring [40].

3. Condition (c) enters only to assure that $Z_{cl} < \infty$ and various technical conditions. It could be dispensed with. If one uses Lebesgues theorem on differentiation of integrals, (b) could be replaced by measurability and (a) by L^1 conditions with $\int_{|x-y| \leq 1} |V(y)| dy \leq Ce^{D|x|^2}$, etc. The bounds (a) are used critically in our proof but not in the bracketing or path integral proofs.

To prove Theorem 3.1, we use the special functions

$$[\psi_\hbar(k, y)](x) = (\hbar\pi)^{-v/4} e^{ik \cdot x/\hbar} \exp(-(x - y)^2/2\hbar) \tag{3.3}$$

which obeys

$$[(p_{\hbar} - k)^2 + (x - y)^2] \psi = \hbar \psi. \tag{3.4}$$

We introduce the \hbar dependent Fourier transforms:

$$(\mathcal{F}_{\hbar} g)(k) = (2\pi\hbar)^{-v/2} \int e^{-ik \cdot x/\hbar} g(x) d^v x.$$

Notice that

$$\mathcal{F}_{\hbar}[\psi_{\hbar}(k, y)] = e^{+iy \cdot k} \psi_{\hbar}(y, k). \tag{3.5}$$

Let $P_{\hbar}(k, y)$ be the rank one projection onto $\psi_{\hbar}(k, y)$.

Lemma 3.2 ([19]).

$$\int P_{\hbar}(k, y) d^v k d^v y / (2\pi\hbar)^v = 1 \tag{3.6}$$

in the weak sense. (i.e. integrals of matrix elements converge)

Proof. Let $\eta \in L^2$. Then

$$(2\pi\hbar)^{v/2} (\psi_{\hbar}(k, y), \eta) = \mathcal{F}_{\hbar}(\eta \psi_{\hbar}(0, y))(k)$$

so, by the Plancherel theorem

$$\int (2\pi\hbar)^{-v} d^v k |\langle \psi_{\hbar}(k, y), \eta \rangle|^2 = \int |\eta(x)|^2 |\psi_{\hbar}(0, 0)|^2 (x - y) d^v x. \tag{3.7}$$

Integrating $d^v y$ and using $\|\psi_{\hbar}\|_{L^2} = 1$, (3.6) results. \square

This lemma shows that the $P_{\hbar}(k, y)$ fall into the scheme of Sect. 2. Actually they fall into the scheme used in Sect. 6: In the natural action of the Heisenberg group on $L^2(\mathbb{R}^v)$, i.e.

$$[U(k, y, \alpha) f](x) = e^{i\alpha} e^{ik \cdot x/\hbar} f(x - y)$$

we have that

$$P_{\hbar}(k, y) = U(k, y, \alpha) P_{\hbar}(0, 0) U(k, y, \alpha)^{-1}$$

and $P_{\hbar}(0, 0)$ is the projection onto what is, in many ways, a “minimal weight vector”.

Proof of Theorem 3.1. We first claim the upper bound

$$Z_Q \leq Z_{cf}. \tag{3.8}$$

For diamagnetic inequalities [31, 32, 16, 33, 36] show that $Z_Q(V, a) \leq Z_Q(V, a=0)$ and the Golden-Thompson inequality [13, 41, 35] then proves (3.8).

Given any function $g(x)$ obeying $|g(x)| \leq C e^{D|x|^2}$, for \hbar fixed and sufficiently small,

$$L(g)(k, y) = \text{Tr}(M_g P_{\hbar}(k, y))$$

with $M_g =$ multiplication by g , exists and is independent of k . Write its value as $\tilde{g}(y)$. Then:

$$\begin{aligned} L(p_{\hbar}^2) &= k^2 + v\hbar/2 \\ L(p_{\hbar} \cdot a) &= k \cdot \tilde{a}. \end{aligned}$$

To see the last result, we calculate first

$$\begin{aligned} L(p_{\hbar} \cdot a)(k=0, y) &= -i\hbar \int \sum_j a_j \psi \partial_j \psi \\ &= -i\hbar \int \sum_j \frac{1}{2} \partial_j (a_j \psi^2) = 0 \end{aligned}$$

since $\psi(k=0, y)$ is real-valued and $\operatorname{div} a = 0$. Similarly,

$$L((p_{\hbar} - k_0) \cdot a)(k=k_0, y) = 0$$

so

$$L(p_{\hbar} \cdot a)(k, y) = L(k \cdot a)(k, y) = k \cdot \tilde{a}(y).$$

Thus

$$L(H_{\hbar}) = \tilde{H}_{c\ell}(k, y) + \frac{v\hbar}{2} + [\tilde{a}^2 - (\tilde{a})^2]$$

where

$$\tilde{H}_{c\ell} = (k - \tilde{a}(y))^2 + \tilde{V}(y).$$

By hypothesis (a), (b), $\tilde{a}, \tilde{V} \rightarrow a, V$ pointwise as $\hbar \rightarrow 0$. Thus, pointwise

$$L(H_{\hbar})(k, y) \rightarrow H_{c\ell}(k, y) \equiv (k - a(y))^2 + V(y)$$

as $\hbar \rightarrow 0$. It is also easy to show that for any compact set Ω in $k - y$ spaces, and \hbar sufficiently small

$$L(H_{\hbar})(k, y) > -c(\Omega) > -\infty$$

all $k, y \in \Omega$. Thus

$$\int_{\Omega} d^v k d^v y \exp(-L(H_{\hbar})) \rightarrow \int_{\Omega} d^v k d^v y e^{-H_{c\ell}(k, y)}$$

so using Theorem 2.3 and Lemma 3.2

$$\lim Z_Q \geq \int_{\Omega} d^v k d^v y (2\pi)^{-v} e^{-H_{c\ell}(k, y)}$$

for any Ω . Taking $\Omega \rightarrow R^{2v}$, we prove (3.2). \square

The proof of the upper bound above is the same as that in [7] and does not use Theorem 2.4. In general, it is somewhat difficult to use Theorem 2.4 because for complicated V 's one cannot write down the upper symbol for V so easily. However, in specific cases, one can write down $U(H_{\hbar})$ and typically the upper bound from Theorem 2.4 is worse than (3.8) but by an amount going to zero as $\hbar \rightarrow 0$. For example, using (3.7), one easily sees that

$$U(y^2 - v\hbar/2) = x^2$$

and, by symmetry (3.5):

$$U(k^2 - v\hbar/2) = p_{\hbar}^2$$

so one finds using only Theorem 3.3 that for $H_{\hbar} = p_{\hbar}^2 + x^2$

$$Z_Q \leq e^{v\hbar} Z_{c\ell}$$

rather than (3.8).

There is of course nothing sacred about the particular choice of coherent vectors we used in the above. For example, given δ , one can imagine partitioning R^v into cubes of side δ and then taking Dirichlet Laplacian eigenfunctions inside these boxes. One will get a complete orthonormal set and in this way a coherent families of projections which was also concentrated near single points in those spaces. In fact, this method will essentially yield Dirichlet bracketing from a different point of view. In the next section, the reader should keep this remark in mind.

4. Particle Systems

In this section, we consider a single species of particles interacting only via a pair of potential V . Let

$$U_N(x_1, \dots, x_N) = \sum_{i < j} V(x_i - x_j) \tag{4.1}$$

and

$$\mathcal{E}_{c\ell}(\beta, z, A) = \sum_{n=0}^{\infty} \frac{z^n}{n!} a(\beta)^n \int_{x_i \in A} \exp(-\beta U_N(x_i)) \prod_{i=1}^n d^3x_i$$

with $a(\beta) = \int d^3p e^{-\beta p^2} = \pi^{3/2} \beta^{-3/2}$, and

$$\mathcal{E}_h^Q(\beta, z, A) = \sum_{n=0}^{\infty} \frac{z^n}{n!} (2\pi\hbar)^{3n} \text{Tr}(\exp[-\beta H_n(\hbar)])$$

where H_n acts on $L^2(A^n)$ by $\sum_{i=1}^n -\hbar^2 \Delta_{A_i} + U_n$ with Δ_{A_i} the Laplacian with vanishing boundary conditions on ∂A and with no restriction on statistics.

To define the pressure, one typically assumes two properties of V :

(1) *Stability*, i.e.

$$U_N \geq -cN$$

(2) *Temperedness*, i.e. for some $\varepsilon > 0, R_0 > 0$

$$|V(x)| \leq C(1 + |x|)^{-3-\varepsilon}; \quad \text{all } |x| > R_0.$$

If both these hypotheses hold, then [29], for any $\beta, z, P_{c\ell}(\beta, z) = \lim_{|A| \rightarrow \infty} |A|^{-1} \ln \mathcal{E}_{c\ell}(\beta, z, A)$ exists say as A run through hypercubes. Similarly, [29, 28], the limit $P_h^Q(\beta, z) = \lim_{|A| \rightarrow \infty} |A|^{-1} \ln \mathcal{E}_h^Q(\beta, z, A)$ exists. Here we want to discuss the question of when (with no statistics)

$$\lim_{\hbar \downarrow 0} P_h^Q = P_{c\ell}. \tag{4.2}$$

We will single out a notion we call microstability and show that (4.2) holds for potentials which are stable, tempered and microstable. We conjecture that any sum of a positive and a positive definite potential is microstable; here, we will settle for proving that a non-empty class of V 's is microstable.

Definition. Given $k \in N = \{1, 2, \dots\}$ and V , define $\Delta_k(x)$ to be the cube of side 2^{-k} centered at a point $2^{-k}\mathbb{Z}$ containing x (for definiteness, Δ_k is a product of intervals open on the left and closed on the right) and let

$$V_k(x, y) = \sup[V(z - w) | z \in \Delta_k(x), w \in \Delta_k(y)].$$

Given $\ell \in N$, let $P_{k,\ell}$ be the classical pressure in a box of size 2^ℓ computed with V_k replacing V . Since $U_{k,N} \cong U_N$, V_k is stable and tempered, so by modifying the usual proof, $\lim_{\ell \rightarrow \infty} P_{k,\ell}$ exists. We say that V is *microstable* if and only if

$$\lim_{k \rightarrow \infty} \left(\lim_{\ell \rightarrow \infty} P_{k,\ell} \right) = P_{c\ell}. \tag{4.3}$$

(4.3) is a statement about interchanging limits since $\lim_{k \rightarrow \infty} P_{k,\ell} = |A_\ell|^{-1} \ln \Xi_{c\ell}(\beta, z, A_\ell)$. Thus microstability is a statement about continuity of the pressure under small local changes which preserve stability.

Theorem 4.1. *Let V be stable, tempered and microstable. Then (4.2) holds.*

Proof. We will prove that

$$P_h^Q \leq P_{c\ell} \tag{4.4}$$

and that

$$\lim_{\hbar \downarrow 0} P_h^Q \geq \lim_{\ell \rightarrow \infty} P_{k,\ell} \tag{4.5}$$

for any k so that (4.3) will yield (4.2).

(4.4) follows from the Golden-Thompson inequality. The easiest way of seeing this is to replace the Dirichlet Laplacian on ∂A by the infinite volume Laplacian, $-\Delta$, plus the potential $M \text{dist}(x, A)$. As in Sect. 3, Golden-Thompson yields a fixed A analog of (4.4) with the above modified P^Q and with the classical integrals over all of R^3 with the potential $M \text{dist}(x, A)$ added. Now take M to infinity and the finite volume analog of (4.4) results. Taking the volume to infinity we obtain (4.4).

Now consider the quantum pressure in a box of side 2^ℓ , $P_{\hbar,\ell}^Q$. Following the idea in [24], we can get a lower bound on this quantity by replacing V by V_k and adding extra Dirichlet boundary conditions on the small boxes $\Delta_k(x)$. Having done this, one can write out the eigenfunctions of the new Hamiltonian explicitly as products of Dirichlet functions in each individual box. The net result is that

$$P_h^Q \geq \lim_{\ell \rightarrow \infty} P_{k,\ell}(\beta, \tilde{z})$$

where:

$$\tilde{z} = z \tilde{a}_\hbar(\beta) [a(\beta)]^{-1}$$

and

$$\tilde{a}_\hbar(\beta) = \sum_{n \in \mathbb{Z}^3_+} e^{-\beta \hbar^2 (nc_k)^2} (2\hbar c_k)^3$$

with $c_k = 2^k \pi$. Now by stability of $V_{k,\ell} \lim_{\ell \rightarrow \infty} P_{k,\ell}(\beta, z)$ is continuous in z and clearly $\lim_{\hbar \downarrow 0} \tilde{z} = z$ so (4.5) holds. \square

The following illustrates that the set of microstable potentials is non empty, e.g. $V = (1+r)^{-\alpha}$; $\alpha > 3$, obeys all the hypotheses.

Theorem 4.2. *Suppose that $V(x) \geq 0$ and*

$$|(\nabla V)(x)| \leq C|V(x)| \tag{4.6}$$

for some C . Then V is microstable.

Proof. $P_{k,\ell} \leq P_\ell$ always. Next, let W_k be the same as V_k but with max over $\Delta_k(x)$, $\Delta_k(y)$ replaced by min. We claim that

$$\theta_k \equiv \| [W_k - V_k] V_k^{-1} \|_\infty \leq D2^{-k}. \tag{4.7}$$

For fix boxes, $\Delta_k(x)$, $\Delta_k(y)$. Then there exists s, t with

$$|s - t| \leq 22^{-k} \sqrt{3} \tag{4.8}$$

so that

$$\begin{aligned} |(W_k - V_k) V_k^{-1}| &= |[V(s) - V(t)] V(t)^{-1}| \\ &= \left| \int_0^1 (s-t) \cdot \nabla V(\theta s + (1-\theta)t) V(t)^{-1} d\theta \right| \\ &= \int_0^1 |s-t| |\nabla V(\theta s + (1-\theta)t)| |V(\theta s + (1-\theta)t)|^{-1} d\theta \\ &\leq |s-t| C \leq D2^{-k} \end{aligned}$$

by (4.7) and (4.6). In the above, we used the definition of V_k and the positivity of V to conclude that $|V(\theta s + (1-\theta)t)| \leq |V(t)|$.

Now define \tilde{V}_k by:

$$\theta_k \tilde{V}_k + (1-\theta_k) V_k = V.$$

Then

$$\theta_k \tilde{V}_k \geq W_k - (1-\theta_k) V_k \geq W_k - V_k + V_k [(V_k - W_k) V_k^{-1}] = 0$$

so there is a uniform bound, B , on the pressure due to \tilde{V}_k . By Hölder's inequality:

$$P_{c\ell} \leq (1-\theta_k) \lim_{\ell \rightarrow \infty} P_{k,\ell} + \theta_k B.$$

Since, by (4.7), $\theta_k \rightarrow 0$ as $k \rightarrow \infty$, we see that

$$P_{c\ell} \leq \lim_{k \rightarrow \infty} \lim_{\ell \rightarrow \infty} P_{k,\ell}. \quad \square$$

Remark. Independently, Baumgartner [43] has proven results with the same thrust as Theorem 4.1, but Baumgartner, unlike us, accommodates statistics.

5. A Concise Review of Compact Lie Group Theory

In this section, we review some of the basic elements of Weyl's theory of the representations of compact Lie groups; see Samelson [30] and Adams [1] for additional details. We include this material partially to establish notation and partially for the reader's convenience.

G will denote a compact Lie group with a discrete center (\equiv semisimple) and g will be its Lie algebra. g^* is the dual of g . There is a natural representation of G on g given by:

$$\exp(tA(x)(X)) = x \exp(tX) x^{-1}$$

for $x \in G, X \in g$. This is called the *adjoint representation* of G . If A is irreducible, we will call G *simple*. (Note: if G is connected, then this definition is equivalent to g being simple in the usual [30] sense, but if G is not connected, then G can be simple with g non-simple, e.g. $O(4)$ is simple but its Lie algebra $SO(4)$ is not. Simplicity is equivalent to G having no invariant Lie subgroups of dimension one or more). Since G is compact, g has an inner product making A unitary: if g is also simple, this inner product is unique up to constants. One can fix the constant (and the inner product in the non-simple case) by taking the inner product to be the negative of the Killing form, i.e.

$$(X, Y) = -\text{Tr}(\text{Ad}(X) \text{Ad}(Y)) \tag{5.1}$$

where $\text{Ad}(X)(Z) = [X, Z]$.

The inner products sets up a natural correspondence between g and g^* which will be useful in concrete situations but in abstract it will be useful to distinguish g and g^* to avoid certain awkwardnesses. On g^* , we define by duality the coadjoint representation, A^* , of G by

$$A^*(x) = A(x^{-1})^*.$$

Under the natural correspondence of g and g^* , A and A^* are equivalent.

Orbits of the action A^* , i.e. $\Gamma = \{A^*(x) \mid x \in G\}$ for $\ell \in g^*$ fixed will play a major role in the classical limit theory in Sect. 6; they are called *coadjoint orbits*. These orbits have a natural invariant symplectic structure [20, 38] making them especially attractive as classical limits and incidentally showing that they are even dimensional. Let us describe this symplectic structure; given $\ell_0 \in \Gamma$, the tangent space $T_{\ell_0}(\Gamma)$ of Γ is associated to a subspace of the tangent space $T_{\ell_0}(g^*) = g$. Thus the cotangent space $T_{\ell_0}^*(\Gamma)$ is naturally associated to a quotient space of $(g^*)^* = g$. Explicitly:

$$T_{\ell_0}^*(\Gamma) = g / \{X \in g \mid \ell_0([X, Y]) = 0 \text{ all } Y \in g\}. \tag{5.2}$$

To see (5.2), note that we have quotiented out all the covectors, X , which vanish on the derivatives of curves, $A^*(\exp(tX))\ell_0$, on Γ through ℓ_0 . On $T_{\ell_0}^*(\Gamma) \times T_{\ell_0}^*(\Gamma)$, define a linear functional ω_{ℓ_0} by:

$$\omega_{\ell_0}([X], [Z]) = \ell_0([X, Z]).$$

By (5.2), the value of $\omega_{\ell_0}([X], [Z])$ is clearly independent of the choice of X in the equivalence class $[X]$ and if $\omega_{\ell_0}([X], [Y]) = 0$ for all $[Y]$, then $[X] = 0$, so ω_{ℓ_0} is non-degenerate. It is clearly anti-symmetric and group invariant. The group invariance implies that ω is closed.

The first part of Weyl's beautiful theory of representations of G is to do the obvious thing, namely diagonalize as much as possible. We therefore pick out a maximal connected abelian subgroup of G , i.e. a *maximal torus*, T . The Lie algebra

$h \subset g$ of T is called the *Cartan subalgebra*. It is a basic fact [1] that all maximal tori are conjugate. Since any element of the connected component, G_0 , of G clearly lies in a maximal torus, every conjugacy class of G_0 intersects T . The dimension of h is called the *rank* of G .

We will illustrate the general theory in this section with three examples: $SU(n)$, $SO(n)$, and $O(2n)$.

$SU(n)$. A maximal torus consists of all diagonal matrices,

$$T = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \right\}; \quad h = \left\{ \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \mid \sum x_i = 0 \right\}.$$

In h^* , take as a basis set $\omega_1, \dots, \omega_n$ with $\omega_i(X) = (2\pi i)^{-1} x_i$. Notice $\omega_1 + \dots + \omega_n = 0$. It is obvious that all maximal tori are conjugate. The rank is $n - 1$.

$SO(2n)$ or $O(2n)$. The maximal torus consists of $n \times 2 \times 2$ diagonal blocks $\begin{pmatrix} \cos \theta_j & \sin \theta_j \\ -\sin \theta_j & \cos \theta_j \end{pmatrix}$ and h has 2×2 blocks $i \begin{pmatrix} 0 & \theta_j \\ -\theta_j & 0 \end{pmatrix}$. We pick a basis $\omega_1, \dots, \omega_n$ in h^* by $\omega_i(X) = (2\pi)^{-1} \theta_i$. G has rank n .

Given a representation, π , of G on a finite-dimensional complex space, \mathcal{H}_π we simultaneously diagonalize $\{\pi(X) \mid X \in h\}$, i.e. we seek $v \in \mathcal{H}_\pi$ and $\ell \in h^*$ so that

$$\pi(X)v = \ell(X)v \quad \text{all } X \in h.$$

The ℓ 's are called the *weights* of π and the v 's the *weight vectors*. The weights of the adjoint representation (more precisely of the complexification of A on $g_\mathbb{C}$, the complexification of g) are called the *roots*. [The corresponding root vectors X_α play an important role in those details of the representation theory that will not concern us much here for the following reason: For $X \in h$, $A(X)(X_\alpha) = \alpha(X)X_\alpha$ i.e. $[X, X_\alpha] = \alpha(X)X_\alpha$. Thus

$$\pi(X) [\pi(X_\alpha)v] = \pi(X_\alpha)\pi(X)v + \pi([X, X_\alpha])v = [\ell(X) + \alpha(X)]\pi(X_\alpha)v$$

so $\pi(X_\alpha)v$ is a weight vector (if non-zero) with weight $\ell + \alpha$. As a result, the root vectors play the role of the familiar raising and lowering operators].

There are basic *integrality conditions* that the weights must obey, i.e. any weights must lie in a certain integral lattice \mathcal{I} . If we look for representation of G , this is because the torus must close, i.e. in the above example, ℓ must be an integral sum of the ω_i 's. If we look at representations of g , then the integrality conditions come from the familiar manipulations with ladder operators. For $SU(n)$, there are no extra weights allowed, i.e.

$$\mathcal{I} = \left\{ \sum n_i \omega_i \mid n_i \in \mathbb{Z} \right\} \tag{SU(n)}$$

but for $SO(2n)$, one finds

$$\mathcal{I} = \left\{ \sum n_i \omega_i \mid n_i \in \frac{1}{2}\mathbb{Z}; n_i - n_1 \in \mathbb{Z} \right\}$$

i.e. all integers or all half integers. [The fact that the quotient of weights for g to weights for G is \mathbb{Z}_2 is expressing the fact that $SO(2n)$ is doubly connected.]

The subtle aspect of Weyl's theory is the special role played by an object called the *Weyl group*. This is the group, $W(T)$, of automorphism of T which arise from

those inner automorphisms of G leaving T set wise invariant, [i.e., we take those elements of G which induce inner automorphisms leaving T invariant i.e. the normalizer, $N(T) = \{x \in G | xTx^{-1} = T\}$] and we associate two elements which induce the same automorphism on T [this is equivalent to quotienting out by these x in $C(T) = \{x \in G | xy = yx \text{ all } y \in T\}$, the centralizer of T , i.e. $W(T) = N(T)/C(T)$].

$SU(n)$. The only elements of G which leave T setwise fixed are those that take the basic eigenvectors of T into multiples of other eigenvectors. The corresponding automorphisms, permute the diagonal matrix elements, i.e.

$$W(T) = \Sigma_n$$

the permutation group on n letters.

$SO(2n)$. Again the fundamental eigenvectors must be permuted but in two elements blocks. However, now there can be a flip of blocks. Thus the automorphisms can permute the θ_i 's and flip some signs. For determinant 1 an even number of signs must be flipped, i.e. $W(T) =$ permutations plus even number of sign flips.

$O(2n)$. The analysis is identical but now any number of sign flips are allowed.

Now let $S \in W(T)$ and let $x \in G$ induce S ; i.e. $S(X) = A(x)(X)$; $X \in h$. Let ℓ be a weight for π and v_ℓ the corresponding weight vectors. Then, for $X \in h$:

$$\begin{aligned} \pi(X) [\pi(x)v] &= \pi(x)\pi(S^{-1}(X))v \\ &= \ell(S^{-1}(X))\pi(x)v \\ &= (S\ell)(X)\pi(x)v \end{aligned}$$

for the obvious dual action of $W(T)$ on h^* . Thus $\pi(x)v$ is a weight vector and $S\ell$ a weight, i.e. the weights of π are a set left invariant by the action of $W(T)$ which focuses attention on the action of $W(T)$ on h^* .

The geometry of the action of $W(T)$ on h^* is rather subtle and beautiful. Here are the basic facts [1, 30]: (i) $W(T)$ is generated by elements of order 2 which act on h^* as reflections (with respect to the natural inner product h^* inherits from g^*) in hyperplanes (ii) Any $\ell \in h^*$ left invariant by some non-trivial $S \in W(T)$, is left invariant by some element of order 2 acting as a reflection. Thus, the set of invariant elements is a family of hyperplanes. (iii) If these hyperplanes are removed from h^* , the remaining points are a union of open polyhedral cones whose closures are called *Weyl chambers*. (iv) The *Weyl chambers* are images of each other under the action of the Weyl group and each non-trivial element of the Weyl group leaves *no* chamber setwise invariant. Thus the number of chambers is exactly the order of $W(T)$.

One chooses one chamber once and for all and calls it the *fundamental chamber*. Those elements of \mathcal{I} , the weight lattice, contained in the chamber are said to lie in \mathcal{I}_d , the *dominant weights* and those in the interior of the fundamental chamber are said to lie in \mathcal{I}_o , the *strongly dominant weights*. Here are our standard examples :

SU(n). Any element of \mathcal{J} is of the form $\sum_{i=1}^n m_i \omega_i$, $m_i \in \mathbb{Z}$. Since $\sum \omega_i = 0$, we can suppose the smallest $m_i = 0$. A fundamental chamber is given by

$$\left\{ \sum_{i=1}^n m_i \omega_i \mid m_1 \geq m_2 \geq \dots \geq m_{n-1} \geq m_n = 0 \right\}.$$

All the geometry is easily checked.

SO(n). By sign flips and permutations we can clearly arrange to have $m_1 \geq m_2 \geq \dots \geq m_{n-1} \geq |m_n|$; so

$$\mathcal{J}_d = \left\{ \sum_{i=1}^n m_i \omega_i \in \mathcal{J} \mid m_1 \geq m_2 \geq \dots \geq m_{n-1} \geq |m_n| \right\}.$$

O(n). Clearly

$$\mathcal{J}_d = \left\{ \sum_{i=1}^n m_i \omega_i \in \mathcal{J} \mid m_1 \geq \dots \geq m_n \geq 0 \right\}.$$

The next important point is that the fundamental chambers are actually simplicial cones, indeed an even stronger result is true, there exist $\lambda_1, \dots, \lambda_r \in \mathcal{J}_d$ so that $\mathcal{J}_d = \left\{ \sum_{i=1}^r n_i \lambda_i \mid n_i \geq 0 \right\}$. The λ_i 's are uniquely determined, e.g. in the order defined by \mathcal{J}_d (i.e. $\lambda > \mu$ if and only if $\lambda - \mu \in \mathcal{J}_d$), they are minimal elements of $\mathcal{J}_d \setminus \{0\}$. The λ_i 's are called *fundamental weights*. A special role is also played by the weight $\delta = \sum_{i=1}^r \lambda_i$. It is the minimal element of \mathcal{J}_0 and is often called the *lowest form*; we will call it the *magic weight* because of its rather spectacular role in Weyl's integration formula, Weyl's character formula (see [1, 30]) and in the formula for the main Casimir operator [see (5.3) below].

SU(n). The fundamental weights are $\lambda_1 = \omega_1$, $\lambda_2 = \omega_1 + \omega_2, \dots, \lambda_{n-1} = \omega_1 + \dots + \omega_{n-1}$.

SO($2n$). The fundamental weights are $\lambda_1 = \omega_1, \dots, \lambda_{n-2} = \omega_1 + \dots + \omega_{n-2}$, $\lambda_{n-1} = \frac{1}{2}(\omega_1 + \dots + \omega_n)$, $\lambda_n = \frac{1}{2}(\omega_1 + \dots + \omega_{n-1} - \omega_n)$.

O($2n$). The fundamental weights are $\lambda_1 = \omega_1, \dots, \lambda_{n-1} = \omega_1 + \dots + \omega_{n-1}$, $\lambda_n = \frac{1}{2}(\omega_1 + \dots + \omega_n)$.

The fundamental theorem of representation theory is described by the following result (see [1, 30] for proofs):

Theorem 5.1. *There is a one-one correspondence between irreducible (unitary) representations of g and elements $\ell \in \mathcal{J}_d$. The representation π_ℓ corresponding to ℓ is uniquely determined by the fact that ℓ is a weight of π_ℓ and among all weights, it is maximal in the \mathcal{J}_d -order. Moreover, $\{v \mid v \text{ is a weight vector for } \ell\}$ is one dimensional.*

ℓ is called the *maximal weight* of π_ℓ . Given π_ℓ it can be located by finding a weight of maximal length and using the Weyl group to move this weight to one of the same length lying in \mathcal{J}_d .

SU(n). For $\ell = \sum m_i \omega_i$, ($m_1 \geq m_2 \geq \dots \geq m_{n-1}$), π_ℓ is the representation associated to the Young tableaux [6] with rows with m_1, \dots, m_{n-1} boxes.

SO($2n$). It is not hard to see that $\pi_{L\lambda_1}$ corresponds to degree L spherical harmonics, $\pi_{\lambda_{n-1}} + \pi_{\lambda_n}$ to basic spinors which are not irreducible on SO($2n$) (see Appendix 3) and $\pi_{2L\lambda_{n-1}} + \pi_{2L\lambda_n}$ to spin L spinors.

O($2n$). $\pi_{2L\lambda_n}$ corresponds to spin L spinors which are irreducible for O($2n$).

In π_ℓ , the Casimir operator $\sum_i \pi_\ell(X_i)^2$ (X_i an orthonormal basis in Killing inner product) has eigenvalue (see [30]):

$$(\ell, \ell) + 2(\ell, \delta) \tag{5.3}$$

in the case where G is connected.

6. Coherent Vectors and the Classical Limit of Spin Systems

Fix a fundamental weight λ of some compact Lie group G . For each $L = 1, 2, \dots$, let \mathcal{H}_L and π_L be the space and representation with maximal weight $L\lambda$. For L fixed, let \mathcal{H}_α be a copy of \mathcal{H}_L for $\alpha \in A$ a finite subset of \mathbb{Z}^v and let $\mathcal{H}_A = \bigotimes_{\alpha \in A} \mathcal{H}_\alpha$. On \mathcal{H}_A define operators $S_\alpha(X)$ ($\alpha \in A, X \in g$) to be the tensor product of $\pi_L(X)$ in the α th factor with 1 in the other factors. Fix a basis X_1, \dots, X_m for g (for convenience) and a function H of $|A|m$ -vectors $S_{\alpha,i}$ ($\alpha \in A, i = 1, \dots, m$) which is multiaffine. $H(S_\alpha(X_i))$ is unambiguously defined since H is a sum of monomials which are products of commuting operators. If $d_L = \dim \mathcal{H}_L$, define

$$Z_Q^L(\gamma) = d_L^{-|A|} \text{Tr}(\exp[-H(\gamma S_\alpha(X_L)/L)]). \tag{6.1}$$

Next, let $\tilde{\lambda}$ be the element of g^* obtained by extending λ to g by setting $\tilde{\lambda} = 0$ on h^\perp . Let Γ be the coadjoint orbit containing $\tilde{\lambda}$ and let $d\tilde{\mu}(\cdot)$ be the probability measure on Γ inherited from Haar measure, $d\gamma$ on G , i.e. $\tilde{\mu}(B \subset \Gamma) = \gamma\{x \in G \mid A^*(x) \tilde{\lambda} \in B\}$. For each $\alpha \in A$, let Γ_α be a copy of Γ and let $\Gamma^{|A|} = \prod_\alpha \Gamma_\alpha$. Define

$$Z_{c\ell}(\gamma) = \int_{\Gamma^{|A|}} \exp[-H(\gamma \ell_\alpha(X_i))] \prod_\alpha d\tilde{\mu}(\ell_\alpha).$$

In this section, we will prove:

Theorem 6.1. *With the above notation:*

$$Z_{c\ell}(\gamma) \leq Z_Q^L(\gamma) \leq Z_{c\ell}(\gamma(1 + aL^{-1})) \tag{6.1}$$

with $a = 2(\lambda, \delta)/(\lambda, \lambda)$ where δ is the magic weight, λ is the basic underlying weight and (\cdot) is the Killing inner product.

Remark. 1. The lower bound in (6.1) holds even if λ is not a fundamental weight and it is possible the upper bound is true also for general $\lambda \in \mathcal{J}_d$ but at a technical point in Appendix 2, we use that λ is a fundamental weight.

2. The result as precisely stated is only true for connected G 's or representations of g 's since we use (5.3). The proof holds for irreducible representations

of G 's which are not connected so long as we take for δ in the formula to be the δ for G_0 , the connected component of G .

Proof. Let φ be a maximal weight vector for π_L and let $P(\tilde{\lambda})$ be the projection onto φ . Notice that $(\varphi, \pi(X)\varphi) = L\tilde{\lambda}(X)$ for all $X \in \mathfrak{g}$; for this is obvious if $X \in \mathfrak{h}$ and any $X \in \mathfrak{h}^\perp$ is a sum of root vectors, X_α , for which $\pi(X_\alpha)\varphi$ is orthonormal to φ (as a weight vector with distinct weight). Moreover, since $L\lambda$ is a maximal weight, any unit vector with $(\eta, \pi(X)\eta) = L\lambda(X)$ for $X \in \mathfrak{h}$ is automatically a multiple of φ since the dimension of the weight space is one. Now:

$$(\pi(x)\varphi, \pi(X)\pi(x)\varphi) = L\tilde{\lambda}(x^{-1}Xx) = L(A^*(x)\tilde{\lambda})(X)$$

so, by the above remark

$$\pi(x)P(\tilde{\lambda})\pi(x)^{-1} = P(\tilde{\lambda})$$

if and only if $A^*(x)\tilde{\lambda} = \tilde{\lambda}$. Now let $\ell \in \Gamma$ and pick $y \in G$ with $A^*(y)\tilde{\lambda} = \ell$. Then, define

$$P(\ell) = \pi(y)P(\tilde{\lambda})\pi(y)^{-1}.$$

By the above remark, $P(\ell)$ is independent of which y is chosen with $A^*(y)\tilde{\lambda} = \ell$.

Notice that

$$\text{Tr}(\pi_L(X)P(\ell)) = L\ell(X) \tag{6.2}$$

since

$$\begin{aligned} \text{Tr}(\pi_L(X)P(\ell)) &= \text{Tr}(\pi_L(X)\pi_L(y)P(\tilde{\lambda})\pi_L(y)^{-1}) \\ &= \text{Tr}(\pi_L(A(y^{-1})X)P(\tilde{\lambda})) \\ &= L\tilde{\lambda}(A(y)^{-1}X) = L(A^*(y)\tilde{\lambda})(X) \\ &= L\ell(X). \end{aligned}$$

Moreover, if γ is Haar measure on G and $\mu = d_L\tilde{\mu}$, then

$$\int_\Gamma P(\ell) d\mu(\ell) = 1 \tag{6.3}$$

since

$$\int_\Gamma P(\ell) d\mu(\ell) = d_L \int \pi(x)P(\tilde{\lambda})\pi(x)^{-1} d\gamma(x) = C$$

clearly obeys $\pi(y)C\pi(y)^{-1} = C$, so by Schur's lemma, $C = (\text{const}) 1$. Taking traces, (6.3) results.

Now, for $\ell = \{\ell_\alpha\} \in \Gamma^{|A|}$, let $P(\{\ell_\alpha\}) = \bigotimes_\alpha P(\ell_\alpha)$. By (6.3),

$$\int_{\Gamma^{|A|}} P(\ell) \bigotimes_\alpha d\mu(\ell_\alpha) = 1$$

so the $P(\ell)$'s are a family of coherent projections. By (6.2), the lower symbol of $H(\gamma S_\alpha(X_i))/L$ is exactly $H(\gamma \ell_\alpha(X_i))$ so the lower bound in (6.1) is just Theorem 2.3. By Corollary A.2.7, proven in Appendix 2, and the fact that $\ell_0 \equiv L\lambda$ is a multiple of a fundamental weight, the upper symbol of $H(\gamma S_\alpha(X_i)/L)$ is $H((1 + aL^{-1})\ell_\alpha(X_i))$. Thus the lower bound in (6.1) follows from Theorem 2.4. \square

Example 1. Lieb’s theorem [23] is a special case of this result with $G = \text{SO}(3)$. This is not surprising since for this case our proof is essentially identical to Lieb’s (except that Lieb proves Corollary A.2.7 in this case by explicit computation rather than the abstract consideration in Appendix 2).

Example 2. Let us consider the spherical harmonic representations of $\text{O}(2n)$. This fits the setup with $\lambda \equiv \lambda_1 = \omega_1$. To compute the coadjoint orbit, notice that under the association of g and g^* , ω_1 corresponds to the matrix

$$\begin{pmatrix} 0 & 1 & \begin{matrix} \parallel \\ \parallel \\ \parallel \end{matrix} \\ -1 & 0 & \parallel \\ \hline & & 0 \end{pmatrix}.$$

The image of this matrix under conjugation with arbitrary elements of $\text{O}(2n)$ is exactly

$$\Gamma = \{M \mid M^t = -M, \text{Tr}(M^t M) = 2, \text{rank}(M) = 2\}$$

i.e. decomposable two-forms or equivalently the Grassman manifold $G(n, 2)$ of 2 planes in n -space. By a direct calculation $(\lambda, \delta) = \frac{1}{2}(2n - 2)(\lambda, \lambda)$. Theorem 6.1 thus includes the result of Fuller-Lenard [9] for $\text{O}(2n)$. A similar calculation yields their result for $\text{O}(2n + 1)$.

Example 3. Let us consider the spin L spinors for $\text{O}(2n)$. To compute coadjoint orbits, notice that under the association of g and g^* , $\lambda = \frac{1}{2}(\omega_1 + \dots + \omega_n)$ corresponds to

$$\frac{1}{2} \begin{pmatrix} A & 0 & \dots & \dots & 0 \\ 0 & A & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & A \end{pmatrix}$$

with $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The image of this matrix under conjugation with arbitrary elements of $\text{O}(2n)$ is exactly

$$\Gamma = \{\frac{1}{2}M \mid M^t = -M, M^t M = M M^t = 1\}$$

i.e. $\tilde{\Gamma} = \{2K \mid K \in \Gamma\}$ is the antisymmetric, orthogonal matrices. By a direct calculation in $\text{SO}(2n)$ (see Remark 2 following the theorem),

$$\langle \delta, \lambda \rangle / \langle \lambda, \lambda \rangle = (n - 1).$$

Thus, if Z_{cl} is defined in terms of $\tilde{\Gamma}$:

$$Z_{cl} \left(\frac{\gamma}{2} \right) \leq Z_{Q, 2L}(\gamma) \leq Z_{cl} \left(\frac{1}{2} \gamma (1 + (n - 1)L^{-1}) \right)$$

for spin L spinors (which have maximal weight $2L\lambda_n$).

7. Towards a Lee-Yang Theorem for the D -Vector Model

As described in the introduction, our original motivation for extending Lieb’s limit theorem to general Lie groups concerned the Lee-Yang theorem for classical spins lying in S^{D-1} for $D \geq 4$. It appears at first sight that we have failed since S^{D-1} is never a classical limit if $D \neq 3$. However, there are classical limits which are fiber bundles over S^{D-1} :

Lemma 7.1. *Let $\tilde{\Gamma}$ be the space of Example 3 of Sect. 6. Map $\tilde{\Gamma}$ to R^{2n-1} by*

$$\tau(M) = (M_{12}, M_{13}, \dots, M_{1,2n}).$$

Then $\text{Ran } \tau = S^{2n-2}$ and the induced measure on S^{2n-2} is the rotation invariant one.

Proof. Since M is orthogonal and $M_{11} = 0$, $\tau(M) \in S^{2n-2}$. Since $\tilde{\Gamma}$ has a measure invariant under rotations leaving $(1, 0, \dots, 0)$ fixed and for such R , $\tau(RM) = R\tau(M)$, the induced measure is rotation invariant. \square

Theorem 7.2. *Consider a general quantum model with spin $\frac{1}{2}$, $O(2n)$ spinors at each site. Suppose that for a Hamiltonian:*

$$-H = \sum_{\alpha, \beta} J_{\alpha\beta} \left(\sum_{i < j} L_{ij}^\alpha L_{ij}^\beta \right) + K_{\alpha\beta} \left(\sum_{i \geq 2} L_{1i}^\alpha L_{1i}^\beta \right) + \sum \mu_\alpha L_{12}^\alpha \tag{6.2}$$

we have that $\text{Tr}(e^{-H}) \neq 0$ in the region $J_{\alpha\beta} \geq 0$, $K_{\alpha\beta} \geq 0$, $\text{Re } \mu_\alpha > 0$. Then for an arbitrary classical model with spins on S^{2n-2} and Hamiltonian

$$-H = \sum K_{\alpha\beta} S_\alpha \cdot S_\beta + \sum \mu_\alpha S_\alpha \cdot \hat{e}$$

(\hat{e} a fixed unit vector), $Z \neq 0$ if $K_{\alpha\beta} \geq 0$, $\text{Re } \mu_\alpha > 0$. If $\text{Tr}(e^{-H}) \neq 0$ remains true if a term

$$\sum_{\alpha, \beta} M_{\alpha\beta} \sum_{i=2}^{2n-1} L_{1i}^\alpha L_{1i}^\beta$$

is added to $-H$ in (6.3) with $M_{\alpha\beta} \geq 0$, then the Lee-Yang result holds for S^{2n-3} spins.

Proof. By the Griffiths trick and the isotropic coupling $J_{\alpha\beta}$ we get $\text{Tr}(e^{-H}) \neq 0$ for spin L spinors. Taking $L \rightarrow \infty$, we get a Lee-Yang theorem for $\tilde{\Gamma}$ classical spins. Now consider such spins with $J_{\alpha\beta} = 0$. Then, since the Hamiltonian is independent of L_{ij}^α , $i > j \geq 2$, we can integrate these variables out. There results a Lee-Yang theorem for S^{2n-2} spins. Using the Dunlop-Newman [8] method of going from S^3 to S^2 we can go from S^{2n-2} to S^{2n-3} if $M_{\alpha\beta}$ terms are allowed. \square

It remains to prove the spinor Lee-Yang theorem which is a kind of generalization of the Asano-Suzuki-Fisher results [2, 39]. Various simple special cases have failed to yield a counter example [22]; it seems likely the results is true but its proof may well require a more group theoretical understanding of Lee-Yang in the Asano-Suzuki-Fisher case.

Appendix 1. More on Upper and Lower Symbols

In this appendix, we establish some additional properties of the maps, U , L defined in Sect. 2. Central to our results is the fact that in some sense U and L are dual. We begin with the finite dimensional case:

Theorem A.1.1. *Let $P(x)$ be a family of coherent projections on a finite dimensional Hilbert space. Then for any $f \in L^\infty(X)$, $A \in \mathcal{L}(\mathcal{H})$, we have that*

$$\text{Tr}(AU(f)) = \int f(x)L(A)(x)d\mu(x). \tag{A.1.1}$$

Proof. Both sides equal

$$\int \text{Tr}(AP(x))f(x)d\mu(x).$$

Since $\int d\mu < \infty$, $f, L(A) \in L^\infty$ and we are taking traces of finite matrices, all the interchanges of sums and integrals are permissible. \square

Theorem A.1.2. *Let $P(x)$ be a family of coherent projections on a separable Hilbert space. Let $A \in \mathcal{S}_1$, the trace class. Then $L(A) \in L^1(X, d\mu)$ and (A.1.1) holds for any $f \in L^\infty$.*

Proof. As we noted in discussing (2.1'), for any unit vectors, φ, ψ , $(\varphi, P(x)\psi) \in L^1(X, d\mu)$ and its L^1 -norm is bounded by 1. Since A is trace class, we know [35] that A has a canonical expansion

$$A = \sum_n \mu_n(A)(\varphi_n, \cdot)\psi_n$$

with φ_n, ψ_n vectors, $\mu_n \geq 0$, and $\sum \mu_n(A) = \|A\|_1 < \infty$. Thus $\sum \mu_n(A)(\varphi_n, P(x)\psi_n)$ converges in L^1 and it converges pointwise to $L(A)$. To check (A.1.1), we note that if $B = U(f)$:

$$\begin{aligned} \text{Tr}(AB) &= \sum_n \mu_n(A)(\varphi_n, B\psi_n) \\ &= \sum_n \mu_n(A) \int f(x)(\varphi_n, P(x)\psi_n) d\mu(x) \\ &= \int f(x)L(A)(x)d\mu(x) \end{aligned}$$

where the interchange of sum and integral is permissible since

$$\sum_n \int d\mu(x) |\mu_n(A)f(x)(\varphi_n, P(x)\psi_n)| < \infty$$

on account of $\int |(\varphi_n, P(x)\psi_n)| d\mu(x) \leq 1$. \square

One consequence of the above is:

Theorem A.1.3. *Let $A \in \mathcal{S}_p$. Then $L(A) \in L^p(X, d\mu)$ and*

$$\int |L(A)(x)|^p d\mu(x) \leq \text{Tr}(|A|^p). \tag{A.1.2}$$

If $f \in L^p(X, d\mu)$, then for all $\varphi, \psi \in \mathcal{H}$,

$$\int (\varphi, P(x)\psi)f(x)d\mu(x)$$

converges and the corresponding operator, denoted $U(f)$, lies in \mathcal{S}_p and obeys:

$$\text{Tr}(|U(f)|^p) \leq \int |f|^p(x)d\mu(x). \tag{A.1.3}$$

Proof. By (2.4) and duality, (A.1.2) holds for $p = 1$. It holds for $p = \infty$ by (2.3). Thus, by interpolation, it holds for all p . (A.1.3) then follows by duality. \square

One of the most interesting consequences of duality involves the notion of completeness [19] [i.e. $\text{Ran } U$ sequentially strongly dense in $\mathcal{L}(\mathcal{H})$]:

Theorem A.1.4. *A family, $P(x)$, of coherent projections is complete if and only if $\text{Ker } L \cap \mathcal{I}_1 = \{0\}$.*

Proof. Let \mathcal{A} be the norm closure of $\mathcal{I}_\infty \cap \text{Ran } U$. Then, by duality, $\mathcal{A} = \mathcal{I}_\infty$, if and only if $\text{Ker } L \cap \mathcal{I}_1 = \{0\}$. But if $\mathcal{A} = \mathcal{I}_\infty$, then any $A \in \mathcal{L}(\mathcal{H})$ is a sequential strong limit of operators in $\text{Ran } U$ since \mathcal{I}_∞ is sequentially strongly dense in $\mathcal{L}(\mathcal{H})$. If $\mathcal{A} \neq \mathcal{I}_\infty$, there is $A \in \text{Ker } L \cap \mathcal{I}_1$. But if $B_n \in \text{Ran } U$ and $B_n \rightarrow B$ strongly, then

$$\text{Tr}(AB) = \lim \text{Tr}(AB_n) = 0.$$

So $\text{Ran } U$ is not sequentially strongly dense. \square

Remarks

1. Thus completeness is equivalent to *norm density* in \mathcal{I}_∞ .
2. Of course, if $\dim(\mathcal{H}) < \infty$, all topologies are equivalent and all subspaces closed so that $\text{Ran } U = \mathcal{L}(\mathcal{H})$ if and only if $\text{Ker } L = \{0\}$.
3. In general, $\text{Ker } U$ is very large even if $\text{Ker } L = \{0\}$, i.e. upper symbols are highly non unique. This is obviously the case if $\dim(\mathcal{H}) < \infty$ but $\dim L^\infty(X, d\mu) = \infty$.

Examples

1. Let φ_n be an orthonormal basis of \mathcal{H} , let (X, \sum, μ) be $\{1, 2, \dots\}$ with counting measure and let

$$P(n) = (\varphi_n, \cdot) \varphi_n.$$

Then $P(n)$ is coherent but not complete; $\text{Ran } U$ is obviously diagonal matrices.

2. For each $\sigma \in S^2$, the unit sphere in \mathbb{R}^3 , let $P(\sigma)$ be the projection onto that vector in C^3 with

$$(\sigma \cdot L)\eta = 0$$

where L is the spin 1 representation of $\text{SO}(3)$. Let $d\mu(\sigma)$ be the usual measure on \mathbb{R}^3 but normalized to total weight 3. Then, as in our discussion coherent projections built on maximal weight vectors:

$$\int P(\sigma) d\mu(\sigma) = 1.$$

In this case, $\text{Ker } L \neq \{0\}$. For if A is a component of the angular momentum, then $L(A) = 0$.

3. We will prove in the next appendix that coherent projections built on maximal weight vectors are complete. Seeing Example 2, one might hope that completeness picks out the maximal weight vector but alas this is not so. For example, if we take the spin 3/2 representation of $\text{SO}(3)$ and $P(\sigma)$ with $(\sigma \cdot L - \frac{1}{2})P(\sigma) = 0$, then these P 's are complete.

As a final result in the general theory, we note a result of Berezin [4] which is trivial in the finite dimensional case (Berezin's infinite dimensional result follows from Kato's strong Trotter product formula [17]).

Proposition A.1.5. *Let $\dim(\mathcal{H}) < \infty$. For any $f \in L^\infty(X, d\mu)$*

$$\exp(U(f)) = \lim_{n \rightarrow \infty} \{U[\exp(f/n)]\}^n.$$

Proof. $\exp(f/n) = 1 + \frac{f}{n} + O\left(\frac{1}{n^2}\right)$ so by the linearity and continuity of U :

$$U[\exp(f/n)] = 1 + \frac{1}{n} U(f) + O\left(\frac{1}{n^2}\right)$$

from which the result follows. \square

Appendix 2. Coherent Projections Built on Maximal Weight Vectors

In this appendix, we discuss the framework of Sect. 6 where we considered a coherent family of projections obtained from a maximal weight vector of an irreducible representation, π , of a compact, simple Lie group, G , and the specific problem of finding the upper symbol for $\pi(X)$, the image of $X \in \mathfrak{g}$ under the representation π . Let $d\mu$ denote Haar measure for G normalized so $\int d\mu = d \equiv \dim \pi$. Let $P(e)$ be the projection onto the maximal weight vector for π and for $x \in G$, let

$$P(x) = \pi(x)P(e)\pi(x)^{-1}.$$

By Schur’s lemma, as in Sect. 6 [see (6.3)]

$$\int P(x)d\mu(x) = 1. \tag{A.2.1}$$

In this appendix, we want to show that

$$\pi(X) = c \int \ell_0(\text{Ad}(x^{-1})X)P(x)d\mu(x) \tag{A.2.2}$$

where

$$c = 1 + 2(\ell_0 \cdot \ell_0)^{-1}(\ell_0 \cdot \delta) \tag{A.2.3}$$

with ℓ_0 the maximal weight in π and δ the “magic” weight. We will only succeed in proving (A.2.2/3) in case ℓ_0 is a multiple of a single fundamental weight. In general, our method below will show that if $\ell_0 = \sum_{i \in J} n_i \lambda_i$ ($n_i > 0$, $J \subset \{1, \dots, r\}$, λ_i fundamental weights), then (A.2.2) holds with $c\ell_0$ replaced by some $\tilde{\ell}$ obeying (i) $\tilde{\ell}$ is in the span of the $\{\lambda_i\}_{i \in J}$ and (ii) $\ell_0 \cdot \tilde{\ell} = \ell_0 \cdot \ell_0 + 2\ell_0 \cdot \delta$. Once one establishes (A.2.2), one notes that $\ell_0(\text{Ad}(x^{-1})X)$ and $P(x)$ only depend on the coset of x modulo the isotropy group of ℓ_0 so that the integral over the group in (A.2.2) can be replaced by an integral over the coset space \equiv coadjoint orbit yielding the formula we used in Sect. 6.

Lemma A.2.1. *If A is an operator on the representation space for π with $\text{Tr}(AP(x)) = 0$ for all x , then $A = 0$ (i.e. the kernel of the map L is zero).*

Proof. (cf. Klauder [19]). Let φ be a maximal weight vector. Then $\text{Tr}(AP(x)) = 0$ for all x is equivalent to

$$(\varphi, \pi(x)A\pi(x)^{-1}\varphi) = 0 \tag{A.2.4}$$

for all $x \in G$. Taking derivatives n times at $x = 0$, (A.2.4) implies

$$(\varphi, [[\pi(x_1), [\pi(x_2), \dots, A]] \dots] \varphi) = 0 \tag{A.2.5}$$

for all $x_i \in g$ and, so by linearity, for all $x_i \in g_c$, the complexification of g . In g_c there are special elements $\{X_\alpha\}_{\alpha \in P_+}$ and $\{X_{-\alpha}\}_{\alpha \in P_+}$, root elements (see [30]) with the following properties.

(i) $\pi(X_\alpha)\varphi = 0$.

(ii) $\{\pi(X_{-\alpha_1}) \dots \pi(X_{-\alpha_n})\varphi \mid \text{all } \alpha_i \in P_+, \text{ all } n\}$ span the underlying representation space.

(iii) $\pi(X_\alpha)^* = \pi(X_{-\alpha})$.

In (A.2.5) take all X_i to be X_α 's and use (i) and (iii) to conclude

$$(\pi(X_{-\alpha_n}) \dots \pi(X_{-\alpha_1})\varphi, A\varphi) = 0$$

so by (ii), $A\varphi = 0$. But (A.2.4) for A implies it also for $\pi(y)A\pi(y)^{-1}$ so the same argument shows that $A\pi(y)^{-1}\varphi = 0$ and thus $A = 0$. \square

Remark. In the context of Sect. 3, a similar result holds. $\psi_h(0, 0)$ plays the role of a “minimal” weight vector and $\text{Ker } L = \{0\}$.

This lemma and the general results of Appendix 1, show that any A is of the form $U(f)$ for some f . We can say much more:

Theorem A.2.3. *Let V be a second irreducible representation of G of dimension m . Let A_1, \dots, A_m for m operators obeying*

$$\pi(y)A_i\pi(y)^{-1} = \sum_j V_{ji}(y)A_j.$$

Then there exist functions f_1, \dots, f_n on G so that

$$A_i = \int f_i(x)P(x)d\mu(x)$$

and

$$f_i(y^{-1}x) = \sum_j V_{ji}(y)f_j(x).$$

Proof. As noted already, there exist g_i so that

$$A_i = \int g_i(x)P(x)d\mu(x).$$

Next notice, that if $(L_y h)(x) = h(y^{-1}x)$, then

$$\begin{aligned} U(L_y h) &= \int h(y^{-1}x)P(x)d\mu(x) \\ &= \int h(x)P(yx)d\mu(x) \\ &= \pi(y)L(h)\pi(y)^{-1}. \end{aligned}$$

Thus

$$U(L_y g_i) = \sum_j V_{ji}(y)A_j.$$

Multiplying by $\overline{V_{jk}(y)}$ and integrating, we see that

$$A_j = U(f_j)$$

with $f_j = \int \overline{V_{jk}(y)}L_y g_i$. The f 's then transform under V the right way by general principles. \square

Now let A be the adjoint representation, and let B be a map from g to operators which is linear and obeys

$$\pi(x)B(X)\pi(x)^{-1} = B(A(x)X). \tag{A.2.6}$$

By the above and the Peter-Weyl theorem [which implies that the only functions on G which transform under L_y as A are of the form $\ell(A(\cdot)^{-1}X)$] we conclude that

$$B(X) = C_\ell(X) \tag{A.2.7}$$

for some $\ell \in g^*$ where

$$C_\ell(X) = \int \ell(A(x^{-1})X)P(x)d\mu(x). \tag{A.2.8}$$

Theorem A.2.3. $\pi(X)$ is of the form (A.2.8) for some $\ell \in g^*$ with

$$\ell \cdot \ell_0 = \ell_0 \cdot \ell_0 + 2\ell_0 \cdot \delta. \tag{A.2.9}$$

Proof. By the above discussion $\pi(X)$ has the form (A.2.8). Thus, we need only evaluate (A.2.9). For later purposes, we note that the calculation of (A.2.9) follows for any ℓ with $\pi(X) = C_\ell(X)$. Our calculation follows that in Fuller-Lenard [9]. [These authors use a Wigner-Eckardt theorem in the spherical harmonic case to conclude that $\pi(X) = \alpha C_\ell(X)$ for a constant α . There is a gap in their proof corrected in an erratum.]

Notice first that

$$\text{Tr}(P(e)\pi(X)) = \ell_0(X)$$

so

$$\ell_0(X) = \int \ell(A(x^{-1})X) \text{Tr}(P(e)P(x))d\mu(x). \tag{A.2.10}$$

On the other hand, if X_i is an orthonormal basis (in Killing form inner product), then, since the eigenvalue of the Casimir operator is $\ell_0 \cdot \ell_0 + 2\ell_0 \cdot \delta$ [see (5.3)]

$$\begin{aligned} (\dim \pi)(\ell_0 \cdot \ell_0 + 2\ell_0 \cdot \delta) &= \text{Tr} \left(\sum_i \pi(X_i)\pi(X_i) \right) \\ &= \iint \sum_i \ell(A(x^{-1})X_i)\ell(A(y^{-1})X_i) \\ &\quad \cdot \text{Tr}(P(x)P(y))d\mu(x)d\mu(y) \\ &= \int \ell(A(y^{-1}x)^{-1}X_\ell) \text{Tr}(P(y^{-1}x)P(e))d\mu(x)d\mu(y) \\ &= (\dim \pi)\ell_0(X_\ell) = (\dim \pi)\ell_0 \cdot \ell \end{aligned}$$

where X_ℓ is the element of g with $\tilde{\ell}(X_\ell) = \tilde{\ell} \cdot \ell$ for all $\tilde{\ell} \in g^*$ and we have used

$$\sum_i \ell_1(X_i)\ell_2(X_i) = \ell_1 \cdot \ell_2$$

$$\int d\mu(z) = \dim \pi,$$

$$\text{Tr}(P(x)P(y)) = \text{Tr}(P(x^{-1}y)P(e))$$

and (A.2.10) with $X = X_\ell$. \square

The following result asserts more-or-less that the only part of the functionals that count are the parts which lift to coadjoint orbits (i.e. average over isotropy subgroup).

Theorem A.2.4. Let C_ℓ be given by (A.2.8). Let x be any element of G with $A^*(x)\ell_0 = \ell_0$. Then

$$C_{A^*(x)\ell}(X) = C_\ell(X). \tag{A.2.10}$$

Proof. Clearly by the uniqueness of maximal weight vectors $U(x)^{-1}P(e)U(x) = P(e)$. Thus

$$P(yx^{-1}) = P(y)$$

for any y . Thus

$$\begin{aligned} C_\ell(X) &= \int \ell(A(y)^{-1}X)P(yx^{-1})d\mu(y) \\ &= \int \ell(A(yx)^{-1}X)P(y)d\mu(y) \\ &= \int (A^*(x)\ell)(A(y)^{-1}X)P(y)d\mu(y) \\ &= C_{A^*(x)\ell}(X). \quad \square \end{aligned}$$

Corollary A.2.5. Let C_ℓ be given by (A.2.8). Let P be the projection from g^* to h^* , those elements of g^* which are zero on h^\perp , the orthogonal complement of the Cartan algebra. Then $C_\ell(X) = C_{P\ell}(X)$.

Proof. Let T be the maximal torus, i.e. $\exp(h)$. Then $x \in T$, and φ a maximal weight vector imply $U(x)\varphi = e^{i\alpha(x)}\varphi$. $A^*(x)\ell_0 = \ell_0$. Thus if dv_T is normalized Haar measure on T , then, by (A.2.10)

$$C_\ell(X) = C_{Q\ell}(X)$$

where

$$Q\ell = \int_T (A^*(x)\ell)dv_T(x).$$

Since the duals of the root elements span h^\perp ([30]), it is easy to see that $Q = P$. \square

Corollary A.2.6. Let $\lambda_1, \dots, \lambda_r$ be the fundamental weights. Let $\ell_0 = \sum n_i \lambda_i$ and let $J = \{i | n_i > 0\}$. Let \mathcal{R} be the projection in h^* onto the span of $\{\lambda_i | i \in J\}$. Then

$$C_{\mathcal{R}\ell}(X) = C_\ell(X)$$

for any $\ell \in h^*$.

Proof. Let T be an element of the Weyl group leaving ℓ_0 fixed. By general structure theory [30]; (i) T leaves each $\lambda_i, i \in J$ fixed and so $\text{Ran } \mathcal{R}$. (ii) There is $y \in G$ with $T = A^*(y)$ and $A^*(y)\ell_0 = \ell_0$. (iii) If $W_{\ell_0} \equiv$ subgroup of the Weyl group leaving ℓ_0 fixed, then

$$\mathcal{R} = \left(\sum_{T \in W_{\ell_0}} T \right) / \#(W_{\ell_0}).$$

By (ii) $C_{T\ell}(X) = C_\ell(X)$ for any $T \in W_{\ell_0}$ so by (iii), the result is proven. \square

Corollary A.2.7. If ℓ_0 is a multiple of a fundamental weight, then

- (i) $\pi \otimes \bar{\pi}$ contains A exactly once.
- (ii) $\pi(X) = c \int \ell_0(\text{Ad}(x^{-1})X)P(x)d\mu(X)$ with c given by (A.2.3).

Proof. By the discussion before Theorem A.2.3, the number of times A occurs in $\pi \otimes \bar{\pi}$ is the dimension of ℓ 's with $C_\ell \neq 0$. By the last two corollaries, for any ℓ ,

$C_\ell = C_{\alpha\ell_0}$ for some α . Thus, there is at most one A . Since $\pi(X)$ transforms as A , there is exactly one and $\pi(X) = C_{\alpha\ell_0}(X)$ for some α . α is evaluated by (A.2.9) yielding (A.2.3). \square

Remarks

1. In general, our proof shows that the number of times A occurs in $\pi \otimes \bar{\pi}$ is at most $\#(J)$ with J given in Corollary A.2.6.

2. It follows [21] from general results of PRV [25] that the number of times A occurs in $\pi \otimes \bar{\pi}$ is precisely $\#(J)$ times. In particular, (i) of the last corollary is not new.

Finally, we want to note a formula of Gilmore [12] (whose proof is unnecessarily complicated). Given a dominant weight ℓ , let π_ℓ be the corresponding representation on G and let $P_\ell(e)$ be the projection onto the corresponding maximal weight vector, ψ_ℓ . Define the function F_ℓ on G by:

$$F_\ell(x) \equiv \text{Tr}(P_\ell(e)\pi_\ell(x)) = (\psi_\ell, \pi_\ell(x)\psi_\ell)$$

so that

$$\text{Tr}(P_\ell(y)P_\ell(e)) = |F_\ell(y)|^2.$$

Proposition A.2.8 ([12]). *For any $\ell, \tilde{\ell}$:*

$$F_{\ell+\tilde{\ell}}(x) = F_\ell(x)F_{\tilde{\ell}}(x).$$

In particular, if $\lambda_1, \dots, \lambda_r$ are fundamental weights, then

$$F_{\sum n_i \lambda_i}(x) = r \prod_{i=1}^r [F_{\lambda_i}(x)]^{n_i}.$$

Proof. In $\pi_\ell \otimes \pi_{\tilde{\ell}}$, the vector $\psi_\ell \otimes \psi_{\tilde{\ell}}$ is a maximal weight vector with weight $\ell + \tilde{\ell}$. The cyclic subspace it generates is precisely $\pi_{\ell+\tilde{\ell}}$ so

$$\begin{aligned} F_{\ell+\tilde{\ell}}(x) &= (\psi_\ell \otimes \psi_{\tilde{\ell}}, \pi_\ell(x) \otimes \pi_{\tilde{\ell}}(x) \psi_\ell \otimes \psi_{\tilde{\ell}}) \\ &= F_\ell(x)F_{\tilde{\ell}}(x). \quad \square \end{aligned}$$

The point of this result is that it implies a classical limit result for a special case for general sequences $\ell, 2\ell, \dots$ (Gilmore [12] notes a closely related result) and this suggests that despite our ability to only the classical limit along fundamental sequences, it may hold more generally.

Theorem A.2.9. *Let ℓ be any dominant weight and let Γ be the corresponding coadjoint orbit with $d\tilde{\mu}$ the corresponding normalized measure. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{d_n} \text{Tr}(\exp([\pi_{n\ell}(X)/n]) = \int_{\Gamma} d\tilde{\mu}(\tilde{\ell}) \exp(\tilde{\ell}(X)) \tag{A.2.11}$$

for any $X \in \mathfrak{g}$, the complication of \mathfrak{g} where $d_n = \dim \pi_{n\ell}$.

Proof. For any A :

$$\frac{1}{d_n} \text{Tr}(A) = \int d\gamma(x) (\pi_{n\ell}(x)\psi_{n\ell}, A\pi_{n\ell}(x)\psi_{n\ell})$$

with $d\gamma$ normalized Haar measure. Thus, using Proposition A.2.8:

$$\begin{aligned} \text{LHS of (A.2.11)} &= \lim_{n \rightarrow \infty} \int F_{n\ell}(\exp(x^{-1}Xx/n))d\gamma(x) \\ &= \lim_{n \rightarrow \infty} \int [F_\ell(\exp(x^{-1}Xx/n))]^n d\gamma(x) \\ &= \lim_{n \rightarrow \infty} \int (\psi_\ell, \pi_\ell[\exp(x^{-1}Xx/n)]\psi_\ell)^n d\gamma(x) \\ &= \int \exp(\psi_\ell, \pi_\ell(x^{-1}Xx)\psi_\ell) d\gamma(x) \\ &= \int \exp(\ell[\text{Ad}(x^{-1}X)])d\gamma(x) \\ &= \text{RHS of (A.2.11)}. \quad \square \end{aligned}$$

Appendix 3. On Spinor Representatives

For the reader's convenience, we define and sketch some properties of the spinor representations of $O(2n)$. The basic $\frac{1}{2}$ spinors are defined in terms of operators $\sigma_1, \dots, \sigma_{2n}$ obeying

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} \mathbb{1}. \tag{A.3.1}$$

There exist such operator on C^m with $m = 2^n$ and every set of such σ 's is unitarily equivalent to a direct sum of this special set. In terms of the 2×2 Pauli matrices, τ_1, τ_2, τ_3 and a tensor decomposition $C^m = C^2 \otimes \dots \otimes C^2$ (n times), we can take:

$$\begin{aligned} \sigma_1 &= \tau_1 \otimes 1 \otimes \dots \otimes 1 \\ \sigma_2 &= \tau_2 \otimes 1 \otimes \dots \otimes 1 \\ \sigma_3 &= \tau_3 \otimes \tau_1 \otimes \dots \\ \sigma_4 &= \tau_3 \otimes \tau_2 \otimes 1 \otimes \dots \otimes 1 \\ \sigma_5 &= \tau_3 \otimes \tau_3 \otimes \tau_1 \otimes \dots \end{aligned}$$

The operators, $L_{ij} = \frac{i}{2} \sigma_i \sigma_j$ ($i < j$) obey the commutation relations of $SO(2n)$ and define a representation of $Spin(2n)$, the two fold cover of $SO(2n)$. It is not irreducible but if one adds parity and asks for a representation of $pin(2n)$ [the group related to $Spin(2n)$ as $O(2n)$ is related to $SO(2n)$], one obtains an irreducible representation.

If one takes the conventional Cartan subalgebra generated by $L_{1,2}, L_{3,4}, \dots, L_{2n-1,2n}$ one can read off the weights since then

$$\begin{aligned} L_{1,2} &= \tau_3 \otimes \dots \otimes 1 \\ L_{3,4} &= 1 \otimes \tau_3 \otimes \dots \otimes 1 \end{aligned}$$

are diagonal. The weights are $\pm \frac{1}{2} \omega_1 \pm \dots \pm \frac{1}{2} \omega_n$ so that the maximal weights are $\frac{1}{2} \omega_1 + \frac{1}{2} \omega_2 + \dots + \frac{1}{2} \omega_{n-1} \pm \frac{1}{2} \omega_n$ which are two fundamental weights λ_{n-1} and λ_n (Warning: do not confuse the direct sum of π_{λ_n} and $\pi_{\lambda_{n-1}}$ with $\pi_{\lambda_n + \lambda_{n-1}}$). The spin L spinors, $L = 1, 3/2, 2, 5/2, \dots$ are defined to be the direct sum of $\pi_{2L\lambda_n}$ and $\pi_{2L\lambda_{n-1}}$ which is again irreducible for $pin(2n)$. Alternatively, it is the subspace of the $2L$ -fold tensor product of the spin $1/2$ spinors where $\sum_{i < j} L_{ij}^2$ has its maximum value.

By Weyl's dimension formula [30], the dimension $d(n, L)$ of the spin L spinors is given by:

$$d(n, L) = 2 \left[\prod_{0 \leq i < j \leq n-1} (i+j+2L)/(i+j) \right].$$

For example:

$$d(2, L) = 2(2L+1)$$

$$d(3, 2) = 2 \binom{2L+3}{3}$$

$$d(4, L) = 2(2L+3)3^{-1} \binom{2L+5}{5}$$

$$d(5, L) = 2(60)^{-1}(2L+3)(2L+4)(2L+5) \binom{2L+7}{7}$$

$$d(n, \frac{1}{2}) = 2^n$$

$$d(n, 1) = 2 \binom{2n-1}{n-1}$$

$$d(n, \frac{3}{2}) = 2^n \binom{2n}{n} (n+1)^{-1}.$$

The value of the Casimir operator $L^2 = \sum_{i < j} L_{ij}^2$ in this representation is

$$\binom{2n}{n} L(L+n-1)(2n-1)^{-1}.$$

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