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Coupling Constant Thresholds in Nonrelativistic Quantum Mechanics. I. Short-Range Two-Body Case

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Consider $-\Delta + \lambda V$ with V short range at a value λ_0 where some eigenvalue $e(\lambda) \to 0$ as $\lambda \downarrow \lambda_0$. We analyze two questions: (i) What is the leading order of $e(\lambda)$, i.e., for what α does $e(\lambda) \sim c(\lambda - \lambda_0)^\alpha$? (ii) Is $e(\lambda)$ analytic at $\lambda = \lambda_0$ and, if not, what is the natural expansion parameter? The results are highly dimension dependent.

1. INTRODUCTION

In this paper we will consider a family of Schrödinger operators, $H = -\Delta + \lambda V$, or more generally, $-\Delta + W + \lambda V$, where V (and W) go to zero at infinity fairly rapidly (in fact, for simplicity we suppose $V \leqslant 0$ and V vanishes outside some sphere for much of the paper). H has continuous spectrum $[0, \infty)$ and some discrete negative eigenvalues, $e_i(\lambda)$. Fix some value of i and consider the λ dependence of $e_i(\lambda)$. In the region $e_i(\lambda) < 0$, the problem is well studied with rigorous results going back to the celebrated work of Rellich [14], Kato [7] and Sz.-Nagy [25]: the $e_i(\lambda)$ are analytic (if e_i is taken to be strictly the ith eigenvalue, then due to level crossing, this is not strictly true, but one can always label the eigenvalues for λ near a fixed λ_0 to get strict analyticity). Here we want to consider a situation where as $\lambda \downarrow \lambda_0$, some eigenvalue $e_i(\lambda) \uparrow 0$, i.e., as $\lambda \downarrow \lambda_0$ an eigenvalue is absorbed into continuous spectrum. Conversely, as $\lambda \uparrow \lambda_0 + \epsilon$, the continuous spectrum "gives birth" to a new eigenvalue. For obvious reasons, we call this phenomenon a "coupling constant threshold."

This paper is the first in a series. In later papers, we intend to say something about long-range and also certain multiparticle cases. Two questions will concern us: (i) Is $e(\lambda)$ analytic at $\lambda = \lambda_0$ and if singular does it have a convergent expansion in some singular quality like $(\lambda - \lambda_0)^\alpha$? (ii) What is the rate at which $e(\lambda)$ approaches 0? In general we discuss here $-\Delta + \lambda V$ for $x \in R^\nu$. There will be considerable ν dependence.

The problem of analyzing perturbation of discrete eigenvalues of $-\Delta + \lambda V$ is solved by thinking of the problem as a special case of the more general family of

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abstract problems $A + \lambda B$. Thus, we begin with a few remarks about very general problems of this sort. We know of only one thing one can say in this great generality.

THEOREM 1.1. *Let $A + \lambda B$ be a family of self-adjoint operators with the following properties for $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$*

(i) *$D(A + \lambda B)$, the domain of $A + \lambda B$ is independent of λ,*

(ii) *$\sigma_{ess}(A + \lambda B)$, the continuous spectrum of $A + \lambda B$, is $[0, \infty)$ independently of λ.*

Let $\mu_1(\lambda) \leqslant \mu_2(\lambda) \leqslant \cdots$ denote the negative eigenvalue of $A + \lambda B$ (counting multiplicity) with the convention that $\mu_n(\lambda) = 0$ if there are at most $n - 1$ negative eigenvalues. Then for any λ_0,

$$\lim_{\lambda \downarrow \lambda_0} [\mu_n(\lambda) - \mu_n(\lambda_0)]/(\lambda - \lambda_0)$$

exists and is finite. In particular, if $\mu_n(\lambda) \sim C(\lambda - \lambda_0)^\alpha$ then $\alpha \geqslant 1$.

Proof. Let $F_k(\lambda) = \mu_1(\lambda) + \cdots + \mu_k(\lambda)$. F_k is the bottom of the spectrum of the k-fermion operator with k independent fermions each with Hamiltonian $A + \lambda B$. Since this is linear in λ, $F_k(\lambda)$ is concave by the minimum principle. Thus

$$\lim_{\lambda \downarrow \lambda_0} [F_k(\lambda) - F_k(\lambda_0)]/(\lambda - \lambda_0)$$

exists, so by induction the theorem is proven. ∎

The other general result is due to Simon [20]; we state it for simplicity in the nondegenerate case even though the result is more general:

THEOREM 1.2. *Let A be an operator with $\operatorname{spec}(A) = [0, \infty)$ and $\operatorname{Ker} A = \{0\}$. Let B be an A-compact operator. Suppose that as $\lambda \downarrow \lambda_0$, a unique negative eigenvalue, $e(\lambda)$, approaches zero. Then*

$$\lim_{\lambda \downarrow \lambda_0} (\lambda - \lambda_0)^{-1} e(\lambda) \neq 0$$

if and only if 0 is an eigenvalue of $A + \lambda_0 B$.

It is hard to imagine saying much more than this in an abstract setting, for consider the following:

EXAMPLE. Let $\mathscr{H} = L^2(R, dx)$ and let k be the operator $k = -id/dx$ and let

$$H_0 = |k|^\beta$$

for $\frac{1}{2} < \beta < 1$. Then a simple analysis shows that the integral kernel $K_\beta(x, y; E)$ of $(H_0 - E)^{-1}$ obeys

$$K_\beta(x, y; E) = c_1 |x - y|^{-(1-\beta)} + c_2(-E)^{(1-\beta)/\beta} + o((-E)^{(1-\beta)/\beta})$$

as $E \uparrow 0$ for $c_1, c_2 > 0$. The methods we develop in this paper prove that if $V \in C_0^\infty(R)$, $V \leqslant 0$ and if $e(\lambda)$ is an eigenvalue of $H_0 + \lambda V$ with $e(\lambda) \uparrow 0$ at $\lambda \downarrow \lambda_0$, then

$$e(\lambda) \sim c_3(\lambda - \lambda_0)^{\beta/1-\beta}.$$

As β runs from $\frac{1}{2}$ to 1, $\beta/1 - \beta$ runs from 1 to ∞ skowing that any power consistent with Theorem 1.1 is possible.

As a result, the analysis at coupling constant thresholds must depend on detailed features of the operator $H_0 = -\Delta$ and its relation to V. This will be seen by the rather strong dependence on dimension we will discuss below.

We should say something about previous work on the subject. Some partial results in $\nu = 3$ dimensions were obtained by Rauch [12], whose work, in parts, motivated ours here. Without being explicit, we should mention that Rauch proves that there are only three possibilities that may occur for $e(\lambda)$ as $\lambda \downarrow \lambda_0$. He does not discuss which actually do occur nor how to simply determine which occurs although he does conjecture that one possibility (his case (i)) is generic. We will show that one of his possibilities (his case (ii)) *never* occurs and, indeed, that his case (i) is generic (Section 2). However, this genericity is rather special: Rauch's abstract arguments hold in $\nu = 5, 7, 9,\ldots$, dimensions but then his "generic" case also *never* occurs and only his case (iii) occurs.

The discussion of Newton [10] for central potentials is illuminating as to the possibilities. The discussions of scattering thresholds in Newton [11] and Jenson and Kato [6] have some bearing on our analysis since in both cases the low-energy behavior of Green's functions is crucial. We also note that there is previous work [2, 4, 9, 19] on the special case of the threshold at $\lambda_0 = 0$ which occurs in $\nu = 1$ and $\nu = 2$ dimensions.

Our results are summarized in Table I, where * indicates the convergent expansion

$$\sum_{n \geqslant 2; m \geqslant 0} c_{nm}(\lambda - \lambda_0)^{n/2}[(\lambda - \lambda_0)^{1/2} \ln(\lambda - \lambda_0)]^m$$

and † indicates the convergent expansion

$$\sum_{n \geqslant 2} c_{nmk}[-(\lambda - \lambda_0)/\ln(\lambda - \lambda_0)]^{n/2}[-1/\ln(\lambda - \lambda_0)]^m\{-\ln[\ln(\lambda - \lambda_0)^{-1}]/\ln(\lambda - \lambda_0)\}^k.$$

The meaning of the columns in the table is the following: in dimension $\nu \geqslant 5$, there is universal behavior (different in odd and even dimension) at each threshold but in 1, 2, 3, 4 dimensions there are several different behaviors. The numbers of possible behaviors is listed in column 2 and then we label the types in column 3. Type 1 is always "generic" in the sense that it will occur unless a certain integral involving V and the solution of $(-\Delta + V)u = 0$ vanishes: Type 2 is generic in those cases where this integral vanishes in the sense that type 3 only occurs if two additional integrals vanish. The next two columns are self-explainatory: we note that $e(\lambda)$ is only analytic in $\nu = 1, 3$ dimensions (type 1 only in 3 dimensions). The column marked "Max. mult." (maximum multiplicity) indicates the following: as $\lambda \downarrow \lambda_0$, it

TABLE I

Dimension	No. of types	Type	Leading order	Convergent expansion	Max. mult.	Aug. mom.
5, 7, 9,...	1	1	$(\lambda - \lambda_0)$	$\sum_2^\infty a_n(\lambda - \lambda_0)^{n/2}$	∞	$l > 0$
6, 8, 10,...	1	1	$(\lambda - \lambda_0)$	*	∞	$l > 0$
3	2	1	$(\lambda - \lambda_0)^2$	$\sum_2^\infty b_n(\lambda - \lambda_0)^n$	1	$l = 0$
		2	$(\lambda - \lambda_0)$	$\sum_2^\infty a_n(\lambda - \lambda_0)^{n/2}$	∞	$l > 1$
4	2	1	$(\lambda - \lambda_0)/\ln(\lambda - \lambda_0)$	†	1	$l = 0$
		2	$(\lambda - \lambda_0)$	*	∞	$l > 1$
1	1	1	$(\lambda - \lambda_0)^2$	$\sum_2^\infty b_n(\lambda - \lambda_0)^n$	1	$l = 0, 1$
2	3	1	$\exp(-c/\lambda - \lambda_0)$		1	$l = 0$
		2	$(\lambda - \lambda_0)/\ln(\lambda - \lambda_0)$	†	2	$l = 1$
		3	$(\lambda - \lambda_0)$	*	∞	$l > 2$

Note. See text for explanation of table.

might happen that several eigenvalues approach 0 at once. The symbol 1 in the max. mult. column for $\nu = 3$, type 1 indicates that if k eigenvalues approach 0 at λ_0, at most 1 is in type 1 and $k - 1$ (or k) must be in type 2. The last column indicates what happens in the *special* case where V is spherically symmetric. Then just knowing the angular momentum suffices to determine the type.

We emphasize that the table gives the behavior for thresholds $\lambda_0 \neq 0$. In one and two dimensions, where λ_0 can equal zero, there is an *additional possibility*; namely in one dimension $e(\lambda) \sim c\lambda^4$ and in two dimensions $e(\lambda) \sim \exp(-c/(\lambda - \lambda_0)^2)$; see [19].

We note that some relations between different dimensions must occur since the radial equation for angular momentum l (= degree l spherical harmonics) in ν-dimensions is

$$-u'' + L(L + 1)\, r^{-2}u + Vu$$

with $L = l + \frac{1}{2}(\nu - 3)$. Thus $l = 0$ in $\nu = 5$ dimensions must be the same as $l = 1$ in $\nu = 3$ dimensions.

We should also remark on the fact that "non-generic" behavior always occurs for spherically symmetric potentials. This is not surprising: the integrals that have to vanish for non-generic behavior happen to vanish for reason of symmetry.

Finally, we should remark on the complicated nature of the results summarized above; if one writes λ as a function of e rather than e as a function of λ things are much simpler. The complications enter in the inversion process. Indeed, we will initially solve for λ as a function of e.

We end this introduction with a summary of the contents: in Section 2 we treat $\nu = 3$ dimensions. We will give full details here and then skimp on some in the later sections. In Section 3, we treat $\nu = 5, 7, 9, \ldots$; in Section 4, $\nu = 1$; in Section 5, $\nu = 6, 8, \ldots$; in Section 6, $\nu = 4$; in Section 7, $\nu = 2$, which is by far the most complex dimension. For simplicity of exposition, we assume $V \leqslant 0$, $V \in C_0^\infty(R^\nu)$ in Sections 2–7. In Section 8, we discuss what happens if V is not non-positive and in Section 9, what happens if V does not have compact support. In Section 10, we solve a problem raised by Newton in [11] which is related to when type 1 behavior occurs in 3 dimensions. Finally in Section 11, we say a few words about $-\Delta + W + \lambda V$.

2. $\nu = 3$

We consider $-\Delta$ on $L^2(R^3)$ and a potential $V \in C_0^\infty(R^3)$ (the smoothness will play no real role; the boundedness only a minimal role and the compact support as opposed to exponential falloff only a minimal role but we wish to ignore inessential technicalities). We also suppose $V \leqslant 0$. We study

$$(-\Delta + \lambda V)u = eu \tag{2.1}$$

with $e < 0$, $\lambda > 0$, we note that if $w = |V|^{1/2}u$, then (2.1) holds if and only if

$$K_e w = \lambda^{-1} w, \tag{2.2}$$

where

$$K_e = |V|^{1/2}(-\Delta - e)^{-1}|V|^{1/2}. \tag{2.3}$$

Explicitly, if $e = -\alpha^2$, K_e has integral kernel

$$K_e(x, y) = e^{-\alpha|x-y|}(4\pi\,|x - y|)^{-1}\,|V(x)|^{1/2}\,|V(y)|^{1/2}. \tag{2.4}$$

The reduction of the bound-state problem to a factorized homogenous Lippmann–Schwinger equation is a standard device going back at least to Birman [3] and Schwinger [16] (see [18] for further discussion).

The point is that since $V \in C_0^\infty$, the integral kernel of (2.4) is square integrable for *any* $\alpha \in C$ and thus it defines a Hilbert–Schmidt operator, L_α for all α. Moreover L_α is obviously analytic in α so it is an entire analytic family. Since it has discrete spectrum away from zero and is self-adjoint for α real, we have, by general theory [8, 5, 15, 13]:

PROPOSITION 2.1. *If μ_0 is a non-zero eigenvalue of L_{α_0} (α_0 real) of multiplicity k, then for α near α_0, L_α has exactly k eigenvalues near μ_0, $\mu_1(\alpha), \ldots, \mu_k(\alpha)$ each analytic in α near $\alpha = \alpha_0$.*

Now let $e(\lambda)$ be an eigenvalue of $-\Delta + \lambda V$ with $e(\lambda) \to 0$ as $\lambda \downarrow \lambda_0$. Then, clearly, λ_0^{-1} is an eigenvalue of $K_0 = L_0$ and the multiplicity, m, of λ_0^{-1} as an eigenvalue exactly equals the number of eigenvalues (counting multiplicity) which are absorbed at $\lambda = \lambda_0$. Let $\mu_0 = \lambda_0^{-1}$. By the proposition $\mu(\alpha)$, and hence, $\lambda(\alpha) \equiv \mu(\alpha)^{-1}$ is analytic near $\alpha = 0$. All the complications must come from inverting to solve for α as a function of λ and then using $e = -\alpha^2$. We therefore write

$$\mu(\alpha) = \mu_0 + a\alpha + b\alpha^2 + \cdots \tag{2.5}$$

and study the coefficients a and b in detail. The following can be viewed as the main technical result in three dimensions:

THEOREM 2.2. *Let ϕ obey $L_0\phi = \mu_0\phi$. Then*

(i) $a = -(\phi, |V|^{1/2})^2/4\pi$,

(ii) *If* $a = 0$, *then* $b < 0$,

(iii) *If* μ_0 is degenerate, then the coefficients a_i of $\mu_i(\alpha) = \mu_0 + a_i\alpha + b_i\alpha^2$ *are either zero for all i or zero for all i but one.*

(iv) *If λ_0 is the absorption point for the ground state (smallest eigenvalue), then μ_0^{-1} is a simple eigenvalue of L_0 and the corresponding a is non-zero.*

Proof. We expand L_α about $\alpha = 0$:

$$L_\alpha = L_0 + \alpha A + \alpha^2 B + O(\alpha^3). \tag{2.6}$$

Then, by (2.4):

$$A = -(4\pi)^{-1}(|V|^{1/2}, \cdot)|V|^{1/2},$$
$$(Bf)(x) = (4\pi)^{-1}|V(x)|^{1/2}\int |x-y||V(y)|^{1/2}f(y)\,dy. \tag{2.7}$$

Conclusion (i) is just first-order perturbation theory and conclusion (iii) follows from degenerate perturbation theory if one notes that A is rank 1.

Now suppose that $a = 0$. Since A is rank 1, $A\phi = 0$ so that the αA term contributes only in fifth order of perturbation theory [8, pp. 77–78]. Thus, *when* $a = 0$ we have that

$$b = (\phi, B\phi)$$
$$= (4\pi)^{-1}\int f(x)|x-y|f(y)\,dx\,dy$$

with $f(x) = |V(x)|^{1/2}\phi(x)$. Since $a = 0$, $\int f(x)\,dx = 0$. Therefore $b < 0$ since $|x-y|$ is conditionally strictly negative definite (to see this one notes that if $\hat{g}(0) = 0$, then

$$\int g(x)|x-y|g(y)\,dx\,dy = -8\pi\int|\hat{g}(k)|^2 k^{-4}\,d^3k$$

is negative). This proves conclusion (ii).

To prove (iv) notice that the ground state μ_0 corresponds to the largest eigenvalue of L_0. Since L_0 has a strictly positive kernel on L^2 (support (V)), this eigenvalue is simple and the corresponding ϕ is strictly positive on supp V. Thus $(\phi, |V|^{1/2}) \neq 0$ so $a \neq 0$. ∎

Remarks. 1. There is anothere is another illuminating way of proving (ii). If $a = b = 0$, then $\mu(\alpha) = \mu_0 + c\alpha^m + \cdots$ for $m \geqslant 3$ so solving for e as a function of λ, $e(\lambda) = d(\lambda - \lambda_0)^{2/m} + \cdots$ violating Theorem 1.1.

2. That $a \neq 0$ for the ground state was trivial since $V \leqslant 0$. In Section 9, we will prove the analogous fact for any $V \in C_0^\infty$.

By inverting the relations $e = -\alpha^2$, $\mu = \lambda^{-1}$, we can immediately read off the major properties of $e(\lambda)$.

THEOREM 2.3. *Let λ_0 be a value of coupling constant at which a unique non-degenerate eigenvalue $e(\lambda)$ approaches 0. Then either:*

(A) 0 *is not an eigenvalue of $-\Delta + \lambda_0 V$ in which case $e(\lambda) = -c(\lambda - \lambda_0)^2 + O((\lambda - \lambda_0)^3)$ with $c \neq 0$ and $e(\lambda)$ is analytic at $\lambda = \lambda_0$, or*

(B) 0 *is an eigenvalue of $-\Delta + \lambda_0 V$ in which case $e(\lambda) = -c(\lambda - \lambda_0) + O((\lambda - \lambda_0)^{3/2})$ with $c \neq 0$ and $e(\lambda)$ has a convergent expansion in $(\lambda - \lambda_0)^{1/2}$. Moreover, $e(\lambda)$ is* not *analytic at $\lambda = \lambda_0$.*

If k eigenvalues (coutting multiplicity) approach zero as $\lambda \downarrow \lambda_0$, then at most one is in case A and, in particular, 0 is an eigenvalue of $-\Delta + \lambda_0 V$ of multiplicity k or k − 1.

In addition, the ground state is always in case (A).

Proof. We must invert $e(\lambda) = -\alpha(\lambda)^2$ and $\mu(\alpha(\lambda)) = \lambda^{-1}$. Since $\lambda_0 \neq 0$, we find

$$\lambda = \lambda_0 + \tilde{a}\alpha + \tilde{b}\alpha^2 + \cdots$$

with $\tilde{a} \neq 0$ only if $a \neq 0$ and $\tilde{b} \neq 0$ if $\tilde{a} = 0$. If $\tilde{a} \neq 0$, by the implicit function theorem, $\alpha(\lambda)$ is analytic in λ at λ_0 and $\alpha(\lambda) = -\tilde{a}^{-1}(\lambda - \lambda_0) + O(\lambda - \lambda_0)$ so we are in Case (A). By Theorem 1.2 (see also below), 0 is not an eigenvalue of $-\Delta + \lambda_0 V$.

If $\tilde{a} = 0$, then there are *two* solutions $\alpha_\pm(\lambda)$ of the implicit equation (two branches of square root)

$$\alpha_\pm(\lambda) = \tilde{b}^{-1/2}(\lambda - \lambda_0)^{1/2} + \cdots.$$

All of (B) is now obvious except for the assertion that $e(\lambda)$ cannot be analytic. A priori it might happen that

$$\alpha(\lambda) = (\lambda - \lambda_0)^{1/2} \cdot (\text{analytic function of } \lambda - \lambda_0). \tag{2.8}$$

To see this cannot happen, we follow Rauch [12] and argue as follows. If (2.8) holds, then for $\lambda < \lambda_0$, and $|\lambda - \lambda_0|$ small, α is pure imaginary so for $\lambda < \lambda_0$, L_α has eigenvalue 1 for α pure imaginary. This implies that $E = -\alpha^2 > 0$ is an eigenvalue of $-\Delta + \lambda V$ [18], but such operators do not have positive eigenvalues [13].

The remaining assertions follow from Theorem 2.2. ∎

There is an alternative way of seeing that 0 is an eigenvalue if and only if $\int |V|^{1/2}\phi\, d^3x = 0$. Namely, given ϕ solving $L_{\alpha=0}\phi = \mu_0\phi$ let

$$u = (-\Delta)^{-1} |V|^{1/2}\phi$$

so that

$$u(x) = (4\pi)^{-1} \mu_0^{-1} \int |x-y|^{-1} V(y)\, u(y)\, d^3y \tag{2.9}$$

and $(-\Delta + \mu_0^{-1}V)u = 0$ in distributional sense. Since V has compact support, u has an expansion:

$$u(x) = c_1 x^{-1} + c_2 x^{-2} + \cdots \tag{2.10}$$

for large x. Clearly, $u \in L^2$ if and only if $c_1 = 0$. But, by (2.9), c_1 is given by:

$$c_1 = (4\pi)^{-1} \mu_0^{-1} \int V(y)\, u(y)\, d^3y$$

$$= -(4\pi)^{-1} \int |V(y)|^{1/2} \phi(y)\, d^3y$$

proving once again the result that $\int |V|^{1/2}\phi\, d^3x = 0$ is necessary and sufficient for zero to be an eigenvalue of $-\Delta + \mu_0^{-1}V$. This analysis also proves the first half of:

THEOREM 2.4. *If V is spherically symmetric, then s-waves are in Case* (A) *and $l \geqslant 1$-waves are always in case* (B).

Proof. If the solution u of (2.9) is s-wave (spherically symmetric), then outside the support of V,

$$u(x) = d_1 + d_2 x^{-1}$$

as a spherically symmetric harmonic function. Since $u \to 0$ at infinity $d_1 = 0$. Then $d_2 (= c_1)$ can only be zero if $u \equiv 0$ outside suppose of V but that implies $u \equiv 0$ since we are dealing with a second-order ordinary differential equation. Since $u \not\equiv 0$, $c_1 \neq 0$ so we are in Case (A).

We can see that $l \geqslant 1$ waves are in Case (B) in several distinct ways:

(i) automatically, solutions of the equation $(-\Delta + \mu_0^{-1}V)u = 0$ with $u \to 0$ at infinity and $u(\mathbf{r}) = f(r)\, Y_{l m}$ have $f(r) \sim cr^{-l-1}$ at infinity so $c_1 = 0$;

(ii) $l \geqslant 1$ waves are $(2l+1)$-degenerate so they cannot be in Case (B) by the degeneracy criterion. ∎

We also remark that Case (A) is generic since if one has a situation with $\int |V|^{1/2}\phi = 0$, under a small perturbation the integral will become non-zero.

One can paraphrase Theorem 2.3 in "physical" terms and realize thereby the connection with Newton's results in the central case [10] by saying that $-\alpha^2$ is an *antibound state* if $\alpha < 0$ is such that L_α has eigenvalue 1. If α complex is such that L_α has eigenvalue 1, we call $E = -\alpha^2$ a *resonance energy* (of necessity, $\operatorname{Re} \alpha < 0$ [18]).

Then if a single eigenvalue, $e(\lambda)$, approaches zero as $\lambda \downarrow \lambda_0$, either, $e(\lambda)$ is $O((\lambda - \lambda_0)^2)$ and for $\lambda < \lambda_0$, the bound state turns into an antibound state or $e(\lambda) = 0(\lambda - \lambda_0)$ in which case an antibound state must approach 0 as $\lambda \downarrow \lambda_0$. At $\lambda = \lambda_0$, the bound and antibound states collide and as λ is made smaller than λ_0 turn into a resonance pair. If k eigenvalues approach 0 as $\lambda \downarrow \lambda_0$, then at least $k-1$ antibound states much come in at the same time producing at least $k-1$ resonance pairs.

Consider now the ground-state energy $e_g(\lambda)$ of $-\Delta + \lambda V$ defined a priori for $\lambda > \lambda_g$, the critical coupling constant for e_g. Naively, one might suppose that e_g is singular at $\lambda = \lambda_g$ but since the ground state is always in Case (A), e_g is analytic at $\lambda = \lambda_g$. In the next three results, we show that e_g is analytic on all of $(0, \infty)$ and investigate the singularities at $\lambda = 0, \infty$. To some extent, our study is motivated by a remark of Stillinger [24] who notes the analyticity on $(0, \infty)$ for the square well.

THEOREM 2.5. *e_g is analytic on $(0, \infty)$. It is negative and strictly monotone increasing (resp. decreasing) as λ increases in the interval $(0, \lambda_g)$ (resp. (λ_g, ∞)).*

Proof. For α real, let $\mu_g(\alpha)$ denote that largest eigenvalue of L_α. Then $\mu_g(\alpha)$ is strictly monotone decreasing since if $L_\alpha\phi_\alpha = \mu_g(\alpha)\,\phi_\alpha$, then $(\eta_\alpha(x) = |V(x)|^{1/2}\phi_\alpha(x))$

$$d\mu_g(\alpha)/d\alpha = -(4\pi)^{-1} \int \eta_\alpha(x)\, \eta_\alpha(y)\, e^{-\alpha|x-y|}\, d^3x\, d^3y$$

and $\phi_\alpha > 0$ on $\operatorname{supp}(V)$. Thus

$$\lambda_g(\alpha) \equiv \mu_g(\alpha)^{-1}$$

is strictly monotone increasing so it has a strictly monotone inverse function $\alpha_g(\lambda)$ which is analytic. Since

$$e_g(\lambda) = -\alpha_g(\lambda)^2$$

for $\lambda > \lambda_g$, we are done. ∎

Remark. This result is special to 3 dimensions. Also, our proof only works if V has strictly compact support or at least falloff faster than any exponential.

THEOREM 2.6. *As $\lambda \downarrow 0$, we have that*

$$e_g(\lambda)/[\ln(\lambda^{-1})]^2 \to -d(V)^2,$$

where $d(V)$ = diameter of the support of V.

Proof. By the inversion procedure, this is equivalent to

$$(-\alpha)^{-1} \ln \mu_g(\alpha) \to d(V)$$

as $-\alpha \to \infty$. But the operator norm of L_α is clearly dominated by $e^{-\alpha d(V)} \| L_0 \|$ for $(-\alpha) > 0$ yielding an upper bound on $\mu_g(\alpha)$ and a lower bound is obtained easily from the maximum principle for μ_g. ∎

Theorem 2.7. *As $\lambda \to \infty$,*

$$e_s(\lambda)/\lambda \to \inf(V).$$

Proof. This is a result of Simon [17]. ∎

Remark. If V has a unique absolute minimum at V_0 and the matrix $\partial^2 V/\partial x_i\, \partial y_j$ at x_0 is strictly positive definite, it is easy to develop large λ asymptotics for $e_s(\lambda)$ in terms of harmonic oscillators placed at x_0. Similar (but more involved) problems have already been treated by Aventini, Combes, Duclos, Grossman and Seiler in their treatment of the Born–Oppenheimer approximation [1].

Finally, returning to the spherically symmetric case, we can say something about the first singular term in $e(\lambda)$:

Theorem 2.8. *Let V be spherically symmetric and let $e(\lambda)$ be an eigenvalue of angular momentum $l \geqslant 1$. Then, if $e(\lambda) \to 0$ as $\lambda \downarrow \lambda_0$, we have that $\alpha(\lambda) \equiv [-e(\lambda)]^{1/2} = a_1(\lambda - \lambda_0) + a_2(\lambda - \lambda_0)^2 + \cdots + b_1(\lambda - \lambda_0)^{l+1/2} + \cdots$, i.e., the first singular term occurs at order $l + \frac{1}{2}$. In particular, for $\lambda < \lambda_0$, the resonance has*

$$\operatorname{Im} e(\lambda)/\operatorname{Re} e(\lambda) = 0[(\lambda - \lambda_0)^{l-1/2}].$$

Proof. This is equivalent to saying that in the expansion of $\mu(\alpha) = \mu_0 + \beta\alpha^2 + \cdots$ the first odd order term is α^{2l+1}. In the expansion of L_α, the term multiplying α^{2k+1} is precisely the operator with integral kernel

$$A_{2k+1} = -(4\pi)^{-1} \mid V(x)|^{1/2}(x - y)^{2k} \mid V(y)|^{1/2}.$$

This operator is finite rank (compare [6]) and identically zero on states of angular momentum $l > k$. Thus the first possible odd order term is α^{2l+1}. Its coefficient is exactly $(\phi, A_{2l+1}\phi)$. If $(\phi, A_{2l+1}\phi) = 0$, then the expansion of (2.9) shows that $u(x) = O(x^{-l-2})$ and this is impossible since $u(x) = O(x^{-l-1})$ (compare with the proof of the first half of Theorem 2.4). ∎

3. $\nu \geqslant 5$, Odd

In odd dimension $\nu \geqslant 5$, the story is very simple.

Theorem 3.1. *Let $\nu = 5, 7, 9, \ldots$, let $V \leqslant 0$, $V \in C_0^\infty(R^\nu)$ and let $e(\lambda)$ be a negative eigenvalue of $-\Delta + \lambda V$ with $e(\lambda) \uparrow 0$ as $\lambda \downarrow \lambda_0$. Then 0 is an eigenvalue of $-\Delta + \lambda_0 V$, $e(\lambda) = a(\lambda - \lambda_0) + O((\lambda - \lambda_0)^{3/2})$ and $e(\lambda)$ is non-analytic in $\lambda - \lambda_0$ at $\lambda = \lambda_0$ but is analytic in $(\lambda - \lambda_0)^{1/2}$.*

The analysis in any dimension parallels that in dimension 3, except that K_e now has integral kernel

$$K_e(x, y) = \mid V(x)|^{1/2}\, G_0(x - y; e) \mid V(y)|^{1/2}, \tag{3.1}$$

where $G_0(x - y; e)$ is the integral kernel of $(-\Delta - e)^{-1}$. If $e = -\alpha^2$ with $\alpha > 0$, then

$$G_0(x; e) = \alpha^m x^{-m} (2\pi)^{m+1} K_m(\alpha x), \tag{3.2}$$

where $m = \frac{1}{2}\nu - 1$ and K_m is the conventional Bessel function of imaginary argument. If $\nu = 2k + 1$ so $m = k - \frac{1}{2}$, then

$$K_m(z) = (\pi/2z)^{1/2} e^{-z} \sum_{l=0}^{k-1} (k - 1 + l)!\,[l!\,(k - 1 - l)!\,(2z)^l]^{-1}. \tag{3.3}$$

We thus have:

Lemma 3.2. *If $\nu \geqslant 5$, odd, then $G_0(x; -\alpha^2)$ is an entire function of α for every x obeying*

$$| G_0(x; \alpha^2)| \leqslant Ce^{-|\alpha|\,|x|}[\,| x |^{-(\nu-2)} + | \alpha |^{(\nu-2)}].$$

Moreover, at $\alpha = 0$,

$$G_0(x; -\alpha^2) = C_1 \mid x \mid^{-(\nu-2)} + C_2\alpha^2 \mid x \mid^{-(\nu-4)} + O(\alpha^3). \tag{3.4}$$

Proof. By (3.2) and (3.3), $G_0(x; -\alpha^2) \equiv F(x, \alpha)$ is an entire function of α of the form $\alpha^{\nu-2}F(\alpha x, 1)$, so the bound is obvious from (3.3) and to prove (3.4) we need to show that

$$x^{-m}K_m(x) = C_1 \mid x \mid^{-(\nu-2)} + C_2 \mid x \mid^{-(\nu-4)} + O(\mid x \mid^{-\nu-5}).$$

But, by (3.3)

$$\begin{aligned} x^{-m}K_m(x) &= (\pi/2)^{1/2}\, x^{-m-1/2}\, e^{-x}(2x)^{-k+1} \left[\frac{(2k-2)!}{(k-1)!}\right] (1 + x + O(x^2)) \\ &= Cx^{-m-1/2-k+1}(1 + O(x^2)) \end{aligned}$$

since the $1 + x$ and e^{-x} cancel to $O(x^2)$. Since $k = m + \frac{1}{2}$, $m - \frac{1}{2} - k + 1 = -2k + 1 = -\nu + 2$. ∎

Remark. There is another way of seeing that if $\nu = 2k + 1$

$$G_0(x; 1) = C_1 \mid x \mid^{-(\nu-2)} + c_2 \mid x \mid^{-(\nu-4)} + \cdots + C_k \mid x \mid^{-1} + O(\mid x \mid^{-1})$$

(i.e., in $G_0(x; -\alpha^2)$, the first $O(\alpha)$ term occurs as $\alpha^{\nu-2}$) which is illuminating. Since

$$(e^{+t\Delta})(x - y) = (4\pi t)^{-\nu/2} \exp(-(x - y)^2/4t)$$

and

$$(-\Delta + 1)^{-1} = \int_0^\infty e^{-t} e^{+t\Delta}\, dt$$

we have that

$$G_0(x, 1) = \int_0^\infty (4\pi t)^{-\nu/2} e^{-t} e^{-|x|^2/4t}\, dt$$

$$= x^{-\nu+2} \int_0^\infty (4\pi y)^{-\nu/2} e^{-y|x|^2} e^{-1/4y}\, dy$$

by changing variables from t to $y = t\,|x|^{-2}$. Expanding the exponential and noting that the only problem is a singularity at $y = \infty$, we see only $|x|^{2k}$ occurs for $k < \nu/2 - 1$.

Proof of Theorem 3.1. Let L_α have the integral kernel:

$$|V(x)|^{1/2} \alpha^m |x - y|^{-m} (2\pi)^{m+1} K_m(\alpha |x - y|) |V(y)|^{1/2}$$

with $m = \frac{1}{2}\nu - 1$. Then, by the lemma, L_α is a compact operator (it is not Hilbert–Schmidt but by standard methods [23], it is in a suitable trace ideal) with

$$L_\alpha = L_{\alpha=0} + A\alpha^2 + B\alpha^3 + \cdots$$

about $\alpha = 0$. Thus analyzing as in Section 2,

$$\mu(\alpha) = \mu_0 + b\alpha^2 + O(\alpha^3)$$

and the theorem in proven as in Section 2. ∎

Remark. That $A \neq 0$ in the above proof follows from general principles (Theorem 1.1) but it also can be proven from the fact that $|x - y|^{-(\nu-4)}$ is a strictly positive definite function.

Remarks. 1. Since angular momentum l in ν dimensions is equivalent to angular momentum $l + (\nu - 3)/2$ in three dimensions, we can read off the first fractional power from Theorem 2.8.

2. Theorems 2.5, 2.6 are replaced by the following: at λ_g the ground-state collides with an antibound state $\tilde{e}_g(\lambda)$. If one continues $\tilde{e}_g(\lambda)$ from λ_g to λ increasing, there is another singular value $\tilde{\lambda}_g$ and if one takes the right branch from these, it is analytic in $(0, \tilde{\lambda}_g)$ and has a logarithmic singularity at 0. This follows from the fact that the largest eigenvalue $\mu_g(\alpha)$ of L_α must be as shown schematically in Fig. 1.

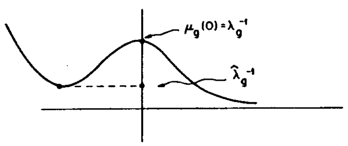

FIGURE 1

4. $\nu = 1$

In one dimension, the result is also very simple:

THEOREM 4.1. *Let $V \leqslant 0$, $V \in C_0^\infty(-\infty, \infty)$ and let $e(\lambda)$ be a negative eigenvalue of $-d^2/dx^2 + \lambda V$ with $e(\lambda) \uparrow 0$ as $\lambda \downarrow \lambda_0$ with $\lambda_0 \neq 0$. Then, 0 is not an eigenvalue of $-d^2/dx^2 + \lambda_0 V$, $e(\lambda) = q(\lambda - \lambda_0)^2 + O((\lambda - \lambda_0)^3)$, $q \neq 0$, and $e(\lambda)$ is analytic about λ_0.*

Remark. The same result is true for the critical value $\lambda_0 = 0$ [19]. We separate out the case $\lambda_0 \neq 0$ because the proof is different (it appears elsewhere [19]) and because the above theorem holds even if V is not nonpositive (see Section 8) while in the case at $\lambda_0 = 0$, $e(\lambda) = a\lambda^4 + \cdots$ is possible [19].

To prove Theorem 4.1, we define the operator L_α on $L^2(-\infty, \infty)$ to have integral kernel

$$L_\alpha(x, y) = |V(x)|^{1/2} e^{-\alpha|x-y|} (2\alpha)^{-1} |V(y)|^{1/2}$$

so that for $\alpha > 0$, $L_\alpha = |V|^{1/2}(-\Delta + \alpha^2)|V|^{1/2}$. The only complication now is that L_α has a pole at $\alpha = 0$ which is actually responsible for $\lambda_0 = 0$ being a critical value.

LEMMA 4.2. *There exists δ sufficiently small and R sufficiently large so that for $|\alpha| < \delta$, L_α has no eigenvalue in the region $\{\mu \mid |\mu| = R\}$ and precisely one eigenvalue $\mu_g(\alpha)$ with $|\mu_g(\alpha)| > R$. Moreover, $\mu_g(\alpha)$ is simple and $\mu_g(\alpha) \to \infty$ as $\alpha \to 0$.*

Proof. Let $M_\alpha = \alpha L_\alpha$. Then M_α is analytic at $\alpha = 0$ and $M_{\alpha=0} = (|V|^{1/2}, \cdot)|V|^{1/2}$ is rank 1. Thus by standard eigenvalue perturbation theory [8, 13], for suitable δ_1 and $|\alpha| < \delta_1$, M_α has exactly one eigenvalue in $\{\gamma \mid |\gamma| \geqslant \frac{1}{2}(|V|^{1/2}, |V|^{1/2})\}$ and it is simple. Put differently, L_α has exactly one eigenvalue $\mu_g(\alpha)$ in $\{\mu \mid |\mu| > \frac{1}{2}\alpha^{-1}(|V|^{1/2}, |V|^{1/2})\}$ and it is simple. We are thus reduced to finding R with no eigenvalue in $\{\mu \mid R \leqslant |\mu| \leqslant \frac{1}{2}\alpha^{-1}(|V|^{1/2}, |V|^{1/2})\}$. Following [19], write

$$L_\alpha = \alpha^{-1} M_{\alpha=0} + Q_\alpha .$$

Then for $|\alpha| < \delta_1$, $\|Q_\alpha\| \leqslant \frac{1}{4}R$ for some R. $\alpha^{-1}M_{\alpha=0}$ is normal and has eigenvalues 0 and $\alpha^{-1}(|V|^{1/2}, |V|^{1/2})$. Thus $(\alpha^{-1}M_{\alpha=0} - z)^{-1}$ exists and has norm

$$\max(|z|^{-1}, |\alpha^{-1}(|V|^{1/2}, |V|^{1/2}) - z|^{-1})$$

for $z \neq 0$, $\alpha^{-1}(|V|^{1/2}, |V|^{1/2})$. In particular, if $R \leqslant |z| \leqslant \{\alpha^{-1}(|V|^{1/2}, |V|^{1/2})$ then

$$\|(\alpha^{-1}M_{\alpha=0} - z(^{-1}Q_\alpha\| \leqslant \max(R^{-1}(\tfrac{1}{4}R), (\tfrac{1}{4}R)[\tfrac{1}{2}\alpha^{-1}(|V|^{1/2}, |V|^{1/2})]^{-1}).$$

By shrinking δ_1 to δ if necessary, we can be sure that

$$\|(\alpha^{-1}M_{\alpha=0} - z)^{-1}Q_\alpha\| \leqslant \tfrac{1}{2}$$

for $|\alpha| \leqslant \delta$, $R \leqslant |z| \leqslant \frac{1}{2}\alpha^{-1}(|V|^{1/2}, |V|^{1/2})$. This implies that

$$(L_\alpha - z) = (\alpha^{-1}M_{\alpha=0} - z)[1 + (\alpha^{-1}M_{\alpha=0} - z)^{-1}Q_\alpha]$$

is invertible. ∎

Proof of Theorem 4.1. For $|\alpha| < \delta$, define

$$P(\alpha) = (2\pi i)^{-1} \oint_{|\mu|=R} (\mu - L_\alpha)^{-1}\, d\mu.$$

Then, $L_\alpha P(\alpha) = N(\alpha)$ is analytic in the punctured neighborhood $\{0 < \alpha < \delta\}$ and bounded as $\alpha \to 0$ so $N(\alpha)$ is analytic at $\alpha = 0$. Moreover, the eigenvalues of $N(\alpha)$ are exactly those of L_α other than $\mu_e(\alpha)$. In particular, $\mu_0 = \lambda_0^{-1}$ is an eigenvalue of $N(\alpha)$. In the usual way, there is a unique eigenvalue $\mu(\alpha)$ of $N(\alpha)$ near μ_0.

$(-\Delta + \lambda_0 V)u = 0$ has no square integrable solution since $u \sim a + bx$ at infinity. Thus, by Theorem 1.2, $e(\lambda) = d(\lambda - \lambda_0) +$ higher order with $d \neq 0$ is not allowed. By the analysis in Section 2, it cannot happen that

$$\mu(\alpha) = \mu_0 + b\alpha^2 + \cdots$$

and thus

$$\mu(\alpha) = \mu_0 + a\alpha + b\alpha^2 + \cdots$$

with $a \neq 0$ from which the theorem follows as in Section 2. ∎

5. $\nu \geqslant 6$, Even

Here the result is

THEOREM 5.1. *Let $V \leqslant 0$, $V \in C_0^\infty(R^\nu)$, $\nu = 6, 8, 10,\ldots$. Let $e(\lambda)$ be a negative eigenvalue of $-\Delta + \lambda V$ with $e(\lambda) \uparrow 0$ as $\lambda \downarrow \lambda_0$. Then, 0 is an eiganvalue of $-\Delta + \lambda_0 V$, $e(\lambda) = q(\lambda - \lambda_0) + O(\lambda - \lambda_0)$ $(q \neq 0)$, $e(\lambda)$ is not analytic at $\lambda = \lambda_0$ and $e(\lambda)$ for λ small has a convergent expansion*:

$$e(\lambda) = \sum_{n=2}^{\infty} \sum_{m=0}^{\infty} c_{nm} [(\lambda - \lambda_0)^{1/2}]^n [(\lambda - \lambda_0)^{1/2} \ln(\lambda - \lambda_0)]^m.$$

Remark. Non-analyticity only asserts that some term with either $m \neq 0$ or n odd occurs. We do not see how to show there is not some kind of miraculous cancellation of all terms with $m \neq 0$.

The analysis is similar to that in Section 3, but now $m = \frac{1}{2}n - 1$ is an integer and thus the Bessel function

$$K_m(z) = \frac{1}{2} \sum_{l=0}^{m-1} \frac{(-1)^l}{l!} \frac{(m-l-1)!}{z^{m-2l}} 2^{-2l+m}$$

$$+ (-1)^{m+1} \sum_{l=0}^{\infty} \frac{2^{2l+m}}{m!(m+l)!\, 2^{2l+m}} \left(\left[\log \frac{1}{2} z - \frac{1}{2}\psi(l+1) - \frac{1}{2}\psi(m+l+1)\right]\right)$$

with

$$\psi(j) = -\lim_{n\to\infty} \left[\sum_{k=j}^{n} \frac{1}{k} - \ln n\right]$$

$$= 1 + \frac{1}{2} + \cdots + 1/j - 1 - C$$

with C Euler's constant.

Proof of Theorem 5.1. By the above analysis, L_α has the form

$$L_\alpha = A_\alpha + (\alpha \ln \alpha)\, B_\alpha$$

with A_α, B_α entire analytic in α and the leading terms in L_α are

$$L_\alpha = L_0 + \alpha^2 C + \cdots$$

with

$$L_0(x, y) = c_1 |V(x)|^{1/2} |x - y|^{-(\nu-2)} |V(y)|^{1/2},$$
$$C(x, y) = c_2 |V(x)|^{1/2} |x - y|^{-(\nu-4)} |V(y)|^{1/2}$$

with $c_1, c_2 \neq 0$. Standard perturbation theory methods easily show that if μ_0 is an eigenvalue of $L_{\alpha=0}$, then for α small and positive, L_α has an eigenvalue $\mu(\alpha)$ given by the convergent series

$$\mu(\alpha) = \sum_{\substack{n\geqslant 2\\ m\geqslant 0}}^{\infty} c_{nm} \alpha^n (\alpha \ln \alpha)^m.$$

Moreover, since C is strictly positive definite on $L^2(\text{supp } V)$, the leading behavior is

$$\mu(\alpha) = \mu_0 + A\alpha^2 + \cdots, \qquad A \neq 0.$$

The required behavior of $e(\lambda)$ follows from a suitable implicit function theorem (Theorem A.4) which we discuss in the Appendix. ∎

6. $\nu = 4$

THEOREM 6.1. *Let $V \leqslant 0$, $V \in C_0^\infty(R^4)$. Let $e(\lambda)$ be a negative eigenvalue of $-\Delta + \lambda V$ with $e(\lambda) \uparrow 0$ as $\lambda \downarrow \lambda_0$. Then either*

(a) 0 is an eigenvalue of $-\Delta + \lambda_0 V$ in which case $e(\lambda)$ obeys all the properties given in Theorem 5.1, *or*

(b) 0 is not an eigenvalue of $-\Delta + \lambda_0 V$ in which case

$$e(\lambda) = q(\lambda - \lambda_0)/[\ln(\lambda - \lambda_0)]^2 + \text{lower order}$$

$(q \neq 0)$ and e is not analytic at $\lambda = \lambda_0$ but is given by a convergent expansion

$$\sum_{\substack{n\geq 2\\ m,k\geq 0}} c_{nmk}[-(\lambda - \lambda_0)/\ln(\lambda - \lambda_0)]^{n/2}[-1/\ln(\lambda - \lambda_0)]^m\{-\ln[\ln(\lambda - \lambda_0)^{-1}]/\ln(\lambda - \lambda_0)\}^k.$$

Moreover, if several eigenvalues approach zero at once, then at most one is in case (b).

Remark. The proof shows that for central potentials, s-waves are in case (b), $l \geq 1$ in case (a).

Proof. The expansion of L_α comes from K_1 which is given in Section 5 but now the leading terms are

$$L_\alpha = L_0 + \alpha^2 \ln \alpha\, D_1 + \alpha^2 D_2 + \cdots,$$

where

$$D_1 = c_1(|V|^{1/2}, \cdot)|V|^{1/2}$$

$$D_2 = c_2(|V|^{1/2}, \cdot)|V|^{1/2} + c_3 \int |V(x)|^{1/2} \ln|x - y|\,|V(y)|^{1/2}.$$

If $(|V|^{1/2}, \phi) \neq 0$ for the unperturbed eigenfunction $L_{\alpha=0}\phi = \mu_0\phi$, then

$$\mu(\alpha) = c\alpha^2 \ln \alpha + \cdots \qquad (c \neq 0)$$

and we are in case (b) given Theorem A.5. Clearly, in the degenerate case, at most one eigenvalue is in this case. If $(|V|^{1/2}, \phi) = 0$, then the leading behavior is

$$\mu(\alpha) = d\alpha^2 + \cdots.$$

That $d \neq 0$ follows either from general principles (Theorem 1.1) or from the conditional strict positive definiteness of $-\ln|x - y|$. (To see this conditional positive definiteness, use $\lim_{\beta\downarrow 0}(x^{-\beta} - 1)/\beta$.) ∎

7. $\nu = 2$

THEOREM 7.1. *Let $V \leq 0$, $V \in C_0^\infty(R^2)$ and let $e(\lambda)$ be a negative eigenvalue of $-\Delta + \lambda V$ with $e(\lambda) \uparrow 0$ as $\lambda \downarrow \lambda_0$ with $\lambda_0 \neq 0$. Then one of three mutually exclusive situations holds:*

(a) *0 is an eigenvalue of $-\Delta + \lambda_0 V$, in which case $e(\lambda)$ obeys the conclusions of Theorem* 5.1;

(b) *0 is not an eigenvalue of $-\Delta + \lambda_0 V$ and $e(\lambda)$ obeys the conclusions of case* (b) *of Theorem* 6.1;

(c) *0 is not an eigenvalue of $-\Delta + \lambda_0 V$ and*

$$e(\lambda) \sim c_1 \exp(-1/c_2(\lambda - \lambda_0)) \tag{7.1}$$

with $c_1, c_2 \neq 0$. [(7.1) *holds in the sense of the ratio going to* 1.]

Moreover, if several eiganvalues approach zero at once, at most one is in case (c) *and at most two are in case* (b).

Remarks. 1. Our proof actually establishes an *asymptotic* series $e(\lambda) \sim \exp(-1/c_2(\lambda - \lambda_0))(c_1 + d_2(\lambda - \lambda_0) + \cdots)$.

2. If V is central, s-waves are in case (c), p-waves in case (b) and $l \geq 2$ in case (a).

3. For $\lambda_0 = 0$, see [19].

$\nu = 2$ not only has the most complicated set of possibilities but has the most complicated analysis. The problem is that as in case $\nu = 1$, L_α has a singularity at $\nu = 0$ but it is $(\ln \alpha)$ rather than α^{-1}. Thus, the trick we used in Section 4 to treat $\nu = 1$ will not work because even after cancelling the leading singularity the precise behavior is not simple; indeed we will find $(\ln \alpha)^{-1}$ terms. If we had used the analysis below in case $\nu = 1$, we would have found corrections of the form $[(\alpha^{-1})^{-1}]^n$ which we could lump together as "analytic corrections" in the operator $P(\alpha)$. We begin our analysis by noting that

$$K_0(z) = -\log\left(\frac{z}{2}\right) I_0(z) + \sum_{n=0}^\infty \frac{z^{2m}}{2^{2m}(m!)^2}\psi(m+1)$$

with

$$I_0(z) = \sum_{l=0}^\infty [l!\,\Gamma(l+1)]^{-1} 2^{-2l} z^{2l}$$

and

$$L_\alpha(x, y) = |V(x)|^{1/2}(2\pi)^{-1}K_0(\alpha|x - y|)\,|V(y)|^{1/2}. \tag{7.2}$$

In particular, we can write

$$L_\alpha = \sigma(\alpha)P + A_\alpha, \tag{7.3}$$

where P is the rank 1 projection

$$P = \left(-\int V\,d^2x\right)^{-1}(|V|^{1/2}, \cdot)\,|V|^{1/2} \tag{7.4}$$

and

$$\sigma(\alpha) = (2\pi)^{-1}\left(\int V\,d^2x\right)\ln \alpha \tag{7.5}$$

and where A_α has a limit, A_0, as $\alpha \downarrow 0$.

We thus begin with a general analysis of the eigenvalue of $(\sigma P + A)$ as $\sigma \to \infty$, where P is rank 1 and A is compact. We will let $Q = (1 - P)$.

LEMMA 7.2. *Let P be a rank 1 projection and A a bounded operator. Then for $\sigma > 2\|A\|$, $A + \sigma P$ has exactly one eiganvalue outside the disc of radius $\|A\|$, it is simple and $O(\sigma)$ as $\sigma \to \infty$. Moreover:*

(1) *For z fixed with $z > \|A\|$*

$$(z - \sigma P - A)^{-1} \to Q(z - QAQ)^{-1}Q$$

in norm as $\sigma \to \infty$.

(2) *If A is self-adjoint and compact, then the limit points of the eigenvalues of $\sigma P + A$ are $\{\infty\} \cup \{$the eigenvalue of QAQ on Ran $Q\}$.*

Proof. As in Section 5, for σ large, spec$(A + \sigma P)$ consists of one large eigenvalue of multiplicity 1 and stuff within a circle of radius R for some fixed R. We will show that for $\sigma > 2\|A\|$, spec$(A + \sigma P)$ is disjoint from $R_\sigma \equiv \{z \mid \|A\| < z < \sigma - \|A\|\}$ from which the first part of the theorem (up to "moreover") follows. For $z \in R_\sigma$,

$$\|(z - \sigma P)^{-1}\| \leqslant \max(z^{-1}, |z - \sigma|^{-1}) < \|A\|^{-1}$$

and thus the perturbation series

$$(z - \sigma P - A)^{-1} = (z - \sigma P)^{-1} + (z - \sigma P)^{-1}A(z - \sigma P)^{-1} + \cdots \tag{7.6}$$

converges.

To prove statement (1), note that $(z - \sigma P)^{-1} \to Qz^{-1}$ in norm as $\sigma \to \infty$. Thus, by the convergence of (7.6) uniformly for σ large and z fixed with $|z| > \|A\|$:

$$\begin{aligned}\text{norm-lim}(z - \sigma P - A)^{-1} &= z^{-1}Q + z^{-2}QAQ + \cdots + z^{-n}Q(AQ)^n \\ &= Q(z - QAQ)^{-1}Q.\end{aligned}$$

Statement (2) now follows from the spectral mapping theorem and the continuity of spectrum under norm limits. ∎

To go further in the analysis of $(z - \sigma P - A)^{-1}$ it is useful to think of P as the perturbation (even if σ is large!). Suppose that $P = (\phi, \cdot)\phi$. Then for z very large (compared to σ)

$$\begin{aligned}(z - \sigma P - A)^{-1} &= (z - A)^{-1} + \sum_{n=0}^{\infty} \sigma(z - A)^{-1}[\sigma P(z - A)^{-1} P]^n P(z - A)^{-1} \\ &= (z - A)^{-1} + \omega_\sigma(z)^{-1} \sigma(z - A)^{-1} P(z - A)^{-1},\end{aligned} \tag{7.7}$$

where

$$\omega_\sigma(z) = 1 - \sigma(\phi, (z - A)^{-1}\phi) \tag{7.8}$$

having proven (7.8) for z large, it follows for all $z \notin \sigma(A) \cup \{z \mid \omega_\sigma(z) = 0\}$ assuming $\sigma(A)$ does not disconnect C. The analysis leading to (7.7), (7.8) is fairly standard, [8, pg. 244 ff] and (7.8) is called the *Weinstein–Aronszajn* determinant.

For large σ and z fixed $|z| \geqslant \|A\|$, we have that

$$\begin{aligned}\sigma\omega_\sigma(z)^{-1} &= [(\phi, (z - A)^{-1} \phi) - \sigma^{-1}]^{-1} \\ &= \sum_{m=0}^{\infty} (\sigma^{-1})^m (\phi, (z - A)^{-1} \phi)^{-m-1}.\end{aligned}$$

Thus

$$\begin{aligned}(z - \sigma P - A)^{-1} = (z - A)^{-1} &+ (\phi, (z - A)^{-1}\phi)^{-1}(z - A)^{-1}P(z - A)^{-1} \\ &+ \sigma^{-1}(\phi, (z - A)^{-2}\phi)(z - A)^{-1}P(z - A)^{-1} + O(\sigma^{-2}).\end{aligned} \tag{7.9}$$

Notice that the first two terms in (7.9) sum up to

$$Q(z - QAQ)^{-1}Q$$

so we recover our earlier result on norm convergence.

LEMMA 7.3. *Let $\mu_0 \neq 0$ be an eigenvalue of QAQ. Then*

(i) *If $\mu_0 \notin \sigma(A)$, then μ_0 is a simple eigenvalue of QAQ.*

(ii) *If $V = \{\eta \mid QAQ\eta = \mu_0\eta\}$, then $\exists W \subset V$ with* co dim$(W) = 0$ *or* 1 *so that for all σ*

$$(z - \sigma P - A)^{-1}\eta \equiv (z - A)^{-1}\eta \tag{7.10}$$

for $\eta \in W$. For $\eta \in W$, $A\eta = \mu_0\eta$.

Proof. (ii) implies (i) since W has codimension at most 1 and thus $\dim(V) \geqslant 2$ implies $\mu_0 \in \sigma(A)$.

To prove (ii) consider the map from V to $\mathbb{C}$ given by

$$l(\eta) = (\phi, (z - A)^{-1}\eta)$$

for some fixed z with $|z| > \|A\|$. Let $W = \operatorname{Ker} l$ so dim W = dim V or dim $V - 1$. By (7.9), we have (7.10) and taking $\sigma \to \infty$,

$$(z - \mu_0)^{-1}\eta = (z - A)^{-1}\eta$$

so that $A\eta = \mu_0\eta$. This later fact shows that W is z-independent. ∎

Proof of Theorem 7.1. We write

$$(z - \sigma_\alpha P - A_\alpha)^{-1} = (z - \sigma_\alpha P - A_0)^{-1} + (z - \sigma_\alpha P - A_0)^{-1}(A_\alpha - A_0)(z - \sigma_\alpha P - A_0)^{-1} + \cdots \tag{7.11}$$

and (since $A_\alpha - A = O(\alpha^2 \ln \alpha)$) conclude that the eigenvalues of L_α which remain finite and non-zero as $\alpha \downarrow 0$ converge to the non-zero eigenvalue of QA_0Q. Let μ_0 be an eigenvalue of QA_0Q and let ψ be the corresponding eigenvector. If

$$P(z - A_0)^{-1}\psi \neq 0 \tag{7.12}$$

then, by the above analysis

$$\mu(\alpha) = \mu_0 + c_1(\ln \alpha)^{-1} + \cdots + c_k(\ln \alpha)^{-1} + O((\ln \alpha)^{-\mu-1})$$

and inverting we are in case (c). By Theorem 1.2, 0 is not an eigenvalue. Moreover, Lemma 7.3 shows that if μ_0 is degenerate at most one dimension is in the situation where (7.12) holds. Now suppose that

$$P(z - A_0)^{-1}\psi = 0. \tag{7.13}$$

By (7.10), we can replace $(z - \sigma_\alpha P - A_0)^{-1}\psi$ by $(z - A\alpha)^{-1}\psi$ wherever it occurs in the systematic perturbation series based on (7.11). In particular the leading term which comes from the α^2ln term is $K_0(\alpha \mid x - y \mid)$ has the form $c\alpha^2 \ln \alpha$ with

$$c = \text{const} \int \eta(y) \mid x - y \mid^2 \eta(x)\, dx\, dy$$

with $\eta = \mid V \mid^{1/2}(z - A_0)^{-1}\psi$. Since (7.13) holds, $\int \eta = 0$ so

$$c = 2(\text{const}) \int (-x \cdot y)\, \eta(x)\, \eta(y)\, dx\, dy$$

$$= 2(\text{const}) \left(- \left[\int x\eta(x)\, dx\right]^2\right)$$

Thus $c \neq 0$ at most on a space of dimension 2. On that space we are in case (b). If $c = 0$ also, then the $O(\alpha^2)$ term is non-zero either by general principles or by the fact that $\ln(x - y)$ is conditional strictly negative definite.

8. Extensions, A. V of Both Signs

We want to discuss here how to deal with the situation where $V \leqslant 0$ is dropped. Of course V must have both signs or else $-\Delta + \lambda V$ has no threshold for $\lambda > 0$. We will indicate how to deal with the case $\nu = 3$. Let $\mid V(x)\mid^{1/2}$ be the obvious function

and $V^{1/2}(x) \equiv V(x) \mid V(x)\mid^{-1/2} = \mid V(x)\mid^{1/2} \operatorname{sgn}(V(x))$. One lets L_α hav integral kernel

$$L_\alpha(x, y) = - \mid V(x)\mid^{1/2} e^{-\alpha|x-y|}(4\pi \mid x - y \mid)^{-1} V^{1/2}(y).$$

Then if $\mu(\alpha)$ is an eigenvalue of L_α for α near zero with $\mu(0) > 0$, then thresholds are obtained by inversion in the usual way.

For non-degenerate eigenvalues, there is no problem since the self-adjointness of L_α plays no role in this case (all perturbation coefficients are real). But if $\mu(0)$ is degenerate we must worry since in the non-self-adjoint case, eigenvalues may be non-analytic in α. Rather, if $\mu(0)$ is k-fold degenerate, then a priori,

$$\mu(\alpha) = \sum_{n=0}^{\infty} c_n(\alpha)^{n/k}. \tag{8.1}$$

Here is a somewhat involved argument showing that $c_n = 0$ if n is not a multiple of k so that the analysis of Section 2 still holds (except for the arguments on the ground state; we do not see why $\mu(\alpha)$ must be monotone in the $\alpha < 0$ region):

Lemma 8.1. *Let $\alpha \geqslant 0$. Then all eigenvalues of L_α are real. If ϕ is any eigenvector of L_α with non-zero eigenvalue, then*

$$(\phi, (\operatorname{sgn} V)\phi) \neq 0. \tag{8.2}$$

If ϕ, ψ are eigenvectors with distinct eigenvalues, then

$$(\phi, (\operatorname{sgn} V)\psi) = 0. \tag{8.3}$$

Proof. Let S = multiplication by (sgn V). Then $SL_\alpha = L_\alpha^* S$. Thus, if $L_\alpha\phi = \mu\phi$, we have that

$$\mu(S\phi, \phi) = (S\phi, L_\alpha\phi) = (L_\alpha^* S\phi, \phi) = (S\mu\phi, \phi)$$

$$= \bar{\mu}(S\phi, \phi)$$

so (8.2) implies also that $\mu = \bar{\mu}$, i.e., μ is real. But

$$(\phi, S\phi) = \mu^{-1}(\phi, SL_\alpha\phi) \neq 0$$

since SL_α is positive definite and strictly positive definite on supp V.

Similarly, if $L_\alpha\phi = \mu\phi$, $L_\alpha\psi = \nu\psi$, the above argument shows that

$$(\mu - \bar{\nu})(\phi, S\psi) = 0$$

implying (8.3) if $\mu \neq \nu = \bar{\nu}$. ∎

LEMMA 8.2. *Let $A(\alpha)$ (α near zero) be an analytic family and let μ_0 be an isolated eigenvalue of finite multiplicity of $A(0)$. Suppose that the eigenvalues $\mu(\alpha)$ which approach μ, are genuinely multivalued analytic functions. Then, there exist unit vectors $\phi_\pm(\alpha)$ and eigenvalues $\{\mu_{+,i}(\alpha)\}_{i=1}^{k}$, $\{\mu_{-,i}(\alpha)\}_{i=1}^{k}$ for $\alpha > 0$ so that*

(i) $\mu_{+,i}(\alpha) \neq u_{-,j}(\alpha)$, *for $\alpha > 0$ all $i, j, \ldots$,*

(ii) *$\phi_\pm(\alpha)$ are sums of eigenvectors of $A(\alpha)$ with eigenvalues $\mu_{\pm,i}(\alpha)$,*

(iii) $\| \phi_+(\alpha) \| = \| \phi_-(\alpha) \| = 1$,

(iv) $\phi_+(\alpha) \to \phi_0$, $\phi_-(\alpha) \to \phi_0$ *in norm where the same ϕ_0 occurs.*

EXAMPLE. To see this really happens, let

$$A(\alpha) = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$$

so $\mu_\pm(\alpha) = \pm\alpha^{1/2}$ *and*

$$\phi_\pm(\alpha) = (1 + \alpha)^{-1} \begin{pmatrix} \pm\alpha^{1/2} \\ 1 \end{pmatrix} \quad \text{and} \quad \phi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Proof. By standard methods [8, 13] one can reduce the problem to $A(\alpha)$ a family on a finite-dimensional space with $\sigma(A(0)) = \{\mu_0\}$. Butler's theorem [8] then asserts that if $P_i(\alpha)$ are the projections onto the eigenvectors for $\mu_i(\alpha)$ (with $AP_i = P_iA$), for $\alpha > 0$, then some $\| P_i(\alpha) \| \to \infty$ as $\alpha \to 0$. This can only happen if the Ran $P_i(\alpha)$ become linearly dependent as $\alpha \to 0$.

THEOREM 8.3. *The series (8.1) is analytic at $\alpha = 0$.*

Proof. If not, pick $\phi_\pm(\alpha)$ according to Lemma 8.2. By Lemma 8.1, with $S = (\operatorname{sgn} V)$

$$(\phi_+(\alpha), S\phi_-(\alpha)) = 0.$$

Taking $\alpha \downarrow 0$, $(\phi_0, S\phi_0) = 0$ which violates Lemma 7.1. ∎

Similarly, there can only be one eigenvalue with an $O(\alpha)$ term. This does not immediately follow from $dL_\alpha/d\alpha \mid_{\alpha=0} =$ rank 1, (e.g.,

$$\begin{pmatrix} 0 & \alpha^2 \\ 1 & 0 \end{pmatrix}$$

has two $O(\alpha)$ eigenvalues) but if first-order terms come from some higher order, Butler's theorem still holds and one can use the above arguments.

9. EXTENSION, B. V MEDIUM RANGE

We deal with successively less short range V's in a series of remarks (for simplicity we consider the case $\nu = 3$).

1. If V does not have compact support but

$$\int e^{a|x|} \mid V(x)|^{3/2} \, d^3x < \infty \tag{9.1}$$

for some $a > 0$, then all arguments involving local behavior at $\alpha = 0$ go through without change since L_α is still analytic. The only global argument concerns following the ground state to $\lambda = 0$ and that argument does not go through.

2. If (9.1) fails, but

$$\int |x|^n \mid V(x)|^2 \, d^3x < \infty \tag{9.2}$$

for all n, then L_α has an asymptotic expansion to all orders and, at least for non-degenerate eigenvalues, we get asymptotic expansions for $\mu(\alpha)$ and then for $e(\lambda)$ (expansions in either $(\lambda - \lambda_0)$ or $(\lambda - \lambda_0)^{1/2}$).

3. If (9.2) fails, but

$$\int (1 + |x|)^{2+\epsilon} \mid V(x)|^2 \, dx < \infty$$

then one can still expand $L_\alpha = L_0 + \alpha A + \alpha^2 B + O(\alpha^{2+\delta})$ so one finds that if 0 is not an eigenvalue of $-\Delta + \lambda_0 V$, then $e(\lambda) = c(\lambda - \lambda_0)^2 + O(\lambda - \lambda_0)^2$ for $c \neq 0$.

10. EXTENSION, C. A PROBLEM OF NEWTON

Our main result in this section is the following:

THEOREM 10.1. *Let $\nu \geqslant 3$ and let $(1 + |x|)^{(\nu-2)} V \in L^{\nu/2+\epsilon} \cap L^{\nu/2-\epsilon}(R^\nu)$. Define $L_0 = -|V|^{1/2}(-\Delta)^{-1}V^{1/2}$. Let μ_0 be the largest eigenvalue of L_0 and let $L_0\psi = \mu_0\psi$. Define*

$$\phi = (-\Delta)^{-1}V^{1/2}\psi.$$

Then for some $c > 0$:

$$\phi(x) \geqslant c(1 + |x|)^{-(\nu-2)}. \tag{10.1}$$

Remarks. 1. Thus ϕ obeys $(-\Delta + \mu_0^{-1}V)\phi = 0$ and ϕ is associated with the ground-state absorbtion.

2. $\phi \geqslant 0$ is discussed in detail in [22]. It is shown there that $\phi \in L^\infty$.

Before proving Theorem 10.1 we want to describe its relevance to the questions under discussion in this paper and then to a problem raised by Newton [11].

COROLLARY 10.2. *If $\nu = 3$ and $e_0(\lambda)$ is the ground state of $-\Delta + \lambda V$ and if*[1] $(1 + |x|)\,V \in L^{3/2+\epsilon} \cap L^{3/2-\epsilon}(R^3)$, *then at threshold,*

$$e(\lambda)/(\lambda - \lambda_0) \to 0 \tag{10.2}$$

as $\lambda \downarrow \lambda_0$.

Proof. By Theorem 1.2, $e(\lambda)/(\lambda - \lambda_0) \not\to 0$ implies that 0 is an eigenvalue of $-\Delta + \lambda_0 V$. It follows that there is a square integrable positive eigenfunction η [13]. But positive, L^∞, solutions of $(-\Delta + \lambda_0 V)\eta = 0$ are unique up to constants [22] and ϕ is not square integrable if $\nu = 3$. This contradiction proves (10.2). ▮

COROLLARY 10.3. *If for some $\delta > 0$, $(1 + |x|)^{\delta + (\nu-2)} V \in L^{\nu/2+\epsilon}$ then (with ϕ the function in Theorem* 10.1)

$$\int V(x)\, \phi(x)\, d^\nu x \neq 0. \tag{10.3}$$

Remarks. 1. Since the proof of the corollary uses lemmas needed in the proof of Theorem 10.1, we defer its proof.

2. Since $(1 + |x|)^{-[\delta + (\nu-2)]} \in L^p$ with $p^{-1} + (\nu/2)^{-1} = 1$, the hypothesis implies that $V \in L^1$ so that the integral in (10.3) converges. Roughly speaking we require that $V \sim (x)^{-\nu-\delta'}$ at infinity. In particular, if $|V(x)| \leqslant C(1 + (x))^{-\nu-\delta'}$ $(\delta' > 0)$, the hypothesis holds.

3. Corollary 10.3 resolves a problem raised by Newton at the end of Section 5 of [11].

We now begin the proof of Theorem 10.1.

LEMMA 10.4. *Let p, q be dual indices $(p^{-1} + q^{-1} = 1)$. Let W be a function with $W \geqslant 0$ and $(-\Delta - \lambda W) \geqslant 0$ for $\lambda \leqslant p + \delta$ (some $\delta > 0$). Let $\eta \in L^{\nu/2 \pm \epsilon}$ with $\eta \geqslant 0$. Then*

$$|(-\Delta)^{-1}\eta(x)| \leqslant \{[[-\Delta - pW)^{-1}\eta](x)\}^{1/p}\{[(-\Delta + qW)^{-1}\eta](x)\}^{1/q}. \tag{10.4}$$

Proof. Let $t > 0$. Then, by the Feynman–Kac formula [21]:

$$\begin{aligned} e^{t\Delta}\eta(x) &= E(\eta(x + b(t))) = E\left(\exp\left(\int_0^t W(x + b(s))\, ds\right) \eta^{1/p}(x + b(t))\right. \\ &\qquad \times \exp\left(-\int_0^t W(x + b(s))\, ds\right) \eta^{1/q}(x + b(t))\Big) \\ &\leqslant E(e^{p\int W}\eta)^{1/p}\, E(e^{-q\int W}\eta)^{1/q} \\ &= [(e^{-t(-\Delta - pW)}\eta)(x)]^{1/p}[(e^{-t(-\Delta + qW)}\eta)(x)]^{t/q} \end{aligned}$$

[1] In [26], this result is extended to $V \in L^{3/2}$.

by Hölder's inequality. Using $(\Delta)^{-1} = \int_0^\infty e^{-t\Delta}\, dt$ and Hölder's inequality again, (10.4) results. ▮

LEMMA 10.5. *Let $G_0 = (-\Delta)^{-1}$. Then*

$$(1 + |x|)^{(\nu-2)} G_0 (1 + |x|)^{-\nu-2}$$

is bounded from $L^{\nu/2+\epsilon} \cap L^{\nu/2-\epsilon}$ to L^∞. The same thing remains true if $\nu - 2$ is replaced by $(\nu - 2) + \delta$ and G_0 by convolution with $|x - y|^{-(\nu-2)-\delta}$ so long as $\delta > 0$ is sufficiently small (how small depends on ϵ).

Proof. We consider the case $\delta = 0$. Since G_0 is, up to a constant, convolution with $|x - y|^{-(\nu-2)}$ we need only show that

$$(1 + |x|)^{\nu-2}(|x - y|^{-(\nu-2)})(1 + |y|)^{-\nu-2} \equiv L(x, y)$$

is the integral kernel of a bounded operator from $L^{\nu/2+\epsilon} \cap L^{\nu/2-\epsilon}$ to L^∞. Since

$$(1 + |x|)^{\nu-2} \leqslant (1 + |y| + |x - y|)^{\nu-2} \leqslant 2^{(\nu-2)}[(1 + |y|)^{\nu-2} + |x - y|^{\nu-2}]$$

we need only show that

$$L_1(x, y) = |x - y|^{-(\nu-2)}$$

and

$$L_2(x, y) = (1 + |y|)^{-(\nu-2)}$$

are bounded integral kernels. The first follows from Young's inequality and the second from Hölder's inequality. ▮

LEMMA 10.6. *Let W lie in $L^{\nu/2+\epsilon} \cap L^{\nu/2-\epsilon}$ with sufficiently small norms. Then $(1 + |x|)^{(\nu-2)}(-\Delta - W)^{-1}(1 + |x|)^{-(\nu-2)}$ is bounded from $L^{\nu/2+\epsilon} \cap L^{\nu/2-\epsilon}$ on L^∞.*

Proof. By the last lemma, $(1 + |x|)^{(\nu-2)}\, G_0 W (1 + |x|)^{-(\nu-2)}$ is bounded from L^∞ to L^∞ with norm less than one if the L^p norms of W are sufficiently small. Thus,

$$[(1 + |x|)^{(\nu-2)}(1 - G_0 W)^{-1}(1 + |x|)^{-(\nu-2)}][(1 + |x|)^{(\nu-2)} G_0 (1 + |x|)^{-(\nu-2)}]$$

is bounded from $L^{\nu/2+\epsilon} \cap L^{\nu/2-\epsilon}$ to L^∞. ▮

Proof of Theorem 10.1. With $V = V_+ - V_-$ with $V_\pm = \max(\pm V, 0)$. *Then* [22], ϕ obeys

$$\phi = (-\Delta + V_+)^{-1} V_- \phi \tag{10.5}$$

with $\phi \in L^\infty$ and $\phi \geqslant 0$. Let $\eta \equiv V_-\phi$. By Lemma 10.4 for suitable p, q, λ

$$(-\Delta)^{-1}\eta \leqslant [(-\Delta + V_+)^{-1}\eta]^{1/p}[(-\Delta - \lambda V_+)\eta]^{1/t}. \tag{10.6}$$

(By [22], pick λ so small that $-\Delta - \lambda V_+$ is subcritical; let $\lambda = q/p$ and then $W = p^{-1}V_+$.) Since $\eta \geqslant 0$,

$$(-\Delta^{-1})\eta(x) \geqslant c(1 + |x|)^{-(\nu-2)}$$

is trivial, so by (10.5) and (10.6), we can show (10.1) by proving

$$[(-\Delta - \lambda V_+)\eta](x) \leqslant c(1 + |x|)^{-(\nu-2)}. \tag{10.7}$$

Since $(1 + |x|)^{\nu+2}\eta \in L^{\nu/2\pm\epsilon}$ by hypothesis, (10.7) follows from Lemma 10.6. ∎

Proof of Corollary 10.3. Since G_0 is convolution with $c_\nu |x - y|^{-(\nu-2)}$

$$\phi(x) = c_\nu |x|^{-(\nu-2)} \left(\int V(y)\,\phi(y)\,d^\nu y\right) + c_\nu R(x), \tag{10.8}$$

where

$$R(x) = \int [|x - y|^{-(\nu-2)} - |x|^{-(\nu-2)}]\,V(y)\,\phi(y)\,d^\nu y. \tag{10.9}$$

Clearly we need only show that

$$|R(x)| \leqslant d\,|x - y|^{-(\nu-2)-\gamma} \tag{10.10}$$

for some $\gamma > 0$ for (10.8), (10.10) and (10.1) clearly imply (10.3). For any $\gamma < 1$

$$\big|\,|x - u|^{-(\nu-2)} - |x|^{-(\nu-2)}\big| \leqslant D_{\gamma,\nu}\,|y|^\gamma[|x - y|^{-(\nu-2)-\gamma} + |x|^{-(\nu-2)-\gamma}] \tag{10.11}$$

by hypothesis, for γ small, $|\gamma|^\gamma V \in L^1$ so the second term obtained from inserting (10.11) in (10.9), obeys (10.10). By Lemma 10.5, the same is true of the first term.

11. Extension, D. $V + \lambda W$

We wish to indicate here that treating $-\Delta + V + \lambda W$ is not much harder than treating V of general sign (Section 8). For one can replace L_α by

$$M_\alpha(V) \equiv e^{-x^2}G_0(x - y; -\alpha^2)\,e^{+y^2}(V(y))$$

and the problem is to look at eigenvalues $E(\lambda, \alpha)$ of

$$M_\alpha(V) + \lambda M_\alpha(W)$$

and solve $E(\lambda, \alpha) = 1$. It is clear that the results of the case $-\Delta + \lambda V$ extend without any significant change.

Appendix: An Inverse Function Theorem

In this appendix, we want to prove results of which the folloling is the simplest case:

Theorem A.1. *Let f, g be functions analytic in a neighbourhood of $\alpha = 0$. Suppose that f, g are real for α real and that*

$$f(0) = g(0) = g'(0) = 0;\ f'(0) > 0.$$

Then for λ sufficiently small and positive there is a unique positive $\alpha(\lambda)$ satisfying

$$\lambda = f(\alpha) + g(\alpha)\ln\alpha. \tag{A.1}$$

Moreover, for λ sufficiently small, α has an expansion

$$\alpha(\lambda) = \sum_{n\geqslant 1;\, m\geqslant 0} c(n, m)\,\lambda^n(-\lambda\ln\lambda)^m \tag{A.2}$$

converging uniformly and absolutely for λ small; $c(1, 0) = (f'(0))^{-1}$.

Since this generalizes a well-known result when $g = 0$ we expect this theorem is not new but we cannot find a proof of it in the literature. By making suitable substitutions, we will reduce the proof to an ordinary implicit function theorem in several variables.

Proof of Theorem A.1. Without loss of generality we can suppose that $f'(0) = 1$, so we write

$$f(\alpha) = \alpha + \sum_{n=2}^{\infty} a_n\alpha^n,$$

$$g(\alpha) = \sum_{n=2}^{\infty} b_n\alpha^n.$$

Since we know that the leading order for α is λ, we write

$$\alpha = \lambda(1 + z) \tag{A.3}$$

and easily find that in the region where $|\alpha|$ is small, (A.1) is equivalent to

$$z = -\lambda\sum_{n=0}^{\infty} a_{n+2}(1 + z^2)^2\,\lambda^n(1 + z)^n + \tau\sum_{n=0}^{\infty} b_{n+2}(1 + z)^2\lambda^n(1 + z)^n$$

$$- \lambda\sum_{n=0}^{\infty} b_{n+2}(1 + z)^{n+2}\ln(1 + z)\,\lambda^n, \tag{A.4}$$

where

$$\tau = -\lambda \ln \lambda. \tag{A.5}$$

Equation (A.4) can be written in the form

$$F(z, \lambda, \tau) = 0,$$

where (i) $z = 0$, $\lambda = 0$, $\tau = 0$ is a solution; (ii) F is analytic for $|z|$, $|\lambda|$, $|\tau|$ small since $\ln(1 + z)$ is analytic at $z = 0$; (iii) $\partial F/\partial z\,(0, 0, 0) = 1 \neq 0$. Thus, by the ordinary implicit function theorem, (A.4) has a unique solution for $|\lambda|$, $|\tau|$ small given by a convergent expansion

$$z = \sum_{n,m\geq 0} d_{nm} \lambda^n \tau^m.$$

Given (A.3) and (A.5), this yields (A.2). ∎

With this warmup, we can be briefer about the more complicated cases we need in the paper.

THEOREM A.2. *Let f, g be functions analytic near $\alpha = 0$ and real-valued for α real so that*

$$f(0) = f'(0) = g(0) = g'(0) = g''(0) = 0; \qquad f''(0) > 0.$$

Then, for λ sufficiently small and positive, there is a unique positive $\alpha(\lambda)$ satisfying (A.1). *For λ sufficiently small, $\alpha(\lambda)$ has a convergent expansion*

$$\alpha(\lambda) = \sum_{n\geq 1, m\geq 0} c(n, m)\, \lambda^{n/2} (-\lambda^{1/2} \ln \lambda)^m \tag{A.6}$$

with $c(1, 0) = [1/f''(0)]^{1/2}$.

Proof. Write, without loss, $f(\alpha) = \alpha^2 + \sum_{n=3}^{\infty} a_n \alpha^n$, $g(\alpha) = \sum_{n=3}^{\infty} b_n \alpha^n$ and try the substtitution

$$\alpha = \lambda^{1/2}(1 + z). \tag{A.7}$$

Then

$$2z + z^2 = -\lambda \sum_{n=0}^{\infty} a_{n+3} (1 + z)^{n+3} \sigma^n$$

$$+ \tau \sum_{n=0}^{\infty} \tfrac{1}{2} b_{n+3} (1 + z)^{n+3} \sigma^n - \sigma \sum_{n=0}^{\infty} b_{n+3} (1 + z)^{n+3} \ln(1 + z)\, \sigma^n,$$

where $\tau = -\lambda^{1/2} \ln \lambda$, $\sigma = \lambda^{1/2}$. Again, we have an equation $F(z, \sigma, \tau) = 0$ which we can solve implicitly:

$$z = \sum_{n,m\geq 0} d_{nm} \sigma^n \tau^m.$$

THEOREM A.3. *Let f, g be functions analytic near $\alpha = 0$ and real-valued for α real so that*

$$f(0) = f'(0) = g(0) = g'(0) = 0; \qquad g''(0) < 0.$$

Then, for λ sufficiently small and positive, there is a unique positive $\alpha(\lambda)$ satisfying (A.1). *For λ sufficiently small, $\alpha(\lambda)$ has a convergent expansion*

$$\alpha(\lambda) = \sum_{n\geq 1, m, k\geq 0} c(n, m, k)\, \sigma^n \tau^m \omega^k$$

with

$$\sigma = (\lambda/-\ln \lambda)^{1/2}, \qquad \tau = -1/\ln \lambda; \qquad \omega = [\ln_2(\lambda^{-1})]\tau$$

and $c(1, 0, 0) = [-\tfrac{1}{2} g''(0)]^{-1/2}$.

Proof. Suppose, without loss, that $f(\alpha) = \sum_{n=2}^{\infty} a_n \alpha^n$, $g(\alpha) = -2\alpha^2 + \sum_{n=3}^{\infty} b_n \alpha^n$. Let

$$\alpha = \sigma(1 + z).$$

Then, by tedious but straightforward substitution,

$$2z + z^2 = (1 + z)^2 \omega + 2\tau(1 + z)^2 \ln(1 + z) - \tau \sum_{n=0}^{\infty} a_{n+2} \sigma^n (1 + z)^{n+2}$$

$$+ [\tfrac{1}{2}\sigma - \sigma\omega - \sigma\tau \ln(1 \times z)] \sum_{n=0}^{\infty} b_{n+3} \sigma^n (1 + z)^{n+3}, \tag{A.8}$$

which, in the usual way, proves the theorem ∎

THEOREM A.4. *Let α and λ be related by*

$$\lambda = \sum_{\substack{n\geq 2\\ m\geq 0}} c_{nm} \alpha^n (\alpha \ln \alpha)^m,$$

where $\sum c_{nm} x^n y^m$ defines an analytic function of two variables with $c_{20} = 1$. Then, there is a unique solution $\alpha(\lambda) = \lambda^{1/2} + O(\lambda^{1/2})$ for λ small with a convergent expansion

$$\alpha(\lambda) = \sum d_{nm} \sigma^n \tau^m; \qquad \sigma = \lambda^{1/2}, \quad \tau = \lambda^{1/2} \ln(\lambda^{-1}).$$

Proof. Let $\alpha = \lambda^{1/2}(1 + z)$ and find

$$z^2 + 2z = - \sum_{n \geq 0, m \geq 0} c_{n+2,m} \sigma^n (1 + z)^{m+n+2} [-\tfrac{1}{2}\tau + \sigma \ln(1 + z)]^m.$$

The usual method yields the desired result. ∎

Similarly, one can extend the analysis of Theorem A.3 to obtain:

THEOREM A.5. *Let α and λ be realted by*

$$\lambda = -2\alpha^2 \ln \alpha + \sum_{\substack{n \geq 2 \\ m \geq 0}} c_{nm} \alpha^n (\alpha \ln \alpha)^m,$$

where $\sum c_{nm} x^n y^m$ defines an analytic function of two variables. Then, there is a unique solution $\alpha(\lambda) = [\lambda / \ln(\lambda^{-1})]^{1/2}(1 + O(1))$ for λ small with a convergent expansion

$$\alpha(\lambda) = \sum_{n \geq 1, m, k \geq 0} c_{nmk} \sigma^n \tau^m \omega^k,$$

where

$$\sigma = (\lambda / -\ln \lambda)^{1/2}; \qquad \tau = -1/\ln \lambda; \qquad \omega = [\ln_2(\lambda^{-1})]\tau.$$

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