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Coupling Constant Thresholds in Nonrelativistic Quantum Mechanics

II. Two Cluster Thresholds in N-Body Systems

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Abstract. We extend the analysis of absorbtion of eigenvalues for the two body case to situations where absorbtion occurs at a two cluster threshold in an N-body system. The result depends on a Birman–Schwinger kernel for such an N-body system, an object which we apply in other ways. In particular, we control the number of discrete eigenvalues in the $\hbar \to 0$ limit.

1. Introduction

This is the second paper in our series on the behavior of discrete eigenvalues, $e(\lambda)$, of Schrödinger operators, $-\Delta + \lambda V$, as $\lambda \to \lambda_0$, a value where $e(\lambda)$ approaches the continuous spectrum. In paper I, [9], we analyzed the general short range two body case, and in a third paper [10] we will analyze certain special long range potentials. In this paper, we want to say something about a certain class of N-body systems.

We recall from [9] that the behavior is highly dependent on the underlying dimension v, i.e. we considered $-\Delta + \lambda V$ on $L^2(\mathbb{R}^v)$ with $V \in C_0^\infty(\mathbb{R}^v)$ and found the behavior varying as v varies; for example, $v = 4$ is characterized by the fact that the ground state has a convergent expansion:

$$e(\lambda) = \sum_{\substack{n \geqq 2 \\ m \geqq 0 \\ k \geqq 0}} a_{nmk}(\lambda - \lambda_0)^{n/2}[\ln(\lambda - \lambda_0)]^{-\alpha}[\ln_2(\lambda - \lambda_0)^{-1}]^k \tag{1}$$

$$\alpha = m + k + \tfrac{1}{2}n.$$

Consider now the three body system on $L^2(\mathbb{R}^{2v})$:

$$H = -\Delta + \lambda V(x, y) \tag{1.1}$$

$$V(x, y) = V_{12}(x) + V_{23}(\alpha y + \beta x) + V_{13}(\gamma y + \delta x) \tag{1.2}$$

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where α, β are parameters depending on the masses and $\alpha \neq 0 \neq \gamma$. Suppose $V_{ij} \leqq 0$ and in $C_0^\infty(\mathbb{R}^\nu)$. One can ask about the behavior of the ground state energy, also, as $\lambda \downarrow \lambda_0$, the point where it gets absorbed. In particular, one would like to know if the behavior is the same as that for two body systems in dimension ν or dimension 2ν. This will depend on the nature of the bottom of the continuous spectrum.

By the HVZ theorem [13, Section XIII.5],

$$\Sigma \equiv \inf \sigma_{\mathrm{ess}}(H) = \min(a_{12}, a_{23}, a_{13}) \tag{1.3}$$

with $a_{ij} = \min \sigma(H_{ij})$ and $H_{ij} = -\Delta + V_{ij}$, with $-\Delta$ the 2ν-dimensional Laplacian. If $\Sigma = 0$, we say that the bottom of the continuum is three cluster and if $\Sigma < 0$, we say that it is two cluster. In addition, if only one $a_{ij} = \Sigma < 0$, we say it is unique two cluster. The intuition is quite simple: if the continuum is two cluster, the threshold is basically due to a ν-dimensional Laplacian, namely the relative kinetic energy of the two clusters, so one expects the coupling constant behavior to be that typical of a ν-dimensional problem. If the continuum is three cluster, one might well expect the behavior to be that characteristic of 2ν-dimension. Unfortunately, we have nothing to report about this interesting three cluster case. Here we will concentrate on the case of unique two cluster thresholds in three, and more generally N-body, systems.

We want to make two remarks about when this kind of eigenvalue absorbtion at two cluster thresholds takes place:

(1) If $V_{ij} \leqq 0$, then for large λ, the $\Sigma(\lambda)$ for $-\Delta + \lambda V$ is negative. If all $V_{ij} < 0$, there will definitely be indefinitely many eigenvalues as $\lambda \to \infty$. If we define

$$a_{ij}^{cl} = \min V_{ij}(x) \tag{1.4}$$

so that if V is continuous $a_{ij}(\lambda)/\lambda \to a_{ij}^{cl}$ as $\lambda \to \infty$ ([16]), and if

$$\Sigma_{cl} = \min_{i,j} a_{ij}^{cl},$$

so $\Sigma(\lambda)/\lambda \to \Sigma_{cl}$, then the threshold will be unique two cluster as $\lambda \to \infty$ so long as $\Sigma_{cl} = a_{ij}^{cl}$ for a unique i, j, say if

$$a_{12}^{cl} < a_{23}^{cl}, a_{13}^{cl}. \tag{1.5}$$

Thus, for systems obeying (1.5) without Effimov effort, all but finitely many eigenvalue absorbtions are of the type we consider.

(2) It can even happen that the ground state absorbtion is at a unique two cluster threshold. For example, if $-\Delta_x + \lambda_0 V_{12}(x)$ has a negative eigenvalue but for all x

$$-\Delta_y + \lambda_0 V_{23}(\alpha y + \beta x) + \lambda_0 V_{13}(\gamma y + \delta x) \geqq 0$$

(which will happen for fixed $\lambda_0, \alpha, \gamma$ so long as $\nu \geqq 3$ and V_{23}, V_{13} have sufficiently small $L^{\nu/2}$ norm), then absorbtion of the ground state will occur at a unique two cluster threshold.

Our analysis of coupling constant thresholds in the two body case depended on using a Birman–Schwinger kernel. In the unique two cluster case, we will introduce a suitable Birman–Schwinger kernel in Sect. 2. Unlike the two body case, this kernel will *not* be compact. However, absorbtion at a threshold will correspond to the kernel having eigenvalue 1 and the spectrum will be discrete at 1. This will allow us to extend our analysis from [9] to the present situation and we do this in Sect. 3.

The original purpose of the Birman–Schwinger kernel was to control the number of bound states. In just the same way, our Birman–Schwinger kernel gives us information on bound states, i.e. eigenvalues in $(-\infty, \Sigma)$ in N-body systems with two cluster thresholds. In the first place, it implies that the number is finite (see Sect. 2). This result has already been obtained by many other methods, see e.g. [24, 26, 15, 19]. Moreover (see Sect. 4) we get in some circumstances fairly explicit bounds on the number of bound states. Here we feel that our results considerably improve the very few existing bounds [6, 7, 25]. Finally, our bounds are good enough to control the classical limit for systems obeying (1.5). Explicitly, we will show in Sect. 5 that when the V's lie in C_0^∞, are negative, (1.5) holds and $\nu \geqq 3$:

$$N(\lambda)/\lambda^\nu \to c \int_{V(x) \leqq a_{12}} (a_{12} - V(x))^\nu d^{2\nu}x \tag{1.6}$$

for an explicit c. Here $N(\lambda)$ is the number of eigenvalues in $(-\infty, \Sigma(\lambda))$. (1.6) for the number of eigenvalues in $(-\infty, \lambda a_{12})$ is standard Dirichlet–Neumann bracketing [13, Sect. 8.15]. The difficulty which we solve with the methods of this paper involve the interval $(\lambda a_{12}, \Sigma(\lambda))$. These problems are discussed in more detail at the beginning of Sect. 5.

2. A Birman–Schwinger Kernel

We consider a general N-body system of ν-dimensional particles so H is an operator on $L^2(\mathbb{R}^{\nu(N-1)})$

$$H = H_0 + V, \tag{2.1}$$

where H_0 is the operator resulting from removing the center of mass from $\sum_{i=1}^{N} -(2m_i)^{-1}\Delta_i$ and

$$V = \sum_{i<j} V_{ij}(r_i - r_j). \tag{2.2}$$

Throughout this paper we suppose

$$V_{ij} \leqq 0 \tag{2.3}$$

and normally we suppose

$$V_{ij} \in C_0^\infty(\mathbb{R}^\nu). \tag{2.4}$$

(2.3) considerably simplifies the arguments since the Birman–Schwinger kernel is self-adjoint when (2.3) holds but most results should hold without (2.3). (2.4) is similarly made for technical convenience. Occasionally, we will indicate which arguments extend to cases where (2.4) fails and what hypotheses are then necessary.

We follow the notation of [13] generally. Specifically, let $D=\{C_1,\dots,C_k\}$ be a partition of $\{1,\dots,N\}$ and let $\sum_{iDj}$ (resp. $\sum_{\sim iDj}$) denote the sum over all pairs with i,j in the same (resp. different) cluster of D. We define

$$V_D=\sum_{iDj} V_{ij};\quad I_D=\sum_{\sim iDj} V_{ij},$$

$$H_D=H_0+V_D=H-I_D$$

$$\Sigma_D=\inf\sigma(H_D).$$

For each D, there is a natural decomposition of $\mathscr{H}=L^2(\mathbb{R}^{\nu(N-1)})$ as $\mathscr{H}_D\otimes\mathscr{H}^D$ with $\mathscr{H}_D=$ functions of r_{ij} with iDj and $\mathscr{H}^D=$ functions if R_q-R_l where $R_q=\sum_{i\in C_q} m_i r_i/\sum_{i\in C_q} m_i$ is the center of mass of C_q. Under this decomposition.

$$H_D=h_D\otimes 1+1\otimes t_D.$$

The hypothesis of unique two cluster threshold is the existence of a D with $\#(D)\equiv k\neq 1$ and

$$\Sigma_D<\Sigma_{D'} \tag{2.5}$$

for all $D'\neq D$ with $\#(D')\neq 1$. Of necessity, (2.5) can only hold if $\#(D)=2$ (since $\Sigma_{D_1}\leqq\Sigma_{D_2}$ if D_2 is a refinement of D_1, written $D_1<D_2$) and if (2.5) holds, then h_D must have an eigenvalue at the bottom of its spectrum (since $\inf\sigma_{\mathrm{ess}}(h_D)=\min(\Sigma_{D'}|D<D',D\neq D')$ by the HVZ theorem) and this eigenvalue will be simple. Thus, we pick once and for all a vector $\eta\in\mathscr{H}_D$ with $\|\eta\|=1$ and

$$h_D\eta=\Sigma_D\eta. \tag{2.6}$$

We let p be the projection in $\mathscr{H}_D$ onto η and $P=p\otimes 1$, the projection in $\mathscr{H}$. We define $q=1-p$, $Q=1-P$. It follows that

$$\sigma(H_D\upharpoonright Q\mathscr{H})=[\Sigma',\infty). \tag{2.7}$$

with

$$\Sigma'>\Sigma\equiv\min_D\Sigma_D=\Sigma_D \tag{2.8}$$

since Σ_D is a simple eigenvalue of h_D.

We define the Birman–Schwinger kernel by:

$$K(E)=|I_D|^{1/2}(H_D-E)^{-1}|I_D|^{1/2} \tag{2.9}$$

for $E<\Sigma$. We begin with the elementary:

Proposition 2.1. *Let $E<\Sigma$. Then $E\in\sigma(H_D+\mu I_D)$ if and only if $\mu^{-1}\in\sigma(K(E))$. This result remains true if σ is replaced by σ_{ess} in both places.*

Proof. Write $H_D+\mu I_D-E=(H_D-E)(1+\mu(H_D-E)^{-1}I_D)$ and conclude that (since $E\notin\sigma(H_D)$), $E\in\sigma(H_D+\mu I_D)$ if and only if $\mu^{-1}\in\sigma(-(H_D-E)^{-1}I_D)=\sigma((H_D-E)^{-1}|I_D|)=\sigma(K(E))$. The last equality follows from the well-known fact (see eg. [4]) that $\sigma(AB)\backslash\{0\}=\sigma(BA)\backslash\{0\}$ for any bounded A,B.

If $E\in\sigma_{\mathrm{ess}}(H_D+\mu_0 I_D)$, then since $\sigma_{\mathrm{ess}}(H_D+\mu I_D)$ is an interval $[\Sigma_\mu,\infty)$ with

Σ_μ decreasing in μ, $E\in\sigma_{\text{ess}}(H_D+\mu I_D)$ for all $\mu\geqq\mu_0$ so $[0,\mu_0^{-1}]\subset\sigma(K(E))$ which implies that $\mu_0^{-1}\in\sigma_{\text{ess}}(K(E))$.

If $E\notin\sigma_{\text{ess}}(H_D+\mu_0 I_D)$ then, we claim that for some $\delta, E\notin\sigma(H_D+\mu I_D)$ for $\mu\in(\mu_0-\delta,\mu_0+\delta)\backslash\{\mu_0\}$. This is obvious if $E\notin\sigma(H_D+\mu_0 I_D)$ and if $E\in\sigma_{\text{disc}}(H_D+\mu_0 I_D)$, $E\notin\sigma_{\text{disc}}(H_D+\mu I_D)$ for μ near μ_0 since discrete eigenvalues of $H_D+\mu I_D$ are *strictly* monotone in μ. It follows that μ_0^{-1} is either not in $\sigma(K(E))$ or is an isolated point of $\sigma(K(E))$. In the latter case, we must show that the multiplicity of μ_0^{-1} as an eigenvalue of $K(E)$ is finite. But if $K(E)\psi=\mu_0^{-1}\psi$, then $(H_D+\mu I_D-E)\phi=0$ with

$$\phi=(H_D-E)^{-1}|I_D|^{1/2}\psi$$

since

$$(H_D+\mu I_D-E)\phi=|I_D|^{1/2}(1-\mu K(E))\psi.$$

Since $(H_D-E)^{-1}|I_D|^{1/2}\upharpoonright\{\psi\,|\,K(E)\psi=\mu_0^{-1}\psi\}$ has no kernel, this shows that the multiplicity of μ_0^{-1} is at most the multiplicity of E as an eigenvalue of $H_D+\mu I_D$. ∎

Notice, at the end of the proof, that if $(H_D+\mu_0 I_D-E)\phi=0$, then

$$\psi=|I_D|^{1/2}\phi$$

obeys $K(E)\psi=\mu_0^{-1}\psi$. Since $|I_D|^{1/2}$ is non-vanishing on such ϕ (since $\operatorname{Ker}(H_D-E)=\{0\}$), we have that

Proposition 2.2. *Let $E<\Sigma$. Let $E\in\sigma_{\text{disc}}(H_D+\mu_0 I_D)$. Then the multiplicity of E as an eigenvalue of $H_D+\mu_0 I_D$ is exactly the multiplicity of μ_0^{-1} as an eigenvalue of $K(E)$.*

Remark. In the first step of the proof of Proposition 2.1, we used the fact that H_D and $H_D+\mu I_D$ have the same domain to be sure that $(H_D+\mu I_D-z)^{-1}(H_D-z)$ is bounded. This is actually critical; if H_0 is the free Dirac Hamiltonian, and if $V=\frac{1}{2}\sqrt{3}|r|^{-1}$, then H_0+V is essentially self-adjoint, $0\notin\sigma(H_0+V)$ but $1\in\sigma(H_0^{-1}V)$.

In the proof of Proposition 2.1, we noted that $\sigma_{\text{ess}}(H_D+\mu I_D)=[\Sigma_\mu,\infty)$ with Σ_μ monotone in μ. It follows that:

Proposition 2.3. *Let $E<\Sigma$. Then $\sigma_{\text{ess}}(K(E))=[0,\Lambda(E)]$ where $\Lambda(E)=\sup_\lambda\{\lambda\,|\,E\in\sigma_{\text{ess}}(H_D+\lambda^{-1}I_D)\}$.*

The applicability of these ideas to threshold phenomena depends on the fact that $K(E)$ has a *norm* limit as $E\uparrow\Sigma$. To see this it is useful to define:

$$K_P(E)=|I_D|^{1/2}(H_D-E)^{-1}P|I_D|^{1/2},$$
$$K_Q(E)=|I_D|^{1/2}(H_D-E)^{-1}Q|I_D|^{1/2},$$

with P, Q the projections introduced before (2.7). On account of (2.7), (2.8) we have:

Proposition 2.4. $K_Q(E)$ *has an analytic continuation to* $\mathbb{C}\backslash(-\infty,\Sigma')$. *In particular, on account of* (2.8), $K_Q(E)$ *has a norm limit as* $E\uparrow\Sigma$.

As a preliminary to analyzing $K_P(E)$, we note the following (the case $m=0$ is well known; [21, Thm. 4.1]):

Lemma 2.5. *Let $\mathscr{H} = L^2(\mathbb{R}^{n+m}) = L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^m)$ and denote a point in $\mathbb{R}^n \times \mathbb{R}^m$ as $(\mathbf{x}, \mathbf{y})$. Let $q = -i\nabla_x$, let T be a fixed affine map from $\mathbb{R}^m$ to $\mathbb{R}^n$ and let $\eta \in L^2(\mathbb{R}^m)$ be a fixed unit vector. Let p = projection onto η and $P = 1 \otimes p$. Then for each $r \in [2, \infty)$, and $f, g \in L^r(\mathbb{R}^n)$, the operator $f(q)Pg(x + T(y)) \equiv A(f, g)$ is in the trace ideal $\mathscr{I}_r(\mathscr{H})$ and*

$$\| A(f, g) \|_r \leqq \| f \|_r \| g \|_r .$$

Proof. For $r = \infty$, A is bounded and its operator norm is clearly bounded by $\| f \|_\infty \| g \|_\infty$. For $r = 2$, the operator is easily seen to have the L^2 integral kernel:

$$(2\pi)^{-n/2} f(x - x') g(x' + T(y')) \eta(y) \overline{\eta(y')},$$

with L^2 norm equal to $(2\pi)^{-n/2} \| f \|_2 \| g \|_2$. The result now follows by interpolation (e.g. [21, Theorem 2.9.]). ■

Proposition 2.6. *Let $v \geqq 3$ and let $r > v/2$. Then for all $E < \Sigma$, $K_P(E)$ is in the trace ideal $\mathscr{I}_r$ and as $E \uparrow \Sigma$, $K_P(E)$ converges to an operator $K_P(\Sigma)$ in $\mathscr{I}_r$-norm.*

Proof. By Holder's inequality, it suffices to prove boundedness and continuity in $\mathscr{I}_{2r}$ of $(H_D - E)^{-1/2} P |I_D|^{1/2}$. Since $|I_D|^{1/2} \left(\sum_{\sim iDj} (V_{ij})^{1/2} \right)^{-1}$ is multiplication by a function bounded by 1, we need only prove the required fact for $(H_D - E)^{-1/2} P |V_{ij}|^{1/2}$ with $\sim iDj$. But since $(H_D - E)^{-1/2} P |V_{ij}|^{1/2} = (t_D + (\Sigma - E))^{1/2} P |V_{ij}(r_i - r_j)|^{1/2}$ and $r_i - r_j = R + \rho$ with $R = R_{c_1} - R_{c_2}$ and ρ an "internal coordinate", the operator in question is exactly of the form to be controlled by Lemma 2.5.

For later purposes it is useful to relate the non-zero eigenvalues of K_P to those associated to the Birman–Schwinger kernel for a two-body problem involving the potential

$$\tilde{V}_{\text{eff}}(R) = \int I_D(R, \rho) |\eta(\rho)|^2 d\rho \tag{2.10}$$

where ρ stands for the internal coordinates and R is the difference of the centers of mass. Thus

$$P I_D P = P \otimes \tilde{V}_{\text{eff}} \tag{2.10'}$$

The tilde is included to distinguish it from the effective potential relevant to scattering theory [12, 5]:

$$P I_D^2 P = P \otimes V_{\text{eff}}^2 ; \quad V_{\text{eff}} \geqq 0 . \tag{2.11}$$

Proposition 2.7. *The non-zero eigenvalues (counting multiplicity) of $K_P(\Sigma)$ are identical to the non-zero eigenvalues of*

$$|\tilde{V}_{\text{eff}}|^{1/2} (t_D)^{-1} |\tilde{V}_{\text{eff}}|^{1/2},$$

as an operator on $\mathscr{H}^D$.

Proof. Since the non-zero eigenvalues (counting multiplicity) of A^*A and AA^* are the same [4] $K_P(\Sigma)$ has the same eigenvalue as $-(t_D)^{-1/2}PI_DP(t_D)^{-1/2} = (t_P)^{-1/2}P|V_{\text{eff}}|(t_P)^{-1/2} = t_D^{-1/2}|V_{\text{eff}}|t_D^{-1/2} \otimes P$. Using the A^*A result again, we have completed the proof. ■

We summarize the last few results and extend them in

Theorem 2.8. *$K(E)$ has a norm limit $K(\Sigma)$ as $E \uparrow \Sigma$ and moreover:*

(i) $$\sigma_{\text{ess}}(K(\Sigma)) = [0, \Lambda(\Sigma)],$$

where $\Lambda(\Sigma) = \sup_\lambda \{\lambda | \sigma_{\text{ess}}(H_D + \lambda^{-1}I_D) \cap (-\infty, \Sigma) \neq \emptyset\} < 1$

(ii) $$K(E) \leqq K(\Sigma)$$

for all $E < \Sigma$.

Proof. Propositions 2.4 and 2.7 imply the existence of the norm limit. The identification of $\sigma_{\text{ess}}(K(\Sigma))$ follows from Proposition 2.3 and Lemma 2.9 below. That $\Lambda(\Sigma) < 1$ follows from the fact that Σ is unique two cluster. Finally (ii) is obvious since

$$(H_D - E)^{-1} \leqq (H_D - \Sigma)^{-1},$$

for $E \leqq \Sigma$. ■

Lemma 2.9. *Let* $0 \leqq A_n, \sigma_{\text{ess}}(A_n) = [0, a_n]$ *and suppose* $A_n \to A$ *in norm. Then* $a = \lim a_n$ *exists and* $\sigma_{\text{ess}}(A) = [0, a]$.

Proof. Let $\bar{a}$ (resp. a) be $\overline{\lim}\, a_n$ (resp. $\underline{\lim}\, a_n$). Let $\lambda < \overline{\lim}\, a_n$. If $\lambda \notin \sigma(A)$, then $\lambda \notin \sigma(A_n)$ for all large n, so $\lambda > a_n$ for all large n violating the assumption. Thus $[0, \bar{a}] \subset \sigma(A)$ and so $[0, \bar{a}] \subset \sigma_{\text{ess}}(A)$. Now let $\lambda > a$. Pick $\delta > 0$ so that $\lambda - \delta > a$. Pick n so large that $\|A - A_n\| \leqq \delta/3$ and that $\lambda - 2\delta/3 > a_n$. Since $[\lambda - (2\delta/3), \lambda + (2\delta/3)] \cap \sigma_{\text{ess}}(A_n) = \emptyset$, we can find F finite rank so that $[\lambda - (2\delta/3), \lambda + (2\delta/3)] \cap \sigma(A_n + F) = \emptyset$. Thus $[\lambda - (\delta/3), \lambda + (\delta/3)] \cap \sigma(A + F) = \emptyset$ so $\lambda \notin \sigma_{\text{ess}}(A)$, i.e. $\sigma_{\text{ess}}(A)\,[0, a]$. ■

One immediate consequence of the machinery of this section is the following:

Theorem 2.9. *Let H be the Hamiltonian of an N-body system with potentials obeying (2.3) and (2.4). Let Σ, the infimum of the essential spectrum be unique two cluster. Then* $\dim \operatorname{ran} E_{(-\infty,\Sigma)}(H) < \infty$ *(i.e. there are finitely many "bound states").*

Proof.
$$\begin{aligned}\dim \operatorname{ran} E_{(-\infty,\Sigma)}(H) &= \lim_{n\to\infty} \dim \operatorname{ran} E_{(-\infty,\Sigma-(1/n))}(H) \\ &= \lim_{n\to\infty} \#\left\{\lambda | \lambda > 1, \lambda \in \sigma\left(K\left(\Sigma - \frac{1}{n}\right)\right)\right\} \\ &\leqq \#\{\lambda | \lambda > 1, \lambda \in \sigma(K(\Sigma))\} < \infty\,.\end{aligned} \tag{2.12}$$

In (2.12) we use the standard Birman [1]-Schwinger [14] argument; in the next step, we use $K(E) \leqq K(\Sigma)$ and, in the last step that $[1, \infty) \cap \sigma_{\text{ess}}(K(\Sigma)) = \emptyset$. ■

We close this section with some remarks about extensions to more general V's. With no change at all, we can allow $V_{ij} \in L^{\nu/2}(\mathbb{R}^\nu)$. Moreover, by a little more argument, we can accommodate V's which aren't negative and even allow cancellations in various V_{ij} in I_D:

Theorem 2.10. *Let Σ, H be as in Theorem 2.9 but replace (2.3), (2.4) with:*

(i) *The V_{ij} are H_0-compact*

(ii) $\tilde{V}_{\mathrm{eff}} \in L^{\nu/2}(\mathbb{R}^\nu)$

(iii) $V^2_{\mathrm{eff}} \in L^{\nu/2}(\mathbb{R}^\nu)$.

Then dim ran $E_{(-\infty,\Sigma)}(H) < \infty$.

Proof. Let $K(E) = (H_D - E)^{1/2} I_D (H_D - E)^{-1/2}$. As above, we need only prove that $K(E)$ has a norm limit as $E \uparrow \Sigma$ since then $\sigma_{\mathrm{ess}}(K(\Sigma)) \cap [1, \infty) = \emptyset$. Introduce four operators $\tilde{K}_{PP}, \tilde{K}_{QQ}, \tilde{K}_{PQ}, \tilde{K}_{QP}$ by

$$\tilde{K}_{AB}(E) = (H_D - E)^{-1/2} A\, I_D B (H_D - E)^{-1/2}.$$

$\tilde{K}_{QQ}$ trivially has a norm limit. By (ii), $(\tilde{E} \equiv E - \Sigma)$

$$\tilde{K}_{PP} = (t_D - \tilde{E})^{-1/2} \tilde{V}_{\mathrm{eff}} (t_D - \tilde{E})^{-1/2} \otimes P$$

has a norm limit. Finally, to show that $\tilde{K}_{PQ}$ has a norm limit we note that $Q(H_D - E)^{-1/2}$ has a norm limit and that

$$C = (H_D - E)^{-1/2} P I_D,$$

has a norm limit since

$$CC^* = (t_D - \tilde{E})^{-1/2} V^2_{\mathrm{eff}} (t_D - \tilde{E})^{-1/2} \otimes P$$

and we have (iii). ∎

Remarks. 1. For reasonable potentials, (iii) is weaker than (ii).

2. The above result is essentially equivalent to that in [19].

3. Coupling Constant Thresholds

In this section, we want to explain how the machinery of Sect. 2 allows one to analyze the coupling constant behavior of eigenvalues in a situation where they are absorbed into unique two-cluster continua. We describe only the case $\nu = 3$. One can extend to general $\nu \geqq 3$ situations by similar methods (although since the Birman–Schwinger kernel is no longer Hilbert–Schmidt, the arguments are slightly more involved): the relevant object is the analyticity of the integral kernel of $(t_D - \alpha^2)^{-1}$ where $\alpha = 0$ so the behavior will be ν-dependent in precisely the way we found in [9]. It should be possible to analyze the case $\nu = 1, 2$ as we did in [9] but we have not worked out the details.

For simplicity, we consider first the family $H_D + \lambda I_D$ with λ varied and then $H_0 + \lambda V$. Obviously, one can consider other variable couplings or even eigenvalues as functions of the $\binom{N}{2}$ coupling constants; the situation will be virtually identical to the $H_D + \lambda I_D$ case.

Lemma 3.1. *Let $\nu = 3$ and let (2.3), (2.4) hold. Then $K(E)$ defined initially for $E < \Sigma$ has an operator valued analytic continuation in the variable $\alpha = \sqrt{\Sigma - E}$*

around $\alpha = 0$. *Moreover:*

$$k(\alpha) \equiv K(\Sigma - \alpha^2) = K(\Sigma) + A_1\alpha + A_2\alpha^2 + 0(\alpha^3) \tag{3.1}$$

with A_1 *rank 1 and* A_2 *strictly negative on* $\mathrm{Ran}(A_1)^{\perp} \cap L^2(\mathrm{Supp}(I_D))$.

Proof. As already noted, K_Q is analytic in E near $E = \Sigma$ and thus certainly in α. Moreover, K_Q contributes only to A_2 and not to A_1 in (3.1). Its contribution to A_2 is

$$A_{2,Q} = -|I_D|^{1/2} Q(H_D - \Sigma)^{-2} |I_D|^{1/2}. \tag{3.2}$$

K_P has an explicit integral kernel in terms of (R, ζ) with R = intercluster coordinates and ζ = intracluster coordinates:

$$K_P(\alpha; R, R'; \zeta, \zeta') = \eta(\zeta)\eta(\zeta')[4\pi|R - R'|]^{-1} e^{-\alpha|R-R'|} |I_D(R,\zeta)|^{1/2} |I_D(R',\zeta')|^{1/2} \tag{3.3}$$

where for notation simplicity we have supposed the masses are such that $t_D = -\Delta_R$. Since the $V_{ij} \in C_0^\infty$ and η has exponential fall off ([13, Sect. XIII.11])

$$\int e^{A|R|} |\int |\eta(\zeta)|^2 I_D(R,\zeta) d\zeta|^{3/4} dR < \infty$$

for some $A > 0$, so using Sobolev's inequality, (3.3) is square integrable for all complex α with $|\alpha| < A$. Thus $K_P(\alpha)$ has a Hilbert–Schmidt valued analytic continuation. A_1 is just the rank 1 one operator:

$$-(4\pi)^{-1}(\gamma, .)\gamma \tag{3.4}$$

with $\gamma = \eta |I_D|^{1/2}$. Finally the contribution, $A_{2,P}$ of K_P to A_2 is strictly conditionally negative in the sense that

$$(\phi, A_{2,P}\phi) \leqq 0$$

and if $(\phi, A_{2,P}\phi) = 0 = (\phi, A_{1,P}\phi)$, then $P|I_D|^{1/2}\phi = 0$. Since $(\phi, A_{2,Q}\phi) = 0$ implies that $Q|I_D|^{1/2}\phi = 0$, we see that if $|I_D|^{1/2}\phi \neq 0$, then either $(\phi, A_1\phi) \neq 0$ or $(\phi, A_2\phi) < 0$ as claimed. ∎

Theorem 3.2. *Suppose that* $H_D + \lambda_0 I_D$ *has a unique two cluster* $\Sigma = \Sigma_D$ *and that for* $\lambda > \lambda_0$, *there is an eigenvalue* $e(\lambda)$ *with* $e(\lambda) \uparrow \Sigma$ *as* $\lambda \downarrow \lambda_0$. *Then, either:*

(a) $$e(\lambda) = \sum_{n=2}^{\infty} a_n(\lambda - \lambda_0)^n; \quad a_2 \neq 0,$$

or

(b) $$e(\lambda) = \sum_{n=2}^{\infty} b_n(\lambda - \lambda_0)^{n/2}; \quad b_2 \neq 0,$$

converging if $\lambda - \lambda_0$ *is small. Moreover, at most one eigenvalue is in Case* (*a*) *and the ground state is always in Case* (*a*).

Proof. Given the lemma and the fact that the vector γ is positive, this follows as in [9]. ∎

Important Remark. We do *not* claim that in Case (b), $e(\lambda)$ is not analytic in $\lambda - \lambda_0$, i.e. that some $b_{2k+1} \neq 0$. Indeed, it can happen for reasons of symmetry, that for all $\lambda > \lambda_0$, the eigenvector $\psi(\lambda)$ with eigenvalue $e(\lambda)$ has $P\psi(\lambda) = 0$. In this case, $e(\lambda) = 0(\lambda - \lambda_0)$ so we are in Case (b), but $e(\lambda)$ is analytic in λ! However, in Case (b), if $e(\lambda)$ is analytic, then $e(\lambda)$ is an eigenvalue for $\lambda < \lambda_0$ imbedded in the continuous spectrum $[\Sigma, \infty)$. Moreover, we *conjecture* that this can only happen in the case given above, i.e., when $P\psi(\lambda) = 0$ for $\lambda > \lambda_0$.

We describe the situation involving absorbtion when all coupling constants are varied in a sequence of remarks.

(1) Since both eigenvalues and the continuum are decreasing functions of λ, it can happen that as λ is *increased*, an eigenvalue gets absorbed because Σ "overtakes" it. One can still describe the behavior with a Birman–Schwinger type analysis.

(2) Consider the two parameter family

$$H = H_0 + \beta V_D + \lambda I_D.$$

For $\beta = \lambda = \lambda_0$, we suppose H has a unique two cluster threshold with breakup D. This will remain true for β, λ both near λ_0 but perhaps unequal. Then, the bottom of the essential spectrum, $\Sigma(\beta)$, and corresponding eigenvectors $\eta(\beta)$ and projection $\boldsymbol{P}(\beta)$ depend on β but in an analytic way.

(3) The natural Birman–Schwinger object is

$$K(E, \beta) = |I_D|^{1/2}(H_0 + \beta V_D - E)^{-1}|I_D|^{1/2}$$

defined initially if $E < \Sigma(\beta)$. The operator

$$k(\alpha, \beta) = K(\Sigma(\beta) - \alpha^2, \beta),$$

has a continuation into a neighborhood of $\beta = \lambda_0, \alpha = 0$.

(4) If an eigenvalue is absorbed at $\beta = \lambda = \lambda_0$, then $k(0, \lambda_0)$ has a discrete eigenvalue $\mu_0 = \lambda_0^{-1}$. *Suppose* this eigenvalue is simple. Then for α near zero, and β near λ_0, $k(\alpha, \beta)$ has a unique eigenvalue, $\mu(\alpha, \beta)$ near μ_0 and μ is analytic. Clearly for λ near λ_0, $e(\lambda)$ is given by

$$e(\lambda) = \Sigma(\lambda) - \alpha(\lambda)^2 \tag{3.5}$$

where $\alpha(\lambda)$ solves

$$\mu(\alpha(\lambda), \lambda) = \lambda^{-1}. \tag{3.6}$$

(5) In any specific case, one can solve (3.6) but the possibilities of the general solution are much more varied than in the situation of Theorem 3.2. For in terms of $\gamma = \lambda^{-1} - \lambda_0^{-1}$, (3.6) has the form

$$\sum_{n=0}^{\infty} b_n(\gamma)\alpha^n = \gamma$$

but $b_0(\gamma)$ may be non-zero so $\gamma - b_0(\gamma)$ might be $0(\gamma^m)$.

(6) Even if a single eigenvalue is absorbed at $\lambda = \lambda_0, \mu_0$ can be a degenerate eigenvalue of $K(0, \lambda_0)$. This can be seen by looking at explicit examples of the form $-\Delta_1 - \Delta_2 - \lambda(V_1(x_1) + V_2(x_2))$. If μ_0^{-1} is degenerate, the μ entering in (3.6) is now only an algebraic function of α, λ with all the possibilities of multivariable algebraic singularities.

To summarize: In principle, one can analyze the case $H_0 + \lambda V$. In general, the possibilities are very varied but in the "generic" situation, things will behave as in the $H_D + \lambda I_D$ case.

4. Bounds on the Number of Discrete Eigenvalues

The methods of this paper provide many ways to obtain fairly explicit bounds on the number, $N(H)$, of discrete eigenvalues of a many body H (counting multiplicity). Here we give one such bound to illustrate these potentialities.

Theorem 4.1. *Let H have a unique two cluster threshold with clustering D. Suppose that Σ' is the lowest point of $\sigma(h_D)\backslash\{\Sigma\}$ (so Σ' obeys 2.7) and that*

$$\beta \equiv \| I_D \|_\infty (\Sigma' - \Sigma)^{-1} < 1 .$$

Then, with $\tilde{V}_{\mathrm{eff}}$ given by (2.10):

$$N(H) \leqq N(t_D + (1-\beta)^{-1} \tilde{V}_{\mathrm{eff}}). \tag{4.1}$$

Remark. Thus $N(H)$ is dominated by a two-body N for which there are many explicit bounds; see [17].

Proof. Write $K = K_P + K_Q$ as in Sect. 2. Clearly:

$$\| K_Q(\Sigma) \| \leqq \| I_D \|_\infty (\Sigma' - \Sigma)^{-1} = \beta .$$

Thus

$$\begin{aligned} N(H) &= \# \{\lambda \,|\, \lambda > 1, \lambda \text{ an e.v. of } K(\Sigma)\} \\ &\leqq \# \{\lambda \,|\, \lambda > 1-\beta, \lambda \text{ an e.v. of } K_P(\Sigma)\} && (4.2) \\ &= \# \{\lambda \,|\, \lambda > 1-\beta, \lambda \text{ an e.v. of } |\tilde{V}_{\mathrm{eff}}|^{1/2} (t_P)^{-1} |\tilde{V}_{\mathrm{eff}}|^{1/2}\} && (4.3) \\ &= N(t_D + (1-\beta)^{-1} \tilde{V}_{\mathrm{eff}}) && (4.4) \end{aligned}$$

In (4.4), we use the Birman–Schwinger principle; in (4.3), we use Proposition 2.7; and, in (4.2), we use the trivial inequality

$$\mu_n(A+B) \leqq \| A \| + \mu_n(B)$$

on singular values of positive operators. ■

5. The Classical (= Large Coupling) Limit

In this section, we want to discuss the small μ behavior of the number of bounded states, $N(H)$, for $H = \mu H_0 + V$ or equivalently for $H_0 + \mu^{-1} V$. Thus, this limit can be thought of alternatively as the large coupling or as the classical ($h \downarrow 0$) limit. For the two body case the result is well-known (see [2, 11, 23, 3] for original work or [13, 22] for further discussion). Here we consider the three-body and then N-body cases in situations where classically the bottom of the continuum is unique two cluster.

Henceforth, we let $N(\mu)$ denote the number of discrete eigenvalues of $\mu H_0 + V$. Then, we will prove that

Theorem 5.1. *Let $\nu \geqq 3$. Let $H_0 = -\Delta$ on $L^2(\mathbb{R}^{2\nu})$ and let V be given by (1.2) with V_{ij} obeying (2.3), (2.4). Suppose that* (1.5) *holds. Then*

$$\lim_{\mu \to 0} \mu^\nu N(\mu) = \tau_{2\nu}(2\pi)^{-\nu} \int_{V(x,y) \leqq a_{12}} [a_{12} - V(x,y)]^\nu d^\nu x d^\nu y \tag{5.1}$$

where $\tau_{2\nu}$ is the volume of the unit sphere in $\mathbb{R}^{2\nu}$.

Before turning to the proof of this result, we make a few remarks. First, given (1.5), (2.3), and (2.4), the assumed form of H_0 is no restriction. We can always change variables so that H_0 has the required form. More generally, in terms of the original masses, we let

$$H(P_1, P_2, P_3, x_1, x_2, x_3) = \sum_{i=1}^{3} P_i^2/2m_j + \sum_{i<j} V_{ij}(x_i - x_j).$$

and $N(\mu)$, the number of discrete eigenvalues of $H(\mu) = \mu H_0 + V$. Then (5.1) holds with the right side replaced by:

$$(2\pi)^{-\nu} \int d^\nu P_1, \dots, d^\nu x_3 \delta(\Sigma P_i) \delta(\Sigma m_i x_i / \Sigma m_i) \theta(a_{12} - H(P,x))$$

with $\theta(y) = 1$ (resp. 0) for $y > 0$ (resp. $y < 0$).

Secondly, we note that while the two-body result extends easily given the right bounds to non-smooth V's, the present results depend heavily on boundedness of the V_{ij}. For example, if $m_1 = \infty$ and $V_{23} = 0$, V_{13} bounded and $V_{12}(x) \sim |x|^{-1}$ for x small, V_{12} of compact support and $V_{12} C^\infty$ away from $x = 0$, then $N(\mu)$ will only grow as $\mu^{-\nu/2}$. For, in this case, the eigenvalues of $H(\mu)$ are sums of eigenvalues of $H_{12}(\mu)$ and $H_{13}(\mu)$. For large μ, the only ones below the ground state of H_{12} are sums of eigenvalues of H_{13} with the ground state of H_{12}. Thus for unbounded V_{ij}, the growth of $N(\mu)$ is a very subtle question on which we have little information.

Write

$$N(\mu) = N_1(\mu) + N_2(\mu) \tag{5.2}$$

with N_1 the number of eigenvalues in $(-\infty, \Sigma_{cl}]$ and N_2 the number in $[\Sigma_{cl}, \Sigma)$ where $\Sigma_{cl} = a_{12}$ is the classical continuum. Notice that

$$\{(x,y) | V(x,y) < \Sigma_{cl}\}$$

is a bounded set, since $V(x,y) < \Sigma_{cl}$ can only happen if two of the $x_i - x_j$ are smaller than R, the maximum range of the V_{ij}. Given this, one easily uses Dirichlet–Neumann bracketing (see e.g. [11, 13, 20]) to prove that:

$$\lim_{\mu \to 0} \mu^\nu N_1(\mu) = \text{right side of (5.1)}. \tag{5.3}$$

Thus, we are reduced to proving

$$\lim_{\mu \to 0} \mu^\nu N_2(\mu) = 0. \tag{5.4}$$

Let $K_\mu(E)$ be the Birman–Schwinger kernel of Sect. 2 for $H(\mu)$. Below, we will prove that for any $\varepsilon > 0$:

$$\mu^\nu [\# \{\text{e.v. of } [K_\mu(\Sigma_\mu) - K_\mu(\Sigma_{cl})] \geqq \varepsilon\}] \to 0 \tag{5.5}$$

as $\mu \to 0$. We first note:

Lemma 5.2. *(5.5) implies (5.4) and hence the theorem.*

Proof. Let $V^\varepsilon(x,y) = V(x,y) + \varepsilon V_{23} + \varepsilon V_{13}$. Then, for ε small enough, $\Sigma^\varepsilon_{cl} = \lim_{|x|+|y|\to\infty} V^\varepsilon(x,y) = \Sigma_{cl} \equiv a_{12}$. It follows that (5.3) continues to hold for V replaced by V^ε in the integral on the right and with $N_1(\mu)$ replaced by $N^\varepsilon_1(\mu)$, the number of eigenvalues for $H(\mu) + \varepsilon I_D$ in $(-\infty, \Sigma_{cl}]$. Thus

$$\lim_{\varepsilon\downarrow 0}\lim_{\mu\downarrow 0}\{\mu^\nu[N^\varepsilon_1(\mu) - N_1(\mu)]\} = 0. \tag{5.6}$$

Since

$$N^\varepsilon_1(\mu) = \#\text{ of e.v. of } K_\mu(\Sigma_{cl}) > (1+\varepsilon)^{-1}$$

and

$$N_2(\mu) = \#(\text{of e.v. of } K_\mu(\Sigma_\mu) > 1) - \#(\text{of e.v. of } K_\mu(\Sigma_{cl}) > 1),$$

we see, using (5.6), that

$$\lim_{\mu\to 0}\mu^\nu N_2(\mu) \leqq \lim_{\varepsilon\downarrow 0}\lim_{\mu\to 0}\mu^\nu[\#(\text{e.v. of } K_\mu(\Sigma_\mu) > 1) - \#(\text{of e.v. of } K_\mu(\Sigma_{cl}) \geqq (1+\varepsilon)^{-1}]$$

since, for $A, B \geqq 0$,

$$(\#\text{ of e.v. of } A+B > 1) - (\#\text{ of e.v. of } A > 1-\delta) \leqq (\#\text{ of e.v. of } B > \delta)$$

(5.5) implies (5.4). □

Lemma 5.3. *(5.5) holds.*

Proof. Let p_μ be the projection onto those eigenvalues of $h_D = -\mu\Delta_x + V_{12}(x)$ less than $(1/2)\Sigma_{cl}$. Let $P_\mu = p_\mu \otimes 1$, $Q_\mu = 1 - P_\mu$. Then

$$[K_\mu(\Sigma_\mu) - K_\mu(\Sigma_{cl})] = \alpha_P + \alpha_Q$$

where

$$\alpha_A = (\Sigma_\mu - \Sigma_{cl})V_D^{1/2}[A(H_D(\mu) - \Sigma_\mu)^{-1}(H_D(\mu) - \Sigma_{cl})^{-1}]V_D^{1/2}.$$

Now, clearly

$$\|\alpha_Q\| \leqq \|V_D\|_\infty(\Sigma_\mu - \Sigma_{cl})((1/2)\Sigma_{cl} - \Sigma_\mu)^{-2} \to 0$$

as $\mu \to 0$ since $\lim_{\mu\downarrow 0}\Sigma_\mu = \Sigma_{cl}$. Thus, for μ large, $\|\alpha_Q\| < \varepsilon/2$ and thus (5.5) follows from

$$\mu^\nu(\#\text{ of e.v. of } \alpha_P \geqq \varepsilon/2) \to 0. \tag{5.7}$$

Next, let R be the range of the potential V_{12}. Then, we will prove below that

$$\|\chi(|x_1| > R+1)P_\mu\| \leqq C_1\exp(-C_2\mu^{-1/2}) \tag{5.8}$$

with $C_2 > 0$. So write (δ will be chosen below)

$$\begin{aligned}\alpha_P &\leqq (\Sigma_\mu - \Sigma_{cl})^\delta V_D^{1/2}P(\mu t_D)^{-1-\delta}V_D^{1/2}\\ &\leqq \alpha_P^{(1)} + \alpha_P^{(2)},\end{aligned} \tag{5.9}$$

where $\alpha_P^{(1)}$ (resp. $\alpha_P^{(2)}$) comes from inserting:

$$\chi(|x_1| \geqq R+1)P + \chi(|x_1| \leqq R+1)P\chi(|x_1| \geqq R+1)$$
$$(\text{resp. } \chi(|x_1| \leqq R+1)P\chi(|x_1| \leqq R+1))$$

in place of P on the right side of (5.9). Incidentally, (5.9) comes from using (with $x = \Sigma_\mu - \Sigma_{cl}$ and $A = H_D(\mu) - \Sigma_\mu$):

$$x(A)^{-1}(A+x)^{-1} \leqq x^\delta A^{-1-\delta}$$

for $x, A \geqq 0$ and

$$P(H_D - \Sigma_\mu)^{-1-\delta} \leqq (\mu t_D)^{-1-\delta} P.$$

Next, notice that, by the one body result,

$$\dim \operatorname{ran} P_\mu \leqq C\mu^{-\nu/2},$$

so $\alpha_P^{(1)}$ is a sum of at most $C\mu^{-\nu/2}$ terms, each of them of the form controlled by Lemma 2.5 if $2+\delta<\nu$. By (5.8), each of these terms has a norm bounded by

$$C\mu^{-1-\delta}\exp(-C_2\mu^{-1/2}),$$

so $\|\alpha_P^{(1)}\| \to 0$ as $\mu \to 0$. Thus, (5.7) follows from

$$\mu^\nu(\#\text{ of e.v. of } \alpha_P^{(2)} \geqq \varepsilon/4) \to 0 \tag{5.10}$$

To prove (5.10), note that

$$\chi(|x| \leqq R+1)V_D^{1/2} \leqq C\chi(|y| \leqq D)$$

for suitable constants C, D and $t_D = -\Delta_y$. Thus:
letting $B = \chi(|y| \leqq D)t_D^{-1-\delta}\chi(|y| \leqq D)$, we see that, for a suitable constant C, C';

$$\begin{aligned}(\#\text{ of e.v. of } \alpha_P^{(1)} \geqq \varepsilon/4) &\leqq \dim(P_\mu)(\#\text{ of e.v. of } B \geqq \varepsilon C\mu^{1+\delta}(\Sigma_\mu - \Sigma_{cl})^{-\delta}) \\ &\leqq C'(\varepsilon C\mu^{1+\delta}(\Sigma_\mu - \Sigma_{cl})^{-\delta})^{-\nu/1+\delta}\dim P_\mu\end{aligned}$$

since B is in the *weak* trace ideal $\mathscr{I}^W_{\nu/1+\delta}$ ([18, 3; 21]). Thus

$$\text{rhs of (5.10)} \leqq (\text{const.})(\Sigma_\mu - \Sigma_{cl})^{\nu\delta/1+\delta}$$

goes to zero as $\mu \to 0$. ∎

Lemma 5.4. *(5.8) holds.*

Proof. It suffices to show that for any η with $(-\mu\Delta_x + V_{12})\eta = e\eta; e < \frac{1}{2}\Sigma_{cl}$, $\|\eta\| = 1$: we have that

$$\|\chi(|x_1| > R+1)\eta\| \leqq C_1\exp(-C_2\mu^{-1/2}). \tag{5.11}$$

Now, pointwise:

$$\begin{aligned}|\eta(x)| &= |[(-\mu\Delta_x - e)^{-1}V_{12}\eta](x)| \\ &\leqq [(-\mu\Delta_x - \tfrac{1}{2}\Sigma_{cl})^{-1}|V_{12}\eta|](x).\end{aligned}$$

Let $G_0(x-y)$ be the integral kernel of $(-\Delta+1)^{-1}$. By scaling. $(-\mu\Delta_x - \frac{1}{2}\Sigma_{cl})^{-1}$

has an integral kernel:

$$\mu^{-\nu/2}(\tfrac{1}{2}\Sigma_{cl})^{(\nu-2)/2}G_0(\mu^{-1/2}(\tfrac{1}{2}\Sigma_{cl})^{1/2}(x-y)).$$

Since $\| V_{12}\eta \|_1 \leqq \| V_{12} \|_2 \| \eta \|_2$, we see that for $|x| > R$, the range of V_{12},

$$|\eta(x)| \leqq C\mu^{-\nu/2}\exp(-\mu^{-1/2}D[|x|-R])$$

from which (5.11) follows. ∎

This completes the three-body case. The N-body case is similar with the following changes: (1) For each D, there is an $a_D = \min V_D$. The hypotheses of unique clustering is $\Sigma_{cl} \equiv \min_{D'} a_{D'} = a_D$ for a unique D. (2) In the three-body case, we took P_μ to be the projection onto states of energy at most $\frac{1}{2}\Sigma_{cl}$. $\frac{1}{2}\Sigma_{cl} \equiv \alpha$ entered as a number between Σ_{cl} and $0 \equiv$ bottom essential spectrum of h_D. In the N-body case, we must pick α between Σ_{cl} and $\tilde{\Sigma}_{cl} = \min(a_{D'} | D' \blacktriangleright D, D' \neq D)$. The control of $\dim P_\mu$ still comes from $D-N$ bracketing. (3) In estimating terms, $\chi(|x_1| > R+1)$ is replaced by $\chi(|\zeta| > \tilde{R}+1)$ where $|\zeta|$ is a measure of the total size of the internal coordinates of D and $\tilde{R}$ is defined by

$$\{\zeta | V_D(\zeta) < \tilde{\Sigma}_{cl}\} \subset \{\zeta | |\zeta| < \tilde{R}\}.$$

(4) In proving the analog of (5.8), we write $V_D = V_D^+ + V_D^-$ with $V_D^- = \min(0, -\tilde{\Sigma}_{cl} + V_D)$ and then if $h_D\eta = e\eta$ with $e \leqq \alpha$ we write

$$|\eta(x)| \leqq |(h_{0,D} + V_D^+ - e)^{-1} V_D^- \eta(x)| \leqq [(h_{0,D} + \tilde{\Sigma}_{cl} - e)^{-1} |V_D^- \eta|](x).$$

since $V_D^+ \geqq \tilde{\Sigma}_{cl}$.

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