J. OPERATOR THEORY
4(1980), 251–270

THE BIRMAN-KREĬN-VISHIK THEORY OF SELF-ADJOINT EXTENSIONS OF SEMIBOUNDED OPERATORS

ALBERTO ALONSO and BARRY SIMON

§1. INTRODUCTION

The von Neumann theory [6] of self-adjoint extensions of a closed symmetric operator, A, on a Hilbert space is given in many textbook presentations. It sets up a one to one correspondence between self-adjoint extensions, $A^{(U)}$, and unitaries U from $\mathrm{Ker}(A^* + \mathrm{i})$ to $\mathrm{Ker}(A^* - \mathrm{i})$ .This parametrization is continuous in a suitable sense (see § 4).

The von Neumann theory is so elegant and complete that it appears to close the subject of the general theory but this is definitely not the case with respect to extensions of A's which are bounded from below. Again from the textbooks, one learns about the Friedrichs' extension [2] but its relation to the von Neumann theory is not clear. Ocasionally mention is made of the theorem of Kreĭn [5] that if A has finite deficiency indices and is bounded from below, then all self-adjoint extensions of A are also bounded from below.

There is a much deeper and most natural analysis possible for the self-adjoint extensions of a semibounded operator. This analysis was presented, in part, in the above paper of Kreĭn and completed in papers of Birman [1] and Vishik [12]. One purpose of the present paper is to make propaganda for this Birman-Kreĭn-Vishik theory which seems to be virtually unknown in the English language literature! In this sense, the present paper is semi-expository.

There is an important difference of emphasis in our presentation. Much of the original spectral theory of *bounded* self-adjoint operators was discussed in terms of quadratic forms. The development of the theory of unbounded operators by von Neumann deemphasized forms and exploited heavily operator ideas especially graphs and Cayley transforms. The papers of Kreĭn-Birman-Vishik extensively use such ideas and base their analysis on them. They are somewhat unusual for their period in that they do consider what we would now call the quadratic form domains but this is subsidiary after the basic analysis. In the past twenty five years, forms have come back into favor and among other reasons has been the realization that the Friedrichs extension is basically a form constructions. The basic objects in our paper are the quadratic forms of the extensions. This difference from emphasis is illustrated by the fact that from our point of view, there is only one natural parametrization A_B of these extensions. In Birman's paper, the extension we associate to B, he associates to B^{-1} (which he calls B).

Since they play a major role, let us recall a few basic facts about quadratic forms (see also [3, 7, 8]) emphasizing the possibility that they are not densely defined, an extension of the usual notion which has proved especially useful of late, [4, 9, 10]

DEFINITION. A *quadratic form* $q(\varphi)$ is a map from a complex Hilbert space H to $(-\infty, \infty]$ (the reals with $+\infty$ added but not $-\infty$) which obeys

(1.1) $$q(\varphi) \geqslant a\|\varphi\|^2 \quad \text{for some fixed } a \text{ in } (-\infty, \infty)$$

(1.2) $$q(\varphi + \psi) + q(\varphi - \psi) = 2q(\varphi) + 2q(\psi)$$

(1.3) $$q(\lambda\varphi) = |\lambda|^2 q(\varphi); \ \lambda \in \mathbf{C}.$$

The *form domain of* q, $Q(q)$, is the set of φ with $q(\varphi) < \infty$. By (1.2) and (1.3), $Q(q)$ is a vector space. q is called *closed* if and only if it is lower semi-continuous, i.e. if $\varphi_n \to \varphi$ in H, then $q(\varphi) \leqslant \underline{\lim}\, [q(\varphi_n)$. (It is known [10] that this is equivalent to $Q(q)$ being complete in the norm $[q(\varphi) + (a+1)\|\varphi\|^2]^{1/2}$). The largest a for which (1.1) holds is called the lower bound, $\gamma(q)$, of q.

DEFINITION. Let B be a self-adjoint operator which is bounded from below. Let $s(x)$ be the function on $(-\infty, \infty)$ which is $+1$ (resp. -1) if $x \geqslant 0$ (resp. <0). Let $s(B)$ be the function given by the functional calculus [9]. Let

$$q_B(\varphi) \begin{cases} = \infty & \varphi \notin D(|B|^{1/2}) \\ = (|B|^{1/2}\varphi, s(B)\, |B|^{1/2}\varphi) & \varphi \in D(|B|^{1/2}); \end{cases}$$

q_B is closed quadratic form called the *form of* B.

If $\varphi \in D(B)$, then $q_B(\varphi) = (\varphi, B\varphi)$, so we will abuse notation and use $(\cdot, B\cdot)$ for q_B.

Now let M be a closed subspace of the Hilbert space H. Given a semi-bounded self-adjoint operator B on M, we can extend q_B from M to H by setting it equal to ∞ on $H \setminus M$. With this extension the fundamental theorem of the theory (see e.g. [3, 7]) becomes

THEOREM 1.1. *There is a one-one correspondence between closed quadratic forms, q, and self-adjoint semi-bounded operators, B, on closed subspaces M of H,* i.e. *every q is a q_B.*

Given a B defined on M and $z \notin \operatorname{spec}(B)$, we extend $(B - z)^{-1}$ to H by writting any $\varphi \in H$ as $\varphi = \eta + \psi$; $\eta \in M$, $\psi \in M^\perp$ and $(B - z)^{-1}\varphi = (B - z)^{-1}\eta$ (i.e. set $(B - z)^{-1} = 0$ on $M^\perp$ and use linearity).

DEFINITION. Given two forms q, p, we write $q \leqslant p$ if and only if $q(\varphi) \leqslant p(\varphi)$ for all φ (i.e. if and only if $Q(p) \subset Q(q)$ and $q(\varphi) \leqslant p(\varphi) < \infty$ for all $\varphi \in Q(p)$).

THEOREM 1.2. (see e.g. Kato [3]) *Let $q = q_B$, $p = q_A$. Let z be a fixed real, smaller than both $\gamma(q)$ and $\gamma(p)$. Then $q \leqslant p$ if and only if $(A - z)^{-1} \leqslant (B - z)^{-1}$ in the ordinary sense for bounded operators.*

We write $A \leqslant B$ if and only if $q_A \leqslant q_B$.

With this result, the fundamental theorem of Kreĭn becomes:

THEOREM 1.3. (Kreĭn [5]). *Let A be a positive symmetric operator. Then among all positive extensions of A there exists two distinguished ones A_∞, and A_0 so that A_∞ is the largest and A_0 is the smallest such extension. The set of all positive extensions is precisely the set of operators C with $A_0 \leqslant C \leqslant A_\infty$.*

In case where $\gamma(A) > 0$ (in which case $A_0 \neq A_\infty$) we prove Theorem 1.3 in § 2. A_∞ is just the Friedrichs' extension (called the "hard" extension in [1, 5]). We will call A_0 the *Kreĭn extension* for obvious reasons.

Let $\gamma(A) > 0$. Then $N = \operatorname{Ker}(A^*)$ is a closed subspace of dimension equal to the deficiency index of A. We will set up, in § 2, a one-one correspondence, $B \leftrightarrow A_B$, between positive quadratic forms, B, on N and positive extensions of A. If $\dim(N) < \infty$, then "positive" can be dropped. This correspondence is given by:

$$Q(A_B) = Q(A_\infty) \dotplus Q(B),$$

(where $X \dotplus Y$ means that $X \cap Y = \{0\}$) and for $\varphi \in Q(A_\infty)$, $\eta \in D(B)$:

$$((\varphi + \eta), A_B(\varphi + \eta)) = (\varphi, A_\infty \varphi) + (\eta, B\eta).$$

The Kreĭn extension corresponds to $B = 0$ and the Friedrichs extension $B = \infty$ (i.e. $q_B(\varphi) = \infty$ for all $\eta \in N$). From this formula, and Theorem 1.2, Kreĭn's theorem will be obvious.

In § 3, we will compute the domain of A_B and find that:

$$D(A_B) = \{\varphi + A_\infty^{-1}(Bf + \eta) + f \mid \varphi \in D(A),\ f \in D(B),\ \eta \in N \cap D(B)^\perp\}$$

with

$$A_B(\varphi + A_\infty^{-1}(Bf + \eta) + f) = A\varphi + Bf + \eta.$$

In particular, $D(A_\infty) = D(A) \dotplus A_\infty^{-1}N$ and $D(A_0) = D(A) \dotplus N$.

In § 4, we begin by noting that the von Neumann parametrization $U \to A^{(U)}$ is continuous when the unitaries and self-adjoints are given the topology of strong resolvent convergence. We then use the results of § 3 to prove a similar continuity of $B \to A_B$.

Finally, in § 5, we consider the Kreĭn extension. We begin by proving Kreĭn's theorem [5] that if A_∞ has purely discrete spectrum, then A_0 restricted to $N^\perp$ has discrete spectrum. We then consider in detail the example of $-\Delta$ on a bounded subset of $\mathbf{R}^n$. We obtain a new *natural* boundary condition which appears not to have been discussed before.

Finally, let us emphasize once again that while we claim some originality of viewpoint, virtually all theorems are already in [5], [12] and/or [1].

§ 2. FORMS

Throughout this section, we suppose that A is a closed symmetric operator with

(2.1) $$(\varphi, A\varphi) \geqslant \|\varphi\|^2.$$

We let

$$N = \operatorname{Ker}(A^*).$$

One defines

$$Q(A_\infty) = \{\varphi \in H \mid \text{there exists } \varphi_n \in D(A),\ \varphi_n \to \varphi \text{ and } ((\varphi - \varphi_n), A(\varphi - \varphi_n)) \to 0\}$$

and sets $(\varphi, A_\infty\varphi) = \lim(\varphi_n, A\varphi_n)$. Then (see, e.g. [3, 8]) $q(\varphi) = (\varphi, A_\infty\varphi)$ (resp $= \infty$) for $\varphi \in Q(A_\infty)$ (resp. $\varphi \notin Q(A_\infty)$) is a well defined closed form.

LEMMA 2.1. $N \cap Q(A_\infty) = \{0\}$.

Proof. Let $\varphi \in N \cap Q(A_\infty)$. Pick $\varphi_n \to \varphi$ in $(\cdot, A_\infty \cdot)^{1/2}$ norm. Then $(\varphi, A_\infty\varphi) = \lim(\varphi, A\varphi_n) = \lim(A^*\varphi, \varphi_n) = 0$ since $\varphi \in N$. Since $(\varphi, \varphi) \leqslant (\varphi, A_\infty\varphi)$, we have that $\varphi = 0$. ▨

Given a quadratic form B on N, set $q^{(B)}$ to be the object with domain

$$Q(q^{(B)}) = Q(A_\infty) \dotplus Q(B)$$

and

$$q^{(B)}(\varphi + \eta) = (\varphi, A_\infty\varphi) + (\eta, B\eta)$$

for $\varphi \in Q(A_\infty)$, $\eta \in Q(B)$. We begin by examining when $q^{(B)}$ is a closed quadratic form.

LEMMA 2.2. *Suppose that* $\dim N < \infty$. *Let* $\varphi_n \in Q(A_\infty)$, $\eta_n \in N$ *so that*

a) $\sup_n \|\varphi_n\| < \infty, \qquad \sup_n \|\eta_n\| < \infty$

b) $\|\varphi_n + \eta_n\| \to 0$

c) $\sup_n\ (\varphi_n, A_\infty\varphi_n) < \infty.$

Then $\varphi_n, \eta_n \to 0$ *in norm.*

Proof. Since N is finite dimensional, we can pass to a subsequence with $\eta_n \to \eta$. Thus $\varphi_n \to -\eta \equiv \varphi$, so by (c) and the lower semicontinuity of $(\cdot, A_\infty \cdot)$ $\varphi \in Q(A_\infty)$. Thus, by Lemma 2.1, $\varphi = 0$. Since η can be any norm limit point, $\|\varphi_n\| \to 0$.

REMARK. Using the weak lower semicontinuity of $(\cdot, A_\infty \cdot)$, one can see that $\varphi_n, \eta_n \to 0$ weakly even if $\dim N$ is infinite.

THEOREM 2.3. *If either* $B \geqslant 0$ *or* $\dim(N) < \infty$, *then* $q^{(B)}$ *is a semibounded, closed quadratic form.*

Proof. Suppose first that $B \geqslant 0$. Then clearly $q^{(B)}(\varphi + \eta) \geqslant 0$, so $q^{(B)}$ is positive and thus semibounded. Moreover,

$$q^{(B)}(\varphi + \eta) + \|\varphi + \eta\|^2 = (\varphi, A_\infty \varphi) + (\eta, B\eta) + \|\varphi + \eta\|^2 \geqslant$$

$$\geqslant \|\varphi\|^2 + \|\varphi + \eta\|^2.$$

Thus Cauchy in $q^{(B)}$ norm implies Cauchy in the norms $\|\varphi\|$, $\|\varphi + \eta\|$, $(\varphi, A_\infty \varphi)$ and $(\eta, B\eta)$. Thus φ converges in A_∞ norm and η in B-norm, so $\varphi + \eta$ converges in $q^{(B)}$ norm since A and B are closed.

Now suppose that $\dim(N) < \infty$. Suppose first that $q^{(B)}$ is not bounded from below. Since A_∞ is bounded below we can find $\varphi_n \in Q(A_\infty)$, $\eta_n \in N$, with $\|\eta_n\| = 1$ and

$$q^{(B)}(\varphi_n + \eta_n) \leqslant -n\|\varphi_n + \eta_n\|^2.$$

Let $b = \gamma(B)$. Then $q^{(B)}(\varphi_n + \eta_n) \geqslant b$ so

$$\|\varphi_n + \eta_n\|^2 \leqslant |b|\, n^{-1}$$

goes to zero. Moreover, $(\varphi_n, A_\infty \varphi_n) \leqslant -(\eta_n, B\eta_n) \leqslant |b|$ so the hypothesis of Lemma 2.2 hold. The conclusion that $\|\eta_n\| \to 0$ is false so we can have a contradiction establishing the fact that $q^{(B)}$ is bounded from below.

Now suppose that $\varphi_n + \eta_n$ is Cauchy in L^2 and in $q^{(B)}(\cdot)$. Suppose that $\|\eta_n\| \to \infty$. Then let $\varphi_n' = \varphi_n/\|\eta_n\|$; $\eta_n' = \eta_n/\|\eta_n\|$. Hypothesis (a), (b) of Lemma 2.2 clearly hold. Moreover, $q^{(B)}(\eta_n' + \varphi_n') \to 0$, so as in the proof of closure, (c) holds. So $\eta_n' \to 0$. This contradiction shows that no subsequence of $\|\eta_n\|$ can diverge. Thus $\sup_n \|\eta_n\| < \infty$. Pass to a subsequence with $\eta_n \to \eta$. Then $\varphi_n \to \varphi$ so

$$(\varphi_n - \varphi_m, A(\varphi_n - \varphi_m)) + (\eta_n - \eta_m, (B - b + 1)(\eta_n - \eta_m)) \to 0$$

and thus $\varphi_n \to \varphi$ in A_∞, $\eta_n \to \eta$ in B-norm. Since limits are unique in A_∞-norm, φ is the unique norm limit point so we have established convergence. ▨

REMARKS 1. Let $0 < \alpha < 1$. Let $\beta = \alpha/(1-\alpha)$. Then

$$\|\varphi\|^2 + \beta\|\varphi + \eta\|^2 - \alpha\|\eta\|^2 \geqslant (1+\beta)\|\varphi\|^2 + (\beta-\alpha)\|\eta\|^2 - 2\beta\|\varphi\|\,\|\eta\| \geqslant 0$$

since $\beta^2 = (1+\beta)(\beta-\alpha)$. It follows that so long as $\gamma(B) > -1$, our proof that $q^{(B)}$ is semibounded and closed will go through. If $\dim(N) = \infty$ and $q^{(B)} \leqslant -1$, we expect that it is possible that $q^{(B)}$ is not semibounded.

2. We emphasize that the theory is more satisfactory when $\dim(N) < \infty$ than when $\dim(N) = \infty$. However, in the later case by replacing A by $A + c$, we can, in principle, describe all semibounded extensions.

PROPOSITION 2.4. *Let B be positive or $\dim(N) < \infty$. Then $q^{(B)}$ is the form of a self-adjoint operator, A_B, which is an extension of A.*

Proof. By construction of A_B ([3,7]) one must show that for $\varphi \in D(A)$, $\psi \in Q(q^{(B)})$:

$$q^{(B)}(\psi, \varphi) = (\psi, A\varphi) \tag{2.2}$$

where $q^{(B)}(\cdot,\cdot)$ is the sequilinear form on $Q(q^{(B)})$ obtained from $q^{(B)}(\cdot)$ by polarization. Write $\psi = \varphi' + \eta'$, $\varphi' \in Q(A_\infty)$, $\eta' \in N$. Since $\eta' \in N$ $(\eta', A\varphi) = (A^*\eta', \varphi) = 0$ so

$$(\psi, A\varphi) = (\varphi', A\varphi) = (\varphi', A_\infty\varphi) = q^{(B)}(\psi, \varphi)$$

proving (2.2). ▨

LEMMA 2.5. ([1, 5, 12]) $D(A^*) = D(A_\infty) \dotplus N$.

Proof. Let $\psi \in D(A^*)$. Then $\varphi = A_\infty^{-1} A^* \psi \in D(A_\infty)$ and $A^*(\psi - \varphi) = A^*\psi - A^*\psi = 0$ so $\eta = \psi - \varphi \in N$ so $D(A^*) = D(A_\infty) + N$. Let $\varphi \in N \cap D(A_\infty)$. Then, by Lemma 2.1, $\varphi = 0$. ▨

PROPOSITION 2.6. *Let $\widetilde{A}$ be a positive self-adjoint extension of A. Then, there exists B on N so that $\widetilde{A} = A_B$.*

Proof. Since $D(A) \subset Q(\widetilde{A})$ and $q_{\widetilde{A}} \mid D(A) = q_{A_\infty}$, we have that $Q(A_\infty) \subset Q(\widetilde{A})$. Let $\psi \in D(\widetilde{A}) \subset D(A^*)$. Then, by Lemma 2.5, $\psi = \varphi + \eta$, with $\varphi \in D(A_\infty)$, $\eta \in N$. It follows that $\eta \in Q(\widetilde{A})$. Let $\widetilde{N} = N \cap Q(\widetilde{A})$ and let

$$p = q_{\widetilde{A}} \,|\widetilde{N}.$$

Since $q_{\widetilde{A}}$ is closed and positive so is p and thus $p = q_B$ for some B on $\widetilde{N}$. We extend p to N by setting it equal to ∞ on $N \setminus \widetilde{N}$ and view B as a partially defined operator.

Now, let $\psi = \varphi + \eta \in D(\widetilde{A})$ as above. Then picking $\varphi_n \in D(A)$ with $\varphi_n \to \varphi$ in A_∞-norm, we see that

$$(\varphi, \widetilde{A}\eta) = \lim(\varphi_n, \widetilde{A}\eta) = \lim(A\varphi_n, \eta) = \lim(\varphi_n, A^*\eta) = 0.$$

Thus

$$(\psi, \widetilde{A}\psi) = (\varphi, \widetilde{A}\varphi) + (\eta, \widetilde{A}\eta) = (\varphi, A_\infty\varphi) + (\eta, B\eta).$$

This establishes that $q_{\widetilde{A}} \mid D(\widetilde{A}) = q^{(B)} \mid D(\widetilde{A})$. But $D(\widetilde{A})$ is clearly a form core of $q_{\widetilde{A}}$. By a simple argument, $D(\widetilde{A})$ is also a form core of $q^{(B)}$. Thus, $\widetilde{A} = A_B$. ▨

LEMMA 2.7. $D(A_\infty) = D(A) \dotplus A_\infty^{-1}N$.

Proof. Let $R = \operatorname{Ran}(A)$ which is closed since A is closed. Then $\psi \in R^\perp$ if and only if $A^*\psi = 0$ so $R^\perp = N$. Thus given any $\psi \in D(A_\infty)$, we write $A_\infty\psi = A\varphi + \eta$ with $\varphi \in D(A)$ and $\eta \in N$. Then $\psi = \varphi + A_\infty^{-1}\eta$, so $D(A_\infty) = D(A) + A_\infty^{-1}N$. If $\varphi \in D(A) \cap A_\infty^{-1}N$, then $(A\varphi, A\varphi) = (AA_\infty^{-1}\eta, A\varphi) = (\eta, A\varphi) = 0$ so $\varphi = 0$. ▨

PROPOSITION 2.8. *If* $\dim(N) < \infty$, *then every self-adjoint extension of A is semibounded and is equal to A_B for some B.*

Proof. Let $\widetilde{A}$ be a self-adjoint extension and suppose that $\dim(N) = m < \infty$. If $\dim \mathscr{E}_{(-\infty,0)}(\widetilde{A}) \geqslant 2m + 1$, we can find $0 \neq \varphi \in D(A) \cap \operatorname{Ran}\mathscr{E}_{(-\infty,0)}(\widetilde{A})$ since $D(A^*) = D(A) \dotplus A_\infty^{-1}N \dotplus N$ by Lemmas 2.5 and 2.8. But then $(\varphi, A\varphi) \geqslant \|\varphi\|^2$ and $(\varphi, \widetilde{A}\varphi) \leqslant 0$, so $\|\varphi\| = 0$. This contradiction shows that $\dim \mathscr{E}_{(-\infty,0)}(\widetilde{A}) \leqslant 2m$ (of course more work shows it is $\leqslant m$) and so $\widetilde{A}$ is bounded below. We can now follow the construction in Proposition 2.6 exploiting the fact that every quadratic form on a finite dimensional space is closed. ▨

We summarize the last three propositions in a theorem which is the fundamental theorem of the theory. We emphasize that our proof is really just chasing one's tail with forms and that the result, proven by other means, is essentially in [1].

THEOREM 2.9. *There is a one to one correspondence between positive self-adjoint extensions, A_B, and positive forms, B, on N. If* $\dim(N) < \infty$, *the word positive may be dropped in both places.*

Of course, while we developed the theory with $\gamma(A) \geqslant 1$, by scaling there is no real difference if $\gamma(A) > 0$.

THEOREM 2.10. $A_B \geqslant A_{B'}$ *if and only if* $B \geqslant B'$.

Proof. Obvious given the basic formula $\psi = \varphi + \eta$:

$$(\psi, A_B\psi) = (\varphi, A_\infty\varphi) + (\eta, B\eta). \tag{2.3}$$

We can now prove Kreĭn's theorem (Theorem 1.3):

THEOREM 2.11. A_∞, *the Friedrichs' extension is the largest s.a. extension of A. A_0, the Kreĭn extension is the smallest positive extension. The set of positive extensions is precisely the set of C's with $A_0 < C < A_\infty$.*

Proof. Clearly $A_\infty \geqslant A_B \geqslant A_0$ if $\infty \geqslant B \geqslant 0$, so the theorem is evident except for the last sentence. If $A_0 \leqslant C \leqslant A_\infty$, then $Q(A_\infty) \subset Q(C) \subset Q(A_0) = Q(A_\infty) \dotplus N$.

In addition, since $A_0 \mid Q(A_\infty) = A_\infty$, $C \mid Q(A_\infty) = A_\infty$. Moreover if $\eta \in N$, $\varphi \in Q(A_\infty)$

$$(\varphi, A_\infty\varphi) = (\varphi + \lambda\eta, A_0(\varphi + \lambda\eta)) \leqslant (\varphi, A_\infty\varphi) + 2\mathrm{Re}(\overline{\lambda}(\eta, C\varphi)) + |\lambda|^2 (\eta, C\eta),$$

so $(\eta, C\varphi) = 0$ and $(\eta, C\eta) \geqslant 0$. Thus $C = A_B$ for some B. ▨

In the remainder of this section we treat a number of special features of the theory. First is a result of Kreĭn:

THEOREM 2.12. ([5]) *Let* (2.1) *hold. Then* A_∞ *is the only self-adjoint extension with lower bound* 1 *if and only if for all* $\eta \in N$, $\eta \neq 0$

$$\sup_{\substack{\varphi \in D(A) \\ (\eta, \varphi) \neq 0}} \frac{|(\eta, \varphi)|^2}{(\varphi, (A-1)\varphi)} = \infty. \tag{2.4}$$

Proof. Let $\varphi \in Q(A_\infty)$, $\eta \in N$. Then

$$((\varphi + \lambda\eta), A_B(\varphi + \lambda\eta)) \geqslant \|\varphi + \lambda\eta\|^2$$

if and only if

$$(\varphi, (A_\infty - 1)\varphi) + |\lambda|^2 (\eta, (B - 1)\eta) \geqslant 2\mathrm{Re}\,(\lambda(\varphi, \eta)).$$

Thus, $A_B \geqslant 1$ if and only if for all $\eta \neq 0$, $(\varphi, \eta) \neq 0$,

$$(\eta, (B-1)\eta) \geqslant |\eta, \varphi)|^2/(\varphi, (A_\infty - 1)\varphi).$$

If (2.4) holds, then this is only consistent with $B = \infty$. Conversely, if the sup in (2.4) is $\alpha < \infty$ for some η_0 then

$$B = (\alpha + 1) P_{\eta_0} + \infty(1 - P_{\eta_0})$$

yields a B with $A_B \geqslant 1$. ▨

REMARK. Notice that if $\gamma(A_B) \geqslant 1$, then $\gamma(A_{\tilde{B}}) \geqslant 1$ for all $B \leqslant \tilde{B}$, so if there are two extensions with $\gamma(\tilde{A}) \geqslant 1$, there are infinitely many.

EXAMPLE 2.1. Let $A = -\dfrac{d^2}{dx^2}$ on $C_0^\infty \subset L^2(0, \pi)$. Then A_∞ has Dirichlet boundary conditions. The lowest eigenvector is $\sin(x)$ so (2.1) holds. The operator with boundary condition $u(0) = -u(\pi)$, $u'(0) = -u'(\pi)$ is another with the same lower bound. There are many others; see the discussion in Example 2.4.

THEOREM 2.13. *Let A obey* (2.1). *Let* $\gamma(B) \geqslant 0$. *Then*

$$\alpha\gamma(B) \leqslant \gamma(A_B) \leqslant \gamma(B)$$

with $\alpha = \dfrac{1}{b+1}$ *and* $b = \gamma(B)$. *In particular,* $\gamma(A_B) = 0$ *if and only if* $\gamma(B) = 0$.

Proof. Clearly, if $\eta \in N$, $(\eta, A_B\eta) = (\eta, B\eta)$ so $\gamma(B) \geqslant \gamma(A_B)$. On the other hand, using $x = \|\varphi\|$, $y = \|\eta\|$, $b = \gamma(B)$

$$\gamma(A_B) \geqslant \min_{x,\, y} \left[(x^2 + by^2) / (x^2 + y^2 + 2xy)\right] = \alpha b. \ ▨$$

COROLLARY 2.14. *If* (2.1) *holds and* $\dim(N) = 1$, *then the Kreĭn extension is the unique one with* $\gamma(A_B) = 0$.

This corollary and the possibility of adding constants leads to a complete analysis of the one dimensional case.

THEOREM 2.15. *Suppose that* $\dim(N) = 1$. *Then, for any extension* A_B *other than* A_∞, *the form domain* $Q(A_B) = Q(A_\infty) \dotplus N$, *is the same. In addition, either*:

a) A *is the unique extension with* $\gamma(A_\infty) = \gamma(\widetilde{A})$. *In that case, there exists a strictly monotone and concave function* c: $(-\infty, \infty)$ *to* $(-\infty, \gamma(A_\infty))$ *with* $\gamma(A_b) = c(b)$.

or

b) *There is a* $b_0 < \infty$ *with* $\gamma(A_b) = \gamma(A_\infty)$ *for* $b \in [b_0, \infty)$ *and a strictly monotone and concave function* c *from* $(-\infty, b_0)$ to $(-\infty, \gamma(A_\infty))$ *with* $\gamma(A_b) = c(b)$.

Proof. By Corollary 2.14, when $c_0 < \gamma(A_\infty)$, there is a unique b with $\gamma(A_b) = c_0$. Concavity of c follows from the fact that it is an inf of linear functions. ▨

EXAMPLE 2.2. Let $A = -\dfrac{d^2}{dx^2} + 1$ on $C_0^\infty(0, \infty)$. Then N = multiples of $\sqrt{2}\, e^{-x}$. $Q(A_\infty)$ consists of functions which are absolutely continuous with $f(0)=0$. Then an arbitrary f in $Q(A_\infty) \dotplus N$ has the form

$$f = (f - f(0)e^{-x}) + f(0)e^{-x}$$

and

$$(f,\ A_b f) = \int_0^\infty |f(x) - f(0)e^{-x}|^2 dx + \int_0^\infty |f'(x) + f(0)e^{-x}|^2 dx + \frac{1}{2} b\, |f(0)|^2 =$$

$$= \int_0^\infty [|f'(x)|^2 + |f(x)|^2]\, dx + \left(\frac{1}{2} b - 1\right)|f(0)|^2.$$

$b = \infty$ corresponds to $f(0) = 0$ b.c. and $b = 2$ to $f'(0) = 0$ b.c. . In general, A_b has $f'(0) = \left(\dfrac{1}{2} b - 1\right) f(0)$ b.c. (the boundary conditions are determined by integrating by parts and demanding symmetry)

$$\gamma(b) = 1 \qquad \text{for} \qquad b \geqslant 2$$

$$\gamma(b) = 1 - \left(\frac{1}{2} b - 1\right)^2 \quad b < 2.$$

EXAMPLE 2.3. Let $A = -\dfrac{d^2}{dx^2}$ on $L^2(0, \pi)$ with the boundary conditions,

$$u(0) = u(\pi) = 0, \;\; u'(\pi) = 0.$$

N consists of multiples of $\sqrt{3}\,\pi^{-3/2}x$. The composition $Q(A_\infty)\dot{+}N$ corresponds to $f = \left[f - \dfrac{f(\pi)}{\pi} x\right] + f(\pi)\dfrac{x}{\pi}$ and all functions in $Q(A_b)$ obey $f(0) = 0$. Moreover

$$(f, A_b f) = \int_0^\pi |f'(x)|^2 \, dx + \left(\frac{b\pi}{3} - \frac{1}{\pi}\right)|f(\pi)|^2$$

and the boundary conditions for A_b are

$$f(0) = 0, \;\; f'(\pi) = -\left(\frac{b\pi}{3} - \frac{1}{\pi}\right)f(\pi)$$

$$\gamma(b) = k(b)^2 \quad b \geqslant 0$$

where $k(b)$ is the inverse function to

$$b = \frac{3}{\pi}\left(\frac{1}{\pi} - \frac{k}{\tan k\pi}\right) \quad 0 \leqslant k \leqslant 1.$$

and

$$\gamma(b) = -K(b)^2 \qquad b \leqslant 0$$

where $K(b)$ is the inverse function to

$$b = \frac{3}{\pi}\left(\frac{1}{\pi} - \frac{K}{\tanh(K\pi)}\right) \quad 0 \leqslant K < \infty$$

Notice this is an example with $b_0 = \infty$.

EXAMPLE 2.4. Let $A = -\dfrac{d^2}{dx^2}$ on $L^2(0, \pi)$ with the boundary conditions

$$u(0) = u(\pi) = 0; \;\; u'(0) = -u'(\pi).$$

N consists of multiples of $\left(x - \frac{\pi}{2}\right)\sqrt{12}\,\pi^{3/2}$ since A^* has the boundary condition $f(0) = -f(\pi)$. As usual $Q(A_\infty)$ consists of functions with $f(0) = f(\pi) = 0$ and thus functions in $Q(A_0)$ obey $f(0) = -f(\pi)$. The basic descomposition of $Q(A_0) = Q(A_\infty) \dotplus N$ is

$$f = \left[f - \frac{2}{\pi} f(\pi)\left(x - \frac{\pi}{2}\right)\right] + \frac{2}{\pi} f(\pi)\left(x - \frac{\pi}{2}\right)$$

and therefore

$$(f,\ A_b f) = \int_0^\pi |f'|^2\, dx + \left[\frac{b\pi}{3} - \frac{4}{\pi}\right]| f(\pi) |^2.$$

The boundary conditions are:

$$f(0) = -f(\pi); \quad f'(\pi) + f'(0) = \left(-\frac{b\pi}{3} + \frac{4}{\pi}\right) f(\pi).$$

$b = \infty$ yields Dirichlet boundary conditions and $b = \frac{12}{\pi^2}$ yields antiperiodic boundary conditions. The lowest eigenfunction even about $x = \frac{\pi}{2}$ has eigenvalue 1. The lowest odd eigenfunction has eigenvalue k^2 or $-K^2$ where k, K solve $(0 < k \leqslant 2)$

$$2k \cos\left(\frac{k\pi}{2}\right) = \left(-\frac{b\pi}{3} + \frac{4}{\pi}\right) \sin\left(\frac{k\pi}{2}\right) \tag{2.5}$$

or the analog

$$2K \cosh\left(\frac{K\pi}{2}\right) = \left(-\frac{b\pi}{3} + \frac{4}{\pi}\right) \sinh\left(\frac{K\pi}{2}\right) \tag{2.6}$$

the lower bound is

$$\begin{aligned} \gamma(b) &= 1 & & b \geqslant \frac{12}{\pi^2} \\ &= k(b)^2 & & 0 \leqslant k \leqslant 1, \quad 0 < b < \frac{12}{\pi^2} \\ &= -K(b)^2 & & 0 \leqslant K < \infty, b \leqslant 0 \end{aligned}$$

where k solves (2.5) and K solves (2.6).

§ 3. GRAPHS

Our main goal in this section is to prove and understand the following result:

THEOREM 3.1. *Under the notation of Section* 2:

$$D(A_B) = \{\varphi + A_\infty^{-1}(Bf + \eta) + f \mid \varphi \in D(A),\ f \in D(B),\ \eta \in N \cap D(B)^\perp\} \tag{3.1}$$

with

$$A_B = A^* \mid D(A_B) \tag{3.2}$$

REMARK. This is equivalent to a rather different looking result in Vishik [12] and Birman [1].

EXAMPLE 3.1. If $D(B) = \{0\}$, i.e. $B \equiv \infty$, then we see that $D(A_\infty) = D(A) \dotplus A_\infty^{-1} N$, a result which is not new (Lemma 2.7).

EXAMPLE 3.2. (Kreĭn [5]). *If* $B = 0$, *then*

$$D(A_0) = D(A) \dotplus N \tag{3.3}$$

and this is the only extension with this property. In particular

$$\mathrm{Ker}(A_0) = N. \tag{3.4}$$

EXAMPLE 3.3. Consider Example 2.2. Then $A_\infty^{-1}\ (\mathrm{e}^{-x}) = \frac{1}{2} x\mathrm{e}^{-x}$ since $g = \frac{1}{2} x\mathrm{e}^{-x}$ obeys $-g'' + g = \mathrm{e}^{-x}$ with the right boundary conditions at 0 and infinity. Thus, an arbitrary function in $D(A_b)$ obeys

$$\eta = \varphi + a\left(\mathrm{e}^{-x} + \frac{1}{2} bx\mathrm{e}^{-x}\right)$$

with $\varphi \in D(A)$. Thus $\eta(0) = a$, $\eta'(0) = \left(\frac{1}{2} b - 1\right) a$ and we obtain the boundary conditions one obtains by trial and error.

LEMMA 3.2. *The right side of* (3.1) *is contained in the domain of* A_B.

Proof. We need only show that for $\eta \in N \cap D(B)^\perp$, $f \in D(B)$, $\varphi \in Q(A_\infty)$ and $\tilde{f} \in Q(B)$, we have that

$$(\varphi + \tilde{f},\ Bf + \eta) = q^{(B)}\ (\varphi + \tilde{f},\ A_\infty^{-1}(Bf + \eta) + f). \tag{3.5}$$

The right side of (3.5) is equal to

$$(\varphi,\ Bf + \eta) + (\tilde{f},\ Bf)$$

which equals the left side since $(\tilde{f}, \eta) = 0$. ▨

All that remains is to show that A^* on the right side of (3.5) defines a maximal symmetric extension of A. Let X be a subspace of $D(A^*) = D(A) \dotplus A_\infty^{-1}N \dotplus N$ and which contains $D(A)$. Let

$$\tilde{\Gamma}(X) = \{ \langle \alpha, \beta \rangle \in N \times N \mid A_\infty^{-1}\alpha + \beta \in X\}.$$

Clearly X is determined by $\tilde{\Gamma}(X)$. Now let

$$\omega(\langle \alpha, \beta\rangle, \langle \alpha', \beta'\rangle) = (\alpha, \beta') - (\beta, \alpha')$$

be the natural symplectic form on $N \times N$. By a direct calculation

$$((A_\infty^{-1}\alpha' + \beta'), A^*(A_\infty^{-1}\alpha + \beta)) - (A^*(A_\infty^{-1}\alpha' + \beta'), (A_\infty^{-1} \alpha + \beta))=$$

$$= -\omega(\langle \alpha', \beta'\rangle, \langle\alpha, \beta\rangle)$$

so we see that if $A^* \mid X$ is symmetric, then $\tilde{\Gamma}(X)$ is a Lagrangian subspace for ω (i.e. $\omega(x, y) = 0$ for all $x, y \in \tilde{\Gamma}(X)$). The converse is also fairly easy to see so we have proven that:

LEMMA 3.3. *$A^* \mid X$ is symmetric if and only if $\tilde{\Gamma}(X)$ is a Lagrangian subspace for ω. Maximal symmetric extensions of A correspond to maximal Lagrangian subspaces.*

Given a quadratic form q_B on N, let

$$\Gamma(B) = \{\langle\varphi, B\varphi + \eta\rangle \mid \varphi \in D(B) \subset Q(B), \eta \in N \cap Q(B)^\perp\}.$$

This is just the natural extension of the notion of the graph of B.

THEOREM 3.4. *Every $\Gamma(B)$ is a maximal Lagrangian subspace. Conversely, if N is finite dimensional, every maximal Lagrangian subspace is of the form $\Gamma(B)$ for some B.*

REMARKS 1. $\dim(N) < \infty$ comes not only because we require B to come from forms but also because of the existence of non-self-adjoint maximal symmetric operators.

2. The converse is not needed to prove Theorem 3.1.

Proof. The symmetry of B shows that $\Gamma(B)$ is Lagrangian. Suppose that $\langle\varphi_0, \psi_0\rangle$ is ω-orthogonal to $\Gamma(B)$. Thus

$$(\varphi_0, B\varphi + \eta) = (\psi_0, \varphi) \tag{3.6}$$

for all $\varphi \in D(B)$, $\eta \in Q(B)^\perp$. Taking $\varphi = 0$, we see that $\varphi_0 \in Q(B)^{\perp\perp} = \overline{Q(B)}$. Moreover (3.6) then implies that $\varphi_0 \in D(B^*)$ and $P\psi_0 = B^*\varphi_0$ where $P =$ proj. onto $\overline{Q(B)}$.

Since B is self-adjoint, we conclude that $\psi_0 = B\varphi_0 + \eta_0$ with $\eta_0 \in D(B)^{\perp}$. Thus $\langle \varphi_0, \psi_0 \rangle \in \Gamma(B)$. It follows that $\Gamma(B)$ is maximal Lagrangian.

Conversely, let $\dim(N) < \infty$ and let M be a maximal Lagrangian subspace. Let

$$D(B) = \{\varphi \in N \mid \langle \varphi, \psi \rangle \in M \text{ for some } \psi \in N\}.$$

Let $\eta \in D(B)^{\perp}$. Then for $\langle \varphi, \psi \rangle, \langle \varphi', \psi' \rangle \in M$:

$$\omega(\langle \varphi, \psi + \eta \rangle, \langle \varphi', \psi' \rangle) = \omega(\langle \varphi, \psi \rangle), \langle \varphi', \psi' \rangle)$$

so by the maximality, $\langle \varphi, \psi + \eta \rangle \in M$ also. Conversely, let $\langle \varphi, \psi \rangle, \langle \varphi, \psi' \rangle \in M$. Then $\langle 0, \psi - \psi' \rangle \in M$ so $\psi - \psi'$ must be orthogonal to $D(B)$. It follows that for $\varphi \in D(B)$, there exists a unique $B\varphi \in D(B)$ with $(\varphi, B\varphi) \in M$. Thus $M = \Gamma(B)$. ▨

Proof of Theorem 3.1. Let X denote the right side of (3.1). Then $\widetilde{\Gamma}(X)$ is a maximal Lagrangian subspace by Theorem 3.4 and the sign flip of ω under $\langle \alpha, \beta \rangle \to \langle \beta, \alpha \rangle$. Thus, by Lemma 3.3, $A^* \mid X$ is a maximal symmetric extension of A. By Lemma 3.2, $A^* \mid X \subset A_B \mid D(A_B)$, so, by the self-adjointness of A_B, we see that $A_B = A^* \mid X$. ▨

REMARK. The constructions show how one can arrive apriori at (3.1). The association $B \to \Gamma(B)$ is natural for non-densely defined B. With it and our convention on $(B + \mathrm{i})^{-1}$, one finds that norm resolvent convergence is equivalent to *graph convergence* in the Kreĭn sense [3] even for non densely defined operators.

§ 4. CONTINUITY OF OUR PARAMETRIZATION

Since the number of elements in $U(n)$ for any $n < \infty$ are the same, one might argue that it is only a matter of convenience that one chooses in the von Neumann theory to parametrize the extensions, in the deficiency index k case, by $U(k)$ and not $U(n)$. However, since $U(n)$ is not homeomorphic to $U(k)$ for $k \neq n$, once one establishes continuity, the labelling is demostrated to be "natural". We begin by noting the continuity of the von Neumann parametrization.

THEOREM 4.1. *The von Neumann map $U \to A^{(U)}$ from unitaries mapping $K_+ = \mathrm{Ker}(A^* + \mathrm{i})$ to $K_- = \mathrm{Ker}(A^* - \mathrm{i})$ to self-adjoint extensions of A is open and continuous if U is given the topology of strong (resp. norm) convergence and the extensions the topology of strong (resp. norm) resolvent convergence.*

Proof. Let $C\colon K_+^{\perp} \to K_-^{\perp}$ by

$$C = (A + \mathrm{i})\ (A - \mathrm{i})^{-1}$$

and extend C to H by setting it equal to 0 on K_+. Extend U: $K_+ \to K_-$ by setting it equal to 0 on $K_+^\perp$. Then the map from U to $A^{(U)}$ is given by:

$$(A^{(U)} + \mathrm{i})\,(A^{(U)} - \mathrm{i})^{-1} \equiv C(U) = C + U$$

since $[C(U) - 1]\,(2\mathrm{i})^{-1} = (A^{(U)} - \mathrm{i})^{-1}$ we see that strong (norm) convergence of U is equivalent to strong (norm) convergence of $(A^{(U)} - \mathrm{i})^{-1}$. ▨

The usefulness of strong graph limits is shown by:

THEOREM 4.2. *If B_n, B are all positive or if* $\dim(N) < \infty$, *and if* $(B_n + \mathrm{i})^{-1} \to$ $\to (B + \mathrm{i})^{-1}$ *strongly, then* $(A_{B_n} + \mathrm{i})^{-1} \to (A_B + \mathrm{i})^{-1}$ *strongly.*

Proof. We begin by noting that $B_n \to B$ in strong graph sense (see [7] for the definition; the proof of Theorem VIII.26 easily extends to non-densely defined B's). By our analysis in § 3, $A_{B_n} \to A_B$ in strong graph sense and thus in strong resolvent sense. ▨

REMARKS 1. If $\dim(N) < \infty$, by compactness, the map is automatically also open. Moreover, norm convergence is equivalent to strong convergence.

2. If $\dim(N) < \infty$ and if B is everywhere defined, this theorem follows also from the results of [10].

§ 5. THE KREĬN EXTENSION

We begin this section by recovering an abstract result of Kreĭn [5] on the discreteness of the spectrum of A_∞ and A_0. Then, we discuss Kreĭn extensions of $-\dfrac{\mathrm{d}}{\mathrm{d}x^2}$ on $C_0^\infty(0,1)$ and of $-\Delta$ on $C_0^\infty(\Omega), \Omega \subset \mathbf{R}^n$.

THEOREM 5.1. *Let* $A \geqslant I$. *Let* $B = A_0 \mid N^\perp$. [*N is an invariant space for A_0 (indeed, it is* $\mathrm{Ker}(A_0)$) *and so $N^\perp$ is also.*] *Let $\mu_k(C)$ be given by the min-max principle (so*

$$\mu_k(C) = \inf\,\{\lambda \mid \dim \mathscr{E}_{(-\infty,\lambda)}\,(C) \geqslant k\}).$$

Then

$$\mu_k(B) \geq \mu_k(A_\infty). \tag{5.1}$$

In particular (Kreĭn [5]), if A_∞ has discrete spectrum, then, except possibly for $\lambda = 0$, A_0 also has discrete spectrum.

Proof. Let Q be the orthogonal projection from H onto $N^\perp$. We claim that Q is a bijection from $Q(A_\infty)$ onto $Q(A_0) \cap N^\perp$ and that

$$(Q(\varphi),\, A_0 Q(\varphi)) = (\varphi,\, A_\infty \varphi). \tag{5.2}$$

For let $P = 1 - Q$ = orthogonal projection onto N. Then given $\varphi \in Q(A_\infty)$, $Q(\varphi) = \varphi - P\varphi \in Q(A_\infty) \dot{+} N = Q(A_0)$ so Q maps into $Q(A_0) \cap N^\perp$. Moreover, if $\varphi \in Q(A_\infty)$ and $Q\varphi = 0$, then $\varphi \in Q(A_\infty) \cap N$, so $\varphi = 0$ by Lemma 2.1. Thus Q is 1-1. Moreover, since

$$Q(\varphi) = \varphi + \eta, \ \eta = -P\varphi \in N,$$

$$(Q(\varphi),\ A_0 Q(\varphi)) = (\varphi,\ A_\infty\varphi) + (\eta,\ 0\eta) = (\varphi,\ A_\infty\varphi)$$

so (5.2) holds. Finally, for any $\psi \in Q(A_0)$, we can write $\psi = \varphi + \eta$ uniquely by Lemma 2.1. Let $T\psi = \varphi$. Then

$$Q\psi = Q(T\psi)$$

so for $\psi \in N^\perp \cap Q(A_0)$, $\psi = Q(T\psi)$ so Q is onto.

Given ε, k, find $\varphi_1, \ldots, \varphi_k$ orthonormal in $\mathscr{E}_{(-\infty,\mu_k(B)+\varepsilon)}(B)$. Given $\psi_1, \ldots, \psi_{k-1} \in H$, we can find $f = \sum \alpha_i \varphi_i \neq 0$ so that $Tf \perp \psi_i$. This is possible since T is one-one and thus $T[\varphi_1, \ldots, \varphi_k]$ has dimension k. Since $(Tf, Tf) \geqslant (f,f)$ ($f = Q(Tf)$), (5.2) implies that

$$\mu_n(B) + \varepsilon \geqslant \frac{(f,\ Bf)}{(f,f)} \geqslant \frac{(Tf,\ A_\infty Tf)}{(Tf,\ Tf)},$$

Therefore:

$$\min_{g \in [\psi_i]^\perp} (g,\ A_\infty g)/(g,\ g) \leqslant \mu_n(B) + \varepsilon.$$

Since ε is arbitrary, we can set ε to zero and then maximize over ψ_i to obtain (5.1). ▨

EXAMPLE 5.1. Let $A = -\dfrac{d^2}{dx^2}$ on $C_0^\infty(0,1)$. Then N consists of the span of 1 and x. The decomposition

$$f = \varphi + \eta, \quad \eta \in N,\ \varphi \in Q(A_\infty)$$

is given by:

$$\eta = f(0) + [f(1) - f(0)]x\ ; \quad \varphi = f - \eta.$$

Thus:

$$(f,\ A_0 f) = \int |\nabla(f - \eta)|^2.$$

But, since $\nabla^2\eta = 0$:

$$\int(\overline{\nabla f})(\nabla\eta) = \int \nabla\cdot(\bar{f}\nabla\eta) =$$

$$= \int_{\partial[0,1]} (\bar{f}\nabla\eta) = \int_{\partial[0,1]} \bar{\eta}\nabla\eta = \int|\nabla\eta|^2 \tag{5.3}$$

where (5.3) follows from the fact that $f = \eta$ on $\partial[0,1]$. Therefore:

$$(f, A_0 f) = \int_0^1 |\nabla f|^2 dx - (f(1) - f(0))^2. \tag{5.4}$$

From this formula, one easily sees that A_0 is smaller than the Neumann form which one might well think was the smallest extension. Moreover, since $|f(1) - f(0)| \leqslant \leqslant \int_0^1 |\nabla f(x)|\, dx$, A_0 is clearly positive.

One can either read off the boundary conditions from (5.4) or from the general theory in § 3. By the general theorem in § 3, $f \in D(A_0)$ is of the form $f = c + dx + g$ with $g(0) = g(1) = g'(0) = g'(1) = 0$. Thus f is given by the boundary conditions:

$$f'(0) = f'(1) = f(1) - f(0). \tag{5.5}$$

Finally, let us compute the spectrum of A_0 and compare it with A_∞ and with A_N, the Neumann extension. By standard calculations:

$$\mu_m(A_\infty) = (m\pi)^2 \tag{5.6}$$

$$\mu_m(A_N) = [(m - 1)\pi]^2. \tag{5.7}$$

To study $\mu_k(A_0)$, we note that $D(A_0)$ is left invariant by $f(x) \to f(1 - x)$ and thus all eigenfunctions obey either $f'\left(\frac{1}{2}\right) = 0$ or $f\left(\frac{1}{2}\right) = 0$. The $f'\left(\frac{1}{2}\right)$ eigenfunctions are $\cos\left(k\left(x - \frac{1}{2}\right)\right)$ and (5.5) becomes $\sin\left(\frac{1}{2}k\right) = 0$ so $k = 0, 2\pi, 4\pi, \ldots$. The $f\left(\frac{1}{2}\right)$ eigenfunctions are $\sin\left(k\left(x - \frac{1}{2}\right)\right)$ so (5.5) becomes

$$\frac{1}{2}k = \tan\left(\frac{1}{2}k\right). \tag{5.8}$$

$k = 0$ solves (5.8) but one must check there is a solution corresponding to this. The function $f(x) = \left(x - \frac{1}{2}\right)$ is such a function. Thus the lowest solution k_1 of (5.8) is $k_1 = 0$. There is no other solution in $\left(0, \frac{\pi}{2}\right)$ and then one each with

$$\frac{1}{2} k_j \in \left(\pi(j-1), \pi\left(j - \frac{1}{2}\right)\right)$$

with $\left| k_j - 2\pi\left(j - \frac{1}{2}\right)\right| \to 0$ as $j \to \infty$. Thus

$$\mu_m(A_0) = [(m-1)\pi]^2 \qquad m \text{ odd}$$

$$\mu_m(A_0) = (k_{\frac{1}{2}m})^2 \qquad m \text{ even.}$$

So for m even

(5.9) $$[\pi(m-2)]^2 \leqslant \mu_m(A_0) < [\pi(m-1)]^2.$$

We see explicitly that $\mu_m(A_0) \leqslant \mu_m(A_N)$ with inequality for $m=2, 4, 6 \ldots$. Moreover, (5.1) reads $\mu_{m+2}(A_0) \geqslant \mu_m(A_\infty)$; notice that the inequality is always strict.

EXAMPLE 5.3. Let $\Omega \subset \mathbf{R}^n$ be an open set which is bounded and for which $\partial\Omega$ has measure zero. Let $A = -\Delta$ on $C_0^\infty(\Omega)$. Then N is precisely the set of functions in $L^2(\Omega)$ which are harmonic on Ω. Thus, if $n > 1$,

$$N = \mathrm{Ker}(A_0)$$

has infinite dimension. Since A_∞ has discrete spectrum, Theorem 5.1 implies that A_0 has discrete spectrum away from zero. If $\partial\Omega$ is smooth, then for any f which is smooth up to the boundary, we can find a harmonic function, $H(f)$, with $H(f) = f$ on $\partial\Omega$. Moreover, $f - H(f) \in Q(A_\infty)$. Thus, by the calculation (5.3)

$$(f, A_0 f) = \int_\Omega |\nabla f|^2 \, dx - \int_\Omega |\nabla(H(f))|^2 \, dx.$$

For f to lie in $D(A_0)$ we need f to obey the special boundary condition

(5.10) $$\frac{\partial f}{\partial n}(x) = \frac{\partial H(f)}{\partial n}(x); \quad x \in \partial\Omega.$$

Notice that the right side only depends on the values of f on $\partial\Omega$ so that (5.10) is a boundary condition, albeit a non-local boundary condition in a natural sense.

Finally, in this example, we note that the map $M: g \to g \mid \partial\Omega$ on N is one-one. Thus, for any boundary conditions the boundary part of $\tilde{q}^{(B)}(f)$ given by

$$q^{(B)}f = \int |\nabla f|^2 \, dx + \tilde{q}^{(B)}(f)$$

$$\tilde{q}^{(B)}f = (M^{-1}(f \mid \partial\Omega), BM^{-1}(f \mid \partial\Omega)) - \int |\nabla(H(f))|^2 \, dx$$

is only a function of $f \mid \partial\Omega$. Thus, *at least formally*,

$$\tilde{q}^{(B)}(f) = \int_{\partial\Omega \times \partial\Omega} f(x) f(y) \, K(x, y) d\sigma(x) d\sigma(y).$$

Of course, in thinking of this formula one must remember that $\tilde{q}^{(B)}(f)$ may be infinite for some f's.

It seems to us that the Kreĭn extension of $-\Delta$, i.e. $-\Delta$ with the boundary condition (5.10), is a natural object and therefore worthly of further study. For example: Are the asymptotics of its non-zero eigenvalues given by Weyl-s formula? What happens to the operator under subdivision [11]? Is there any method of images formula for the Green's function in simple cases?

Added in proofs

W. Faris and H. Kalf have kindly pointed out to us some other English language papers on the BKV theory:

1. There is discussion of the BKV theory in T. Ando and K. Nishio, Positive self-adjoint extensions of positive symmetric operators, *Tohoku Math. J.*, **22** (1970), 65–75 and in Section 15 of W. Faris, *Self-Adjoint Operators*, Springer Math Lecture Notes, **437** (1975). Our presentation turns out to be very close in point of view to that of Faris. See also C. Skau, *Math. Scand.*, 44 (1979), 171–195.

2. What we call the "Kreĭn extension" is the extension introduced by von Neumann on pg. 102 of his paper, our ref. 6. Ando-Nishio call it the "von-Neumann extension".

3. Ref. 12 is available in an English translation in *AMS Translations*, (2) **24** (1963), 107–172.

Research partially supported by U.S.N.S.F. under grant MCS—78—01885.

REFERENCES

1. Birman, M. S., On the self-adjoint extensions of positive definite operators (Russian), *Math. Sb.*, **38** (1956), 431–450.
2. Friedrichs, K., Spectraltheorie halbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differential-operatoren, *Math. Ann.*, **109** (1934), 465–487.

3. KATO, T., *Perturbation theory for linear operators*, Springer Verlag, Berlin/Heidelberg/New York, 1966; 2nd ed., 1976.
4. KATO, T., Trotter's product formula for an arbitrary pair of self-adjoint contraction semigroups, *Topics in Functional Analysis*. (I. Gohberg and M. Kac, Eds.), Academic Press, New York, 1978; pp. 185–195.
5. KREĬN, M., The theory of self-adjoint extensions of semibounded Hermitian transformations and its applications. I, *Rec. Math. (Math. Sb.)*, **20** (62) (1947), 431–495.
6. VON NEUMANN, J., Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren, *Math. Ann.*, **102** (1929), 49–131.
7. REED, M.; SIMON, B., *Methods of modern mathematical physics, I. Functional Analysis*, Academic Press, 1972.
8. REED, M.; SIMON, B., *Methods of modern mathematical physics, II. Fourier analysis, self-adjointness*, Academic Press, 1975.
9. SIMON, B., A canonical descomposition for quadratic forms with applications to monotone convergence theorems, *J. Functional Analysis*, **28** (1978), 377–385.
10. SIMON, B., Lower semicontinuity of positive quadratic forms, *Proc. Roy. Soc. Edinburgh Sect. A*, **29** (1977), 267–273.
11. SIMON, B., Classical boundary conditions as a technical tool in modern mathematical physics, *Advances in Math.*, **30** (1978), 268–281.
12. VISHIK, M., On general boundary conditions for ellyptic differential equations (Russian), *Trudy Moskov. Mat. Obšč.*, **1** (1952), 187–246.

ALBERTO ALONSO
Department of Mathematics
and Mechanics
Ciudad University
IIMAS-UNAM
Mexico (20 DF), Mexico.

BARRY SIMON
Departments of Mathematics
and Physics,
Princeton University
Princeton, N.J. 08540
U.S.A.

Received December 3, 1979.