A COUNTEREXAMPLE TO THE PARAMAGNETIC CONJECTURE

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We provide a counterexample to the universal paramagnetism conjecture of Hogreve, Schrader and Seiler. The counterexample is based on the Bohm–Aharonov effect.

Several years ago, one of us [1] proved an inequality expressing the universal diamagnetic tendency of spinless bosons. For a single particle in external (local) electric potential V and magnetic potential $\boldsymbol{a}$, the inequality can be expressed as follows: Let

$$H_1(\boldsymbol{a}, V) \equiv (-\mathrm{i}\boldsymbol{\nabla} - \boldsymbol{a})^2 + V \tag{1}$$

and let

$$E_1(\boldsymbol{a}, V) \equiv \inf \operatorname{spec}[H_1(\boldsymbol{a}, V)] \, . \tag{2}$$

Then [1]:

$$E_1(\boldsymbol{a}, V) \geqslant E_1(\boldsymbol{a} = 0, V) \, , \tag{3}$$

for *any* $\boldsymbol{a}$, V. Subsequently, motivated by remarks of Nelson, Simon [2] extended (3) to a finite temperature result:

$$\operatorname{Tr}(\exp(-\beta H_1(\boldsymbol{a}, V))) \leqslant \operatorname{Tr}(\exp(-\beta H_1(\boldsymbol{a} = 0, V))) \tag{4}$$

(see ref. [3] for further developments). Of course (4) implies (3) by taking $\beta \to \infty$.

Roughly one year ago, Hogreve et al. [4] put forward a very attractive conjecture about the situation when spin is taken into account. Let $\boldsymbol{\sigma}$ be the conventional Pauli matrices

$$H_2(\boldsymbol{a}, V) \equiv (-\mathrm{i}\not{\nabla} - \not{a})^2 + V \tag{5}$$

$$= (-\mathrm{i}\boldsymbol{\nabla} - \boldsymbol{a})^2 + V + \boldsymbol{\sigma} \cdot \boldsymbol{B} \, , \tag{5'}$$

where $\not{A} = \Sigma_i A_i \sigma_i$ as usual. Then ref. [4] conjectures (eq. (18) of ref. [4]) that

$$\operatorname{Tr}(\exp(-\beta H_2(\boldsymbol{a}, V))) \geqslant \operatorname{Tr}(\exp(-\beta H_2(\boldsymbol{a} = 0, V))) \tag{6}$$

and, in particular, that

$$E_2(\boldsymbol{a}, V) \leqslant E_2(\boldsymbol{a} = 0, V) \, . \tag{7}$$

In ref. [4] a number of arguments are given in favor of this conjecture and since the apperance of ref. [4] a number of interesting developments have tended to support the conjecture. First, an inequality on functional determinants which follows from (6) (and which was the main concern of ref. [4]) has been proven even for suitable Yang–Mills fields [5]. Moreover, for the special case $\boldsymbol{a} = \frac{1}{2}(\boldsymbol{B}_0 \times \boldsymbol{r})$ (i.e. $\boldsymbol{B}$ = a constant, $\boldsymbol{B}_0$), where (7) had been independently conjectured (in an equivalent form) by Avron et al. [6], Lieb [‡1] proved that (7) held (it is still unknown whether (6) holds in this case). Subsequently, Avron and Seiler [7] extended Lieb's result to certain polynomial B's.

Our goal here is to provide a counterexample to (7) and thus to (6). We will deal with two dimensions and allow V to be infinite in certain regions but given that (7) is false in that case it is easy to conclude (7) will be false for suitable three-dimensional cases with V everywhere finite. Indeed, in the three-dimensional case, consider a potential $W_{a,l}(x,y,z) = V_a(x,y)\, \chi_l(z)$, where $V_a(x,y) = \min(a, V(x,y))$ and χ is the function which is 1 (respectively 0) for $|z| < l$ (respectively $|z| \geqslant l$). Then as $a, l \to \infty$, the ground state energy of the three-dimensional system approaches that of the two-dimensional system so if (7) holds for all three-dimensional systems

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[‡1] Lieb's proof appears as an appendix of ref. [6].

with finite potentials, it will hold for two-dimensional systems with potentials allowed to be infinite.

We will take a magnetic field $B(x, y)$ [‡2] which is axially symmetric under rotations in the plane centered at $(x, y) = (0, 0)$. In this case a convenient gauge for $\boldsymbol{a}$ is

$$\boldsymbol{a}(\boldsymbol{\rho}) = (2\pi\rho^2)^{-1}\boldsymbol{\Phi}(\rho)\times\boldsymbol{\rho}\ , \tag{8}$$

where $\Phi(\rho)$ is the total flux through the circle of radius ρ [‡3]. The gauge (8) has $\operatorname{div}\boldsymbol{a} = 0$. Thus [‡4]:

$$\begin{aligned} H_2(\boldsymbol{a}, V) &= (-\mathrm{i}\boldsymbol{\nabla} - \boldsymbol{a})^2 + V + \boldsymbol{\sigma}\cdot\boldsymbol{B} \\ &= \boldsymbol{p}^2 + \boldsymbol{a}^2 + 2\boldsymbol{a}\cdot\boldsymbol{p} + V + \boldsymbol{\sigma}\cdot\boldsymbol{B} \\ &= \boldsymbol{p}^2 + \boldsymbol{a}^2 - (2\pi\rho^2)^{-1}\,\Phi(\rho)\,L_z + V + \sigma_z B\ . \end{aligned} \tag{9}$$

Now take $B = \lambda\widetilde{B}$ where λ is a coupling constant which we will vary and $\widetilde{B}$ is the field which is 1 in a disc of radius 1 and 0 outside the disc. Now let $V = \rho^2 + \mu W$ where W is 1 (respectively 0) inside (respectively outside) the disc and take $\mu\to\infty$. If (7) holds for all finite μ it will hold in the limit. In this limit:

$$H_2(\lambda) = \boldsymbol{p}_{\mathrm{D}}^2 + \lambda^2\widetilde{a}^2 - \lambda(2\pi\rho^2)^{-1}\,\widetilde{\Phi}(\rho)\,L_z + \rho^2\ , \tag{10}$$

where $\boldsymbol{p}_{\mathrm{D}}^2$ indicates the vanishing boundary condition on the circle of radius 1. The B term has dropped out since $B \neq 0$ only in the region where $V = \infty$. When $\lambda = 0$, $H_2(\lambda)$ has a ground state with $L_z = 0$. Since $H_2(\lambda)$ is rotationally invariant, and the ground state has a finite distance from all other states, the ground state of $H_2(\lambda)$ will have $L_z = 0$ for small λ. But then, since $\widetilde{a}^2$ is strictly positive,

$$E_2(\lambda) > E_2(\lambda = 0)$$

for λ small, violating (7).

Clearly our counterexample is based on the old idea of Bohm–Aharonov [8]. We came upon it since in trying to verify (6) by writing the trace as a Wiener integral there are two terms which enter in the action when B is turned on: $\int \boldsymbol{\sigma}\cdot\boldsymbol{B}(\omega(t))\,\mathrm{d}t$, which tends to increase the trace and $\mathrm{i}\int \boldsymbol{a}(\omega(t))\,\mathrm{d}\omega$ (Ito stochastic integral) which tends to decrease it. Since only closed paths enter the trace one is tempted to write $\int \boldsymbol{a}(\omega)\times\mathrm{d}\omega$ = flux within ω [‡5] and it is clear that one would have to cancel effects of the field within ω by the field on ω which leads naturally to Bohm–Aharonov considerations. It is also clear from this point of view that for the V we discuss and β finite that (6) fails [‡6].

Finally we remark that Lieb's proof [‡1] in the case $B = B_0$ depends on the infinite degeneracy of the ground state of $H_1(\boldsymbol{a}, V = 0)$. For the B we pick there is no normalizable ground state for λ small (see ref. [9]).

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‡2 The field in two dimensions is a scalar but it is convenient to think of it as pointing in a fictitious third dimension and using three-vector notation.

‡3 Φ is a vector in the fictitious third dimension.

‡4 We abuse notation and use $\Phi(\rho)$ in eq. (9) as the magnitude of Φ.

‡5 Since ω is a general brownian path which is *not* rectifiable (see, e.g., Simon [9]), one cannot really talk about the flux through ω for general ω but it is a useful intuition.

‡6 Since $V = \infty$ in the region $B \neq 0$, the flux is just a winding number which can be defined for any continuous ω which avoids the region where $V = \infty$. If $A_n \geqslant 0$ is the contribution to the trace when $\boldsymbol{a} = 0$ at paths with winding number n, then the right side of eq. (6) $= A_n$ and the left side of eq. (8) $= \Sigma\, \mathrm{e}^{\mathrm{i}n\Phi}A_n$ and eq. (6) is obviously false.

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