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Distributions and Their Hermite Expansions*

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We present a self-contained treatment of the technical parts of distribution theory needed in quantum field theory. The treatment is particularly suited for physicists since an absolute minimum of abstract functional analysis is used: In fact, only the Baire category theorem is needed. The simple nature of some proofs depends on extensive use of the expansion of a distribution as a sum of harmonic oscillator wavefunctions. While this Hermite expansion is not new, the fact that it provides elementary proofs of several theorems does appear to be new.

1. INTRODUCTION AND NOTATION

Schwartz's theory of tempered distributions is basic to the Gårding–Wightman axiomatation of relativistic quantum field theory.[1–3] Field theory requires technical results from distribution theory and not merely the "classical" differential calculus and Fourier analysis of distributions—in particular, the kernel (or nuclear) theorem is needed to define the Wightman functions as distributions in many variables (Ref. 2, p. 106). The purpose of this article is to present a proof of the kernel theorem particularly suited for the physicist—not only is a minimum amount of real analysis used, but the basic tool is the harmonic oscillator wavefunctions, a familiar friend to any physicist.

The approach we use also provides a simple proof of the regularity theorem and several other results mentioned in Streater and Wightman.[2] By adding short sections on the Baire category theorem and on convergence in $\mathcal{D}'$, we are able to provide a complete treatment of the distribution theory used in Ref. 2. We have thus used the discovery of simple proofs of the kernel and regularity theorems to present a general pedagogic presentation to the reader who wishes to study axiomatic field theory without an extensive detour into the functional analysis texts.

Because we will be dealing with several sets of infinitely many norms and with objects in many-dimensional real spaces, an extensive set of notational conventions seems imperative. The letters s and l will refer to the dimension of the underlying real space. $\mathcal{S}(\mathbb{R}^l)$ and $\mathcal{S}^{(l)}$ will be used interchangeably for the functions of rapid decrease in $\mathbb{R}^l$. The letters m, n, α, and β will be used to refer to multi-indices, i.e., l-tuples of nonnegative integers $m = (m_1, \cdots, m_l)$. We adopt the standard notation

$$m! = m_1!\, m_2! \cdots m_l!,$$
$$|m| = m_1 + m_2 + \cdots + m_l,$$
$$(m)^n = m_1^{n_1} \cdots m_l^{n_l},$$
$$m + 1 = (m_1 + 1, \cdots, m_l + 1),$$
$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_l^{\alpha_l}},$$
$$x^\alpha = x_1^{\alpha_1} \cdots x_l^{\alpha_l}.$$

In one place, we will slightly abuse this notation as follows: We will have l-tuples m and n and s-tuples α and β and will write

$$(m + 1)^n(\alpha + 1)^\beta = ((m, \alpha) + 1)^{(n, \beta)}. \quad (1.1)$$

The letters r, s, i, and j will be used for nonnegative integers. We will use the letters a, b, and c to refer to multisequences, i.e., $a = a_n$ where n runs over N^l, that is, all multi-indices of l entries. We will, of course, use other symbols and, while we will use them systematically, we will not introduce them at this point.

We will suppose that the reader is familiar with the basic notions of distribution theory, as presented, for example, in Ref. 2 (for additional background, see Refs. 4 and 5 on an elementary level and 6–10 on a more advanced level).

In Sec. 2, we present without proofs the basic identification of $\mathcal{S}$ and $\mathcal{S}'$ with sequence spaces, an identification which is basic to the simple proofs we present in Secs. 3 and 4. We return in Sec. 5 to the proof of the identifications of Sec. 2. After a section on miscellaneous results which follow from the sequence approach, we derive the remaining distribution theory needed for (axiomatic) quantum field theory as consequences of one general result—the Baire category theorem.

2. THE n-REPRESENTATION

Functions in $\mathcal{S}(\mathbb{R})$ are in L^2 and thus have expansions $\sum a_n\phi_n$, where the ϕ_n are the harmonic oscillator wavefunctions

$$\phi_n(x) = \pi^{-\frac{1}{4}}2^{-\frac{1}{2}n}(n!)^{-\frac{1}{2}}e^{\frac{1}{2}x^2}\left(\frac{d}{dx}\right)^n e^{-x^2}.$$

The simple proofs in Secs. 3 and 4 depend on the characterization of the Hermite coefficients of functions in $\mathcal{S}$:

Theorem 1: Suppose that $f \in \mathcal{S}$ and

$$a_n = \int \phi_n(x) f(x)\, dx.$$

Then, for any m

$$\sum |a_n|^2 (n+1)^m \equiv \|a\|_m^2 < \infty.$$

Conversely, if $\|a\|_m < \infty$ for all m, then $\sum a_n \phi_n$ converges (in the topology of $\mathcal{S}$) to a function in $\mathcal{S}$.

This theorem, which we prove in Sec. 5, establishes an isomorphism between $\mathcal{S}$ and a sequence space. We will call the representation of $\phi \in \mathcal{S}$, as a sequence a_n, the n-space representation.

Not only do the a_n that arise from functions in $\mathcal{S}$ have a simple description, but they also provide a simple form for the notion of convergence in $\mathcal{S}$. $\mathcal{S}$ has convergence defined by an infinite set of norms $\| \ \|_k$, specifically the norms

$$\|f\|_{r,s,\infty,\Sigma} = \sum_{\substack{|\alpha| \le r \\ |\beta| \le s}} \|x^\alpha D^\beta f\|_\infty,$$

where

$$\|g\|_\infty = \sup_x |g(x)|.$$

If x is any countably normed space, one says $x_n \to x$ if $\|x_n - x\|_k \to 0$ for each fixed k. Equivalently, convergence can be described by the metric

$$\rho(f, g) = \sum_k 2^{-k} \min(1, \|f - g\|_k).$$

If x is given two sets of norms $\| \ \|_i$ and $\| \ \|_j$, we say the sets are equivalent if and only if, for any i, there is a C and $j_1, \cdots, j_r$ so that

$$\|f\|_i \le C(|||f|||_{j_1} + \cdots + |||f|||_{j_r})$$

and, for any j, there is a D and $i_1, \cdots, i_s$ so that $|||f|||_j \le D(\|f\|_{i_1} + \cdots + \|f\|_{i_j})$. It is easy to see equivalent sets of norms provide identical notions of convergence, open set, etc., and that "equivalent" is an equivalence relation.

For example, the norms $\| \ \|_{r,s,\infty,\Sigma}$ on $\mathcal{S}$ are "equivalent" to the norms

$$\|f\|_{\alpha,\beta,\infty} = \|x^\alpha D^\beta f\|_\infty.$$

More to the point:

Theorem 2: For $f \in \mathcal{S}$ define

$$\|f\|_m = (\sum (n+1)^m |a_n|^2)^{\frac{1}{2}},$$

where $\{a_n\}$ are the Hermite coefficients for f. The norms $\| \ \|_m$ and $\| \ \|_{r,s,\infty,\Sigma}$ are equivalent.

This characterization of convergence in $\mathcal{S}$ (also proven in Sec. 5) allows us to find an n-space representation of distributions $T \in \mathcal{S}'$. Let us first point out a useful property of the $\| \ \|_m$.

Definition: A countable family of norms $\| \ \|_k$ is called *directed* if for any finite set $k_1, \cdots, k_r$ there is a k and a C so that $\|f\|_{k_1} + \cdots + \|f\|_{k_r} \le C \|f\|_k$.

The families $\{\| \ \|_{r,s,\infty,\Sigma}\}$ and $\{\| \ \|_m\}$ are directed but the family $\{\| \ \|_{\alpha,\beta,\infty}\}$ is not.

Directed families are very useful because they provide a simple description of open sets and continuous functionals. If one looks at the metric ρ, it is not hard to see that, for any family of countable norms, every neighborhood of 0 contains a canonical neighborhood of the form

$$\{x \mid \|x\|_{k_1} \le A_1, \cdots, \|x\|_{k_r} \le A_r\}.$$

If, in addition, the family of norms is directed, every canonical neighborhood contains a simpler neighborhood

$$\{x \mid \|x\|_k \le A\}.$$

Finally, using the fact that the inverse image of $\{z \mid |z| < 1\}$ under a continuous linear functional is open, one finds:

Lemma 1: A linear map $T: X \to \mathbb{C}$ with X a countably normed space *with a directed family of norms* $\{\| \ \|_j\}$ is continuous if and only if $\exists\ C, k$ such that

$$|Tx| \le C \|x\|_k.$$

This fact and the directed nature of the $\| \ \|_m$ allows us to prove:

Theorem 3: Suppose that $T \in \mathcal{S}'(\mathbb{R})$. Let $b_n = T(\phi_n)$. Then $|b_n| \le C(1+n)^m$ for some C and m, and $T(f) = \sum a_n b_n$ if a_n is the n-space representative of f. Conversely, if $|b_n| \le C(1+n)^m$, then $f \to \sum a_n b_n$ defines a tempered distribution.

Proof: Since $T \in \mathcal{S}'$ and $\| \ \|_m$ is directed, $|Tf| \le C \|f\|_m$ for some m. But $\|\phi_n\|_m = (1+n)^{\frac{1}{2}m}$ so that $|Tf| \le C(1+n)^{\frac{1}{2}m} \le C(1+n)^m$. To complete the proof of the first half of the theorem, we use Theorem 1, which tells us $\sum a_n \phi_n$ converges in $\mathcal{S}$ to f. For the converse, we merely compute the following:

$$\begin{aligned}|\sum a_n b_n|^2 &\le [\sum |a_n|^2 (n+1)^{2m+2}] \\ &\quad\times [\sum |b_n|^2 (n+1)^{-2m-2}] \\ &\le C \|f\|^2_{2m+2} \sum (n+1)^{-2} \\ &\le \tfrac{1}{6} C\pi^2 \|f\|^2_{2m+2},\end{aligned}$$

so that $f \to \sum a_n b_n$ is a continuous linear functional on $\mathcal{S}$. QED

We remark that, while we have stated the results for $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$, identical results hold for $\mathcal{S}(\mathbb{R}^l)$ and $\mathcal{S}'(\mathbb{R}^l)$. We need only interpret n and m as multi-indices, and

$$\phi_n(x) = \phi_{n_1}(x_1) \cdots \phi_{n_l}(x_l).$$

To summarize, we have seen that, in the n-representation, $\mathcal{S}$ represents just the sequences of fast falloff and $\mathcal{S}'$ represents just the sequences of polynomial growth.

3. THE REGULARITY THEOREM

The regularity theorem for tempered distributions says that any tempered distribution is the derivative of a continuous function of polynomial growth. The usual proof (Ref. 6, pp. 239–43) uses the Hahn–Banach and Reisz–Markov theorems plus a detailed analysis of tempered measures. It might seem a little strange that a theorem that never mentions measures needs measure theory in its proof. In fact, it does not: Using the n-space realization, we present a scandalously elementary proof of this theorem. This proof is a distant relative of the proof given by Zerner.[11]

The basic idea behind the proof is that we expect $\frac{1}{2}[-(d^2/dx^2) + x^2 + 1]$ to act as multiplication by $n + 1$ in the n space. In fact:

Lemma 2: Let $T \in \mathcal{S}'(\mathbb{R})$ have Hermite coefficients $b_n = T(\phi_n)$. Then $2^{-m}[-(d^2/dx^2) + x^2 + 1]^m T$ has Hermite coefficients $(n + 1)^m b_n$.

Proof:

$$\begin{aligned}2^{-m}\left(-\frac{d^2}{dx^2} + x^2 + 1\right)^m T(\phi_n) &= T\left[2^{-m}\left(-\frac{d^2}{dx^2} + x^2 + 1\right)^m \phi_n\right] \\ &= (n+1)^m T(\phi_n).\end{aligned}$$

The second input to the proof is that $\sum a_n\phi_n$ is "nice" if a_n falls off fast enough. This follows from:

Lemma 3: $\|\phi_n\|_\infty \le C(n + 1)^M$ for some C and M (independent of n).

Proof: By Theorem 2 and the directed nature of the $\| \ \|_m$, $\|f\|_\infty \le C\|f\|_M$ for some C and M. QED

Remarks: (1) The arithmetic of Sec. 5 actually shows that we can take $M = \frac{1}{2}$. (2) Detailed studies of the generating function for the ϕ_n show that $\|\phi_n\|_\infty \sim C(n + 1)^{-\frac{1}{12}}$ as $n \to \infty$.[12]

We are thus ready to prove the regularity theorem.

Theorem 4: Suppose that $T \in \mathcal{S}'$. Then $\exists\, m$ and a continuous bounded function f such that

$$T = \left(-\frac{d^2}{dx^2} + x^2 + 1\right)^m f.$$

Proof: Let b_n be the n-space representative of T. Then $|b_n| \le C(n + 1)^k$ for some k. Let $m = k + M + 2$, where M is given in Lemma 3. Let $a_n = (n + 1)^{-m} b_n$. Then

$$\sum a_n \|\phi_n\|_\infty < \infty$$

by Lemma 3, and so $\sum a_n\phi_n$ converges in L^∞ norm (and thus in $\mathcal{S}'$) to a bounded continuous function, say $2^m f$. By Lemma 2,

$$T = \left(-\frac{d^2}{dx^2} + x^2 + 1\right)^m f. \qquad \text{QED}$$

It is now straightforward to obtain the alternate form $T = D^k F$, where F is of polynomial growth. By using multi-indices, we can prove the regularity theorem for $\mathcal{S}(\mathbb{R}^l)$.

4. THE KERNEL THEOREM

Most proofs of the kernel theorem for $\mathcal{S}$ rely heavily on the theory of "nuclear" spaces (see Ref. 10, p. 530 or Ref. 13, pp. 73–84). We present here a proof of the kernel theorem on $\mathcal{S}$ which relies only on the n-space representation. As we will discuss in Sec. 8, this is a relative of existing proofs for $\mathcal{D}$.

In its "normal" form, the kernel theorem is a statement about separately continuous bilinear functionals. We divide it into two parts: that any separately continuous functional is jointly continuous and that jointly continuous functionals have the requisite form. In this section, we consider only the latter part. This part is the crucial half of the kernel theorem—in particular, the kernel theorem fails to hold for, say, $L^2(\mathbb{R}^l)$ because the analog of this half breaks down. We will prove the other part of the kernel theorem in Sec. 7.

Let us first establish the form we will need for joint continuity.

Definition: A bilinear map $B(x,y)$ from pairs $x \in X$, $y \in Y$ into $\mathbb{C}$ is jointly continuous if it is continuous as a map of $X \times Y$ into $\mathbb{C}$, i.e., if and only if for any ϵ, x_0, and y_0 there are neighborhoods N of x_0 and M of y_0 such that $x \in N$, $y \in M$ implies $|B(x,y) - B(x_0, y_0)| < \epsilon$.

Lemma 4: Let X and Y be two countably normed spaces with directed families of norms $\{\| \ \|_r\}$ and $\{\| \ \|_\lambda\}$. Let B be a bilinear form on $X \times Y$. Then the following are equivalent:

(a) B is jointly continuous.
(b) B is jointly continuous at $(0, 0)$.
(c) If $x_n \to 0$, $y_n \to 0$, then $B(x_n, y_n) \to 0$.
(d) For some r, λ, and C

$$|B(x,y)| \le C \, \|x\|_r \, \|y\|_\lambda .$$

Proof: (a) $\Rightarrow$ (b) $\Rightarrow$ (d) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b) can be proven by "standard" methods such as those used for linear functionals in Banach space. We only remark that (c) $\Rightarrow$ (b) depends essentially on the fact that we are in metric spaces where the open sets are describable in terms of *sequential* convergence (for example, the analogous result is false for $\mathfrak{D}$). (b) $\Rightarrow$ (d) depends on the fact that the norms are directed. QED

Theorem 5: Let B be jointly continuous bilinear functional on $\mathcal{S}(\mathbb{R}^l) \times \mathcal{S}(\mathbb{R}^s)$. Then there is a unique distribution T in $\mathcal{S}'(\mathbb{R}^{l+s})$ so that

$$B(f,g) = T(f \otimes g),$$

where

$$(f \otimes g)(x,y) = f(x)g(y).$$

Proof: Let C, m, and β be chosen such that

$$|B(f,g)| \le C \, \|f\|_m \, \|g\|_\beta . \tag{4.1}$$

Suppose that

$$t_{(n;\alpha)} = B(\phi_n, \phi_\alpha), \quad n \in N^l, \quad \alpha \in N^s.$$

Since B is jointly continuous and $f = \sum a_n \phi_n$ and $g = \sum b_\alpha \phi_\alpha$, we have that $B(f,g) = \sum t_{n\alpha} a_n b_\alpha$. On the other hand, by (4.1),

$$\begin{aligned} |t_{n,\alpha}| \le \|\phi_n\|_m \, \|\phi_\alpha\|_\beta &= (n+1)^m(\alpha+1)^\beta \\ &= [(n,\alpha)+1]^{(m,\beta)}. \end{aligned}$$

Thus the sequence $t_{(n,\alpha)}$ defines an element $\sum t_{(n,\alpha)} \phi_{(n,\alpha)}$ of $\mathcal{S}'(\mathbb{R}^{s+l})$,

$$T(h) = \sum t_{n\alpha} c_{(n,\alpha)},$$

where

$$h = \sum c_{(n,\alpha)} \phi_{(n,\alpha)} .$$

Since $f \otimes g$ has the Hermite coefficients $a_n b_\alpha$, we have that

$$T(f \otimes g) = \sum_{n,\alpha} t_{n_\alpha} a_n b_\alpha = B(f,g).$$

This proves existence. Since T is completely determined by the $T(\phi_{(n,\alpha)})$ (its Hermite coefficients) and we must have

$$T(\phi_{(n,\alpha)}) = T(\phi_n \otimes \phi_\alpha) = B(\phi_n, \phi_\alpha) = t_{n\alpha},$$

T is unique. QED

Theorem 6: Let M be a jointly continuous multilinear functional on $\mathcal{S}(\mathbb{R}^{l_1}) \times \cdots \times \mathcal{S}(\mathbb{R}^{l_r})$. Then there is a unique distribution T in $\mathcal{S}'(\mathbb{R}^{l_1+\cdots+l_r})$ such that

$$M(f_1, \cdots, f_r) = T(f_1 \otimes \cdots \otimes f_r).$$

Proof: The proof is analogous to Theorem 5.

5. PROOFS OF THEOREMS 1 AND 2

We prove Theorems 1 and 2 through a sequence of lemmas.

Lemma 5: Suppose that $f \in \mathcal{S}(\mathbb{R})$ and $a_n = \langle f, \phi_n \rangle$. Then

$$\sum |a_n|^2 (n+1)^m < \infty$$

for all m.

Proof: Since $f \in D[(p^2 + x^2 + 1)^m]$ for all m, $\sum |a_n|^2 (n+1)^m = 2^{-m} \langle f, (p^2 + x^2 + 1)^m f \rangle < \infty$ for all m.

To complete the proofs of Theorems 1 and 2, we must first establish the equivalence of the $\| \ \|_{r,s,\infty,\Sigma}$ and $\| \ \|_m$. We do this by establishing the equivalence of each of these families of norms with several families of intermediate norms. First we show the $\| \ \|_{\alpha,\beta,\infty}$ are equivalent to the norms

$$\|f\|_{\alpha,\beta,2} = \|x^\alpha D^\beta f\|_2,$$

with

$$\|f\|_2^2 = \int |f(x)|^2 \, dx.$$

Lemma 6:

$$\|f\|_2 \le \pi^{\frac{1}{2}} (\|f\|_\infty + \|xf\|_\infty),$$

so that

$$\|f\|_{\alpha,\beta,2} \le \pi^{\frac{1}{2}} (\|f\|_{\alpha,\beta,\infty} + \|f\|_{\alpha+1,\beta,\infty}).$$

Proof:

$$\begin{aligned} \|f\|_2^2 &= \int_{-\infty}^{\infty} dx (1+x^2)^{-1} [(1+x^2)\,|f|^2] \\ &\le \|(1+x^2)\,|f|^2\|_\infty \int_{-\infty}^{\infty} \frac{dx}{1+x^2} \\ &\le \pi (\|f\|_\infty + \|xf\|_\infty)^2. \end{aligned}$$

QED

To bound the $\| \ \|_\infty$'s by the $\| \ \|_2$ norms, the above trick does not work. However, the Fourier transform "reverses" the ordering of L^p spaces, explicitly:

Lemma 7: Let

$$\hat{f}(p) = (2\pi)^{-\frac{1}{2}} \int e^{-ipx} f(x)\, dx.$$

Then, for $f \in \mathcal{S}$,

$$\|f\|_2 = \|\hat{f}\|_2, \quad \|f\|_\infty \leq (2\pi)^{-\frac{1}{2}} \|\hat{f}\|_1.$$

Proof: The L^2 inequality is, of course, well known to any physicist (see, e.g., Ref. 14, pp. 355–62). The L^∞ inequality is trivial. QED

Lemma 8:

$$\|\hat{f}\|_1 \leq \pi^{\frac{1}{2}}(\|\hat{f}\|_2 + \|p\hat{f}\|_2)$$

so that (by Lemma 7)

$$\|f\|_\infty \leq 2^{-\frac{1}{2}}(\|f\|_2 + \|Df\|_2)$$

and

$$\|f\|_{\alpha,\beta,\infty} \leq 2^{-\frac{1}{2}}(\|f\|_{\alpha,\beta,2} + \|f\|_{\alpha,\beta+1,2} + \alpha \|f\|_{\alpha-1,\beta,2}).$$

Proof: As in the proof of Lemma 6,

$$\begin{aligned}\|\hat{f}\|_1 &= \int_{-\infty}^{\infty} dp[(1+p^2)^{\frac{1}{2}} |\hat{f}|][(1+p^2)^{-\frac{1}{2}}] \\ &\leq \left[\int_{-\infty}^{\infty} dp(1+p^2)^{-1}\right]^{\frac{1}{2}} \left(\int dp\, |\hat{f}|^2 + |p\hat{f}|^2\right)^{\frac{1}{2}} \\ &\leq \pi^{\frac{1}{2}}(\|\hat{f}\|_2 + \|p\hat{f}\|_2).\end{aligned}$$

QED

N.B.: (1) Lemma 8 is known as a Sobolev inequality in the mathematics literature. (2) Thus,

$$\begin{aligned}\|\phi_n\|_\infty &\leq \|\phi_n\|_2 + \|p\phi_n\|_2 \\ &= 1 + 2^{-\frac{1}{2}}[n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}}] \\ &\leq 3(n+1)^{\frac{1}{2}},\end{aligned}$$

as stated in Sec. 2.

We thus see that the $\| \ \|_{\alpha,\beta,\infty}$ and $\| \ \|_{\alpha,\beta,2}$ families are equivalent. Finally:

Lemma 9: The $\| \ \|_{r,s,\infty,\Sigma}$ and $\| \ \|_m$ norms are equivalent.

Proof: All we need prove is that the $\| \ \|_m$ and the $\| \ \|_{\alpha,\beta,2}$ norms are equivalent because we already have proven the $\| \ \|_{\alpha,\beta,2}$ and $\| \ \|_{r,s,\infty,\Sigma}$ equivalent. Let $\eta^\dagger$ and η be the usual creation and annihilation operators. Since $\eta, \eta^\dagger$ are linear combinations of x, p, and vice versa, and since any polynomial in x's and p's is equal to a polynomial with only $x^\alpha D^\beta$ terms, the $\| \ \|_{\alpha,\beta,2}$ norms are equivalent to the norms $\|(\eta^\#)^k f\|_2$, where $(\eta^\#)^k$ is a generic symbol for a monomial of degree k in η and $\eta^\dagger$. Since $\|f\|_m = |(\eta\eta^\dagger)^m f\|_2$, the $\| \ \|_m$ are a subset of the $\|(\eta^\#)^k \cdot \|_2$ norms. But it is easy to see that $\|(\eta^\#)^k f\|_2 \leq (2k)^k \|f\|_m$ with $m = k$ (crude estimate) so that the $\| \ \|_m$ norms are equivalent to the $\|(\eta^\#)^m \cdot \|_2$. This completes the proof. QED

To complete this section, we need only show that $\|a\|_m < \infty$ implies that $\sum a_n \phi_n$ converges in $\mathcal{S}$. This is a consequence of the equivalence of the norms and the fact that $\mathcal{S}$ is complete. (For a proof of this last fact, see Ref. 10, pp. 92–94.)

6. OTHER THEOREMS IN THE n-REPRESENTATION

In this section we point out several theorems whose proofs are also simple in the n-representation.

Theorem 7: $\mathcal{S}$ is separable; i.e., it has a countable dense set.

Proof: Since $\sum_{n<N} a_n\phi_n$ converges to f in $\mathcal{S}$ if $a_n = \langle \phi_n, f\rangle$, the finite linear combinations of the ϕ_n with rational coefficients are dense in $\mathcal{S}$ and are countable. QED

Theorem 8: $\mathcal{S}$ is dense in $\mathcal{S}'$ in the weak topology on $\mathcal{S}'$.

Proof: If $b_n = T(\phi_n)$, $\sum_{n<N} b_n\phi_n \to T$ in the weak topology on $\mathcal{S}'$. But $\sum_{n<N} b_n\phi_n \in \mathcal{S}$. QED

The next result is a little surprising:

Theorem 9[15]*:* For any l, $\mathcal{S}^{(l)}$ and $\mathcal{S}^{(1)}$ are isomorphic as topological vector spaces. Thus, for any s and l, $\mathcal{S}(\mathbb{R}^l)$ and $\mathcal{S}(\mathbb{R}^s)$ are isomorphic.

Proof: We prove the result for $l = 2$. The proof is similar for $l > 2$. Consider the map u of N^2 onto N by $u(0,0) = 0$, $u(1,0) = 1$, $u(0,1) = 2$; $u(2,0) = 3, \cdots$; i.e.,

$$u(r,s) = \tfrac{1}{2}(r+s)(r+s+1) + s.$$

We map $\mathcal{S}^{(1)}$ onto $\mathcal{S}^{(2)}$ by $(F(a))_{r,s} = a_{u(r,s)}$. Because $u(r,s)$ obeys the relations

$$r \leq u(r,s), \quad s \leq u(r,s),$$
$$u(r,s) + 1 \leq (r+1)^2(s+1)^2,$$

we immediately have

$$\sum_{n=1}^{\infty}(n+1)^m |a_n|^2 \leq \sum_{r,s=1}^{\infty}(r+1)^{2m}(s+1)^{2m} |F(a)_{r,s}|^2$$

and

$$\sum_{r,s=1}^{\infty}(r+1)^{m_1}(s+1)^{m_2} |F(a)_{r,s}|^2 \leq \sum_{n=1}^{\infty}(n+1)^{m_1 m_2} |a_n|^2.$$

Thus the norms $a \to \|F(a)\|_{m_1,m_2}$ and the $\| \ \|_m$ are equivalent. QED

This identity of $\mathcal{S}^{(l)}$ and $\mathcal{S}^{(s)}$ is not so useful as one might think at first. It says that we only have to prove theorems for $\mathcal{S}^{(1)}$ *if the theorem only refers to the "internal" structure of* $\mathcal{S}$. However, theorems like the regularity and kernel theorem refer to "external" structure, i.e., the realization of some distributions as functions and the map of $\mathcal{S}^{(l_1)} \times \mathcal{S}^{(l_2)}$ into $\mathcal{S}^{(l_1+l_2)}$.

7. THE BAIRE CATEGORY THEOREM AND APPLICATIONS

There are four results mentioned in Ref. 2 which we have not yet proven:

(1) The completeness of $\mathcal{S}'$.

(2) The nature of bounded sets in $\mathcal{S}$ (equivalence of weak and norm boundedness).

(3) The fact that separate continuity implies joint continuity for bilinear forms.

(4) The uniform convergence on bounded sets of ordinary distributions.

One is able to prove (1)–(3) from one abstract principle (Theorem 10); we will also be able to prove a weak form of (4) sufficient for the application in Ref. 2. The material in this section is rather standard. We only present it here because it is usually difficult to cull only the results needed for Ref. 2 from the texts.

Theorem 10 (*Baire Category Theorem*)*:* Let X be a complete metric space and suppose that

$$X = \bigcup_{i=1}^{\infty} A_i.$$

Then some i, $\bar{A}_i$ has a nonempty interior.

Proof: The argument is quite simple. See Ref. 16.

As a simple consequence:

Theorem 11 (*Principle of Uniform Boundedness*)*:* Let X be a countably normed space with $\| \ \|_r$ a directed sequence of norms. Let $\mathcal{F}$ be a set in X', the dual of X. If $\{F(f) \mid F \in \mathcal{F}\}$ is bounded for each $f \in X$, then there is a C and an r so that, for all f and all $F \in \mathcal{F}$,

$$|F(f)| \leq C \|f\|_r.$$

Proof: Suppose that $S_N = \{f \in X \mid |F(f)| < N$ for all $F \in \mathcal{F}\}$. Then each S_N is closed and, by the hypothesis of the theorem, $X = \bigcup S_N$. Thus, for some N, S_N has a nonempty interior. Therefore there exist an N, r, f_0, and ϵ such that $\|g - f_0\|_k \leq \epsilon$ implies $g \in S_N$. Suppose that $a = \sup |F(f_0)|$. Then $\|h\|_r < \epsilon$ and $F \in \mathcal{F}$ imply that

$$|F(h)| \leq |F(f_0 + h)| + |F(f_0)| \leq N + a.$$

Therefore,

$$|F(h)| \leq [(N + a)/\epsilon] \, \|h\|_r \quad \text{for all } h \text{ and } F \in \mathcal{F}.$$

QED

Corollary 1: $\mathcal{S}'$ is weakly sequentially complete.

Proof: Let T_n be a weak Cauchy sequence of tempered distributions; i.e., for each $f \in \mathcal{S}$, let $T_n(f)$ be a Cauchy sequence of numbers. Since this is Cauchy, $\lim T_n(f) = T(f)$ as $n \to \infty$ exists. T defined this way is linear. We must only show it is continuous. But since $\lim T_n(f)$ as $n \to \infty$ exists, $\{T_n(f)\}_{n=1,2,\cdots}$ is bounded. Thus, by Theorem 11, for some C and m, $|T_n(f)| \leq C \|f\|_m$. Therefore, $|T(f)| \leq C \|f\|_m$, and so T is continuous.

Corollary 2: Let B be a separately continuous bilinear form on $\mathcal{S}(\mathbb{R}^{l_1}) \times \mathcal{S}(\mathbb{R}^{l_2})$. Then it is jointly continuous.

Proof: Let $f_n \to 0, g_n \to 0$, where $f_n \in \mathcal{S}^{(l_1)}$, $g_n \in \mathcal{S}^{(l_2)}$. We need only show that $B(f_n, g_n) \to 0$. Let $F_n(g) = B(f_n, g)$. By continuity for fixed f, $F_n \in \mathcal{S}^{(l_2)\prime}$. By continuity for fixed g, $F_n(g) \to 0$ for each g, and thus $\{F_n(g)\}$ is bounded for each g. Thus, for some C and m, $|F_n(g)| \leq C \|g\|_m$ for all n. Since $g_n \to 0$ in $\mathcal{S}^{(l_2)}$, $\|g_n\|_m \to 0$. QED

Corollary 3: Let M be a separately continuous r-linear form on $\mathcal{S}(\mathbb{R}^{l_1}) \times \cdots \times \mathcal{S}(\mathbb{R}^{l_r})$. Then it is jointly continuous.

Proof: We use induction on r. $r = 2$ has been proved in Corollary 2. Assuming the results for $r = R$, we let $M(f^{(1)}, \cdots, f^{(R+1)})$ be given, and let $f_n^{(j)}$ be sequences in $\mathcal{S}^{(l_j)}$ with $f_n^{(j)} \to 0$. For each $g \in \mathcal{S}^{(l_{R+1})}$, $M(-, -, \cdots, g)$ is jointly continuous as an R-linear form by the induction hypothesis, and thus

$$M(f_n^{(1)}, \cdots, f_n^{(R)}, g) \to 0.$$

Proceeding as in the proof of Corollary 2, we see that

$$M(f_n^{(1)}, \cdots, f_n^{(R+1)}) \to 0$$

so that M is jointly continuous. QED

We can also discuss bounded sets by using Theorems 10 and 11.

Theorem 12: For a set $A \subset \mathcal{S}$, the following are equivalent:

(a) For any neighborhood N of 0, there is a real number λ with $\lambda A \subset N$.

(b) For each m, $\{\|f\|_m \mid f \in A\}$ is bounded.

(c) For each $F \in \mathcal{S}'$, $\{F(f) \mid f \in A\}$ is bounded.

Proof: (a) $\Leftrightarrow$ (b) is quite simple, as is (b) $\Rightarrow$ (c). To prove (c) $\Rightarrow$ (b), we proceed as follows: For each $g \in L^2(\mathbb{R}^s)$ and fixed $(\eta^\#)^m$,

$$\left\{\int (\eta^\#)^m \bar{g}(f) = \int \bar{g}[(\eta^\#)^m f] \,\middle|\, f \in A\right\}$$

is bounded. By Theorem 11 with $X = L^2$, the $\{\|(\eta^\#)^m f\|_2 \mid f \in A\}$ are bounded. Thus (b) follows. QED

Definition: A set $A \subset \mathcal{S}$ obeying the conditions in Theorem 12 is called bounded.

Theorem 13: for a set $B \subset \mathcal{S}'$, the following are equivalent:

(a) For any $f \in \mathcal{S}$, $\{F(f) \mid F \in B\}$ is bounded.

(b) There is a C and an m such that, for all $F \in B$, $f \in \mathcal{S}$, $|F(f)| \le C \|f\|_m$.

(c) For any bounded set $A \subset \mathcal{S}$, $\{F(f) \mid F \in B, f \in A\}$ is bounded.

Proof: (c) $\Rightarrow$ (a) is easy. (a) $\Rightarrow$ (b) is Theorem 12. (b) $\Rightarrow$ (c) is proven as follows: Given (b) and a bounded set A, $\sup\{\|f\|_m f \in A\} = k < \infty$. Thus, for any $f \in A$, $F \in B$, $|F(f)| \le Ck$. QED

Definition: A set $B \subset \mathcal{S}'$ obeying the conditions in Theorem 13 is called bounded.

Theorem 14: A sequence $f_n \to f$ in the $\| \ \|$-topology on $\mathcal{S}$ if and only if, for any bounded set $B \subset \mathcal{S}'$, $F(f_n) \to F(f)$ uniformly for $F \in B$.

Proof: One direction of the proof (only if) is a simple consequence of Theorem 13(b). The other direction is an interesting exercise; since this theorem is purely motivational, we do not provide a complete proof.

Theorem 14 suggests the following definition:

Definition: A sequence of distribution F_n is said to converge strongly to F if and only if, for any bounded subset $A \subset \mathcal{S}$, $F_n(f) \to F(f)$ uniformly for $f \in A$.

The analog for $\mathcal{S}$ of statement (4) at the beginning of the section is the following theorem, which we will not prove.

Theorem 15: A sequence of distributions F_n converges strongly to F if and only if it converges weakly.

Remark: In Theorem 14, we could replace "sequence" by the more general notion of net necessary for the complete description of a topology by convergence. However, the word sequence is essential in Theorem 15 and cannot be replaced by net.

Theorem 15 is implied by two other statements.

Theorem 16: Suppose that $F_n \to F$ weakly with F_n, $F \in \mathcal{S}'$. Let $A \subset \mathcal{S}$ be a compact subset. Then $F_n(f) \to F(f)$ uniformly for $f \in A$.

Theorem 17: If $A \subset \mathcal{S}$ is bounded, then A is compact.

We will not prove Theorem 17, but Theorem 16 will follow from results in Sec. 8.

8. A THEOREM FOR ORDINARY DISTRIBUTIONS

In Ref. 2, Wightman and Streater state and use the analog of Theorem 15 for $\mathfrak{D}$. Actually, one only needs a weak analog of Theorem 16 in his application, and we will prove this weak form in this section and show that it suffices in the application.

Lemma 10: Let X be a countably normed space and suppose that F_n, $F \in X'$, the dual of X. Suppose that $F_n(x) \to F(x)$ for each x in X. Let A be a compact subset of X. Then $F_n(x) \to F(x)$ uniformly for $x \in A$; that is, given ϵ, we can find N such that $n > N$ and $x \in A$ implies $|F_n(x) - F(x)| < \epsilon$.

Proof: Let $\| \ \|_r$ be a directed sequence of norms for X. By Theorem 11, there is a C and an r such that $|F_n(x)| \le C \|x\|_r$ and $|F(x)| \le C \|x\|_r$. For each $x \in A$, let $B_x = \{y \mid \|x - y\|_r < \epsilon/3C\}$. The $\{B_x \mid x \in A\}$ cover A and so, by compactness, we find $x_1, \cdots, x_m$ such that $\bigcup_{i=1}^m B_{x_i} \supset A$. Since $F_n(x_i) \to F(x_i)$, we can find N such that $n > N$ implies $|F_n(x_i) - F(x_i)| < \epsilon/3$ for $i = 1, \cdots, m$. Let $x \in A$. Find i such that $x \in B_{x_i}$, i.e., $\|x - x_i\|_r < \epsilon/3G$. Then, for any n,

$$|F_n(x) - F_n(x_i)| < \epsilon/3 \quad \text{and} \quad |F(x) - F(x_i)| < \epsilon/3.$$

Thus, if $n > N$,

$$|F(x) - F_n(x)| \le |F(x) - F(x_i)| + |F(x_i) - F_n(x_i)| + |F_n(x_i) - F_n(x)| < \epsilon.$$

This proves the result since ϵ is arbitrary.

Remarks: (1) The proof of Lemma 10 is really a classical equicontinuity argument. (2) Thus Theorem 16 is proven.

Theorem 18: Let $A \subset \mathfrak{D}(\mathcal{O})$ be a family of functions such that

(i) A is compact in $\mathfrak{D}(\mathcal{O})$.

(ii) For some fixed compact $C \subset \mathcal{O}$, $f \in A$ implies $\operatorname{supp} f \subset C$. Let $F_n, F \in \mathfrak{D}(\mathcal{O})'$ and let $F_n(g) \to F_n(g)$ for all $g \in \mathfrak{D}$. Then $F_n(f) \to F(f)$ uniformly for $f \in A$.

Remarks: (1) It is not difficult to prove that (i) implies (ii), but in the application (ii) can be directly verified. (2) One can actually show that any closed bounded set in $\mathfrak{D}$ obeys (i) and (ii), and so the analog of Theorem 17 follows.

Proof of 18: Let $\mathfrak{D}_c(\mathcal{O})$ be the subspace of $\mathfrak{D}(\mathcal{O})$ of functions with support in C. The topology of $\mathfrak{D}(\mathcal{O})$ restricted to $\mathfrak{D}_c(\mathcal{O})$ is described by the norms $\|f\|_\alpha = \|D^\alpha f\|_\infty$, so that $\mathfrak{D}_c(\mathcal{O})$ is a countably normed space. Since the F_n and F are continuous on $\mathfrak{D}(\mathcal{O})$, they are continuous on $\mathfrak{D}_c(\mathcal{O})$, and thus we are in the conditions of Lemma 10. QED

The application in Ref. 2 of the uniform convergence idea is to the case (Ref. 2, p. 83) $A = \{f_{-x}\}$ where $f \in \mathfrak{D}(\mathcal{O})$ and x lies in some small compact, so small that we can assume (ii) without thought. To verify (i) is easy: The map $x \to f_{-x}$ is continuous, and so the image of a compact set of x is compact in $\mathfrak{D}(\mathcal{O})$.

9. RELATION TO OTHER APPROACHES

The crucial element in the proofs of Secs. 5 and 6 is the realization of $\mathcal{S}$ as a sequence space and the realization of topology in terms of the norms $\| \ \|_m$. The systematic use of Hermite expansions goes back at least as far as Weiner.[17] Our realization of $\mathcal{S}$ is certainly not new; there is a short discussion of it in Schwartz's book (Ref. 6, pp. 271–83). A set of norms closely related to the $\| \ \|_m$ is implicit in Schwartz and a similar set of norms is discussed by Kristensen *et al.*[18] The proof of the kernel theorem in their norms

$$\|a\|_{r,K}^2 = \sum_n (|n| + l)^r \, |c_n|^2$$

is not as direct as in the $\| \ \|_m$ since the multiplicative property (1.1) avoids messy arithmetic. The only "new" result which we can possibly claim is the fact that the n-space realization of $\mathcal{S}$ provides a simple proof of the nuclear theorem—but this proof is clearly related to the various proofs of the nuclear theorem for $\mathfrak{D}$ which depend on Fourier series (Ref. 13, pp. 11–18; Refs. 19 and 20); in fact, our proof must be the "analogous proof for $\mathcal{S}$" alluded to by Gel'fand and Vilenkin (Ref. 13, p. 19). However, for the student of Ref. 2 faced with the statement "there does not seem to be an analogous elementary proof available for $\mathcal{S}$" (p. 43), it seems useful to have the details spelled out.

It is interesting to notice the close connection with Bargmann's beautiful and complete treatment of tempered distributions.[21] He realizes $\mathcal{S}$ as a family of entire functions and finds that $\mathcal{S}'$ can also be so realized. Up to the factors of $\sqrt{n!}$ the Taylor coefficients for his entire functions are just the Hermite coefficients of the elements of $\mathcal{S}$ and $\mathcal{S}'$. His Hilbert spaces F_r are just the multisequences with $\|a\|_r < \infty$ (although his inner product is not quite that given by $\| \ \|_r$ and he has r run over the all reals). Bargmann's results that $\mathcal{S}$ is "essentially" $\bigcap_{k=-\infty}^{+\infty} \mathcal{F}_k$ and $\mathcal{S}'$ is essentially $\bigcup_{k=-\infty}^{+\infty} \mathcal{F}_k$ (Ref. 21, p. 4) is evident from our Theorems 1 and 3. Bargmann's proofs of the regularity and kernel theorems (Ref. 20, pp. 70 and 68) are more or less our proofs in a complex function theoretic guise. In one sense, then, our simple proof is based on the observation that *for these two theorems* Bargmann's proofs do not require the elaborate constructions he uses. However, the treatment of the wide array of problems he considers uses analytic function theory (particularly variants of the maximum modulus principle) in an essential way. [Perhaps the relation of our approach to Bargmann's can be illustrated by remarking that it is identical to the relation of Schwinger's creation operator treatment of angular momentum,[22] to Bargmann's approach[24] for $SU(2)$.]

To the reader who wishes to use this note as a jumping off point for a more detailed study of tempered distributions, we can recommend Bargmann's approach most emphatically. Alternately, sequence spaces have been studied extensively by Köthe.[25,26]

We should also mention to the student of axiomatic field theory that, while he can avoid delving into the theory of nuclear spaces in studying Ref. 2, Jaffe's important work on "strictly localizable fields"[27] introduces a large class of test function spaces for which the kernel theorem is needed and for which the Hermite expansion method does not work.

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Can Local Gauge Transformations Be Implemented?

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One studies conditions under which a gauge transformation can be implemented by a unitary operator in some representations of the canonical (anti-) commutation relations; an application is then given to local gauge transformations in field theory.

INTRODUCTION

This paper originates from an attempt to understand the source of the difficulties which one faces in constructing generators of "canonical" local transformations of relativistic fields. We shall be mainly concerned with gauge transformation of the second type, but most of what will be said goes over almost verbatim to, e.g., "internal symmetry" groups.

The fact that such generators cannot be constructed in otherwise simple cases has been known for some time,[1] although domain problems plague the nonexistence proofs (these generators, when they exist, are expected to be unbounded operators). We shall discuss the existence of a continuous group of unitary operators that induce the group of transformations considered, a problem equivalent to the previous one, via Stone's theorem. We shall consider only some representations of the canonical fields, selected for having a structure particularly well suited for our purposes, and probably of not much physical interest,[2] and we shall show that in most of them (in a sense to be made precise later) such a weakly continuous group of unitary operators cannot be found.

Also, while the nonexistence of the generators of gauge groups (equal-time currents) as bona fide operators may cast some doubts on their formal manipulations, meaningful results can be obtained by giving them a meaning as bilinear forms.[1] Our results have no bearing on such an approach.

The content of the paper is as follows: In Sec. 1 we pose the problem and fix our notation, and actually generalize the previous setup in a rather natural way. Section 2 will be devoted to the solution of the generalized problem. In Sec. 3 the case of relativistic free fields will be considered, in the light of the preceding results, and the corresponding statement about local "charges" will be explicated.

1. THE PROBLEM

A. Canonical Anticommutation Relations

We shall start posing our problem in the case of canonical anticommutation relations (CAR's).

Let $\{a_i, a_i^*\}$, $i = 1, \cdots, n$, be a countable set of