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Local Ward Identities and the Decay of Correlations in Ferromagnets

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Abstract. Using local Ward identities we prove a number of correlation inequalities for N-component, isotropically coupled, pair interacting ferromagnets; some for all $N \geqq 2$ and some for $N = 2, 3, 4$. These are used to prove a mass gap above the mean field temperature, for all $N \geqq 2$. For $N = 2, 3, 4$ we prove an upper bound on a critical exponent, and a lower bound on the susceptibility which diverges as $m \to 0$.

1. Introduction

Recently, Dobrushin and Pecherski [1] announced some new results about the possible rates of clustering in equilibrium states of lattice systems with finite range interactions. One of the results is that if the clustering falls-off faster than a certain (dimension dependent) power of the separation, then it is exponential. In the above work, the clustering is expressed by a rather strong condition, which measures the independence of the statistical distribution of spins in any region from all the other spins which are further then a given distance away. Subsequently, in a work published in this issue, Simon [2] formulated, and proved for ferromagnetic Ising models, a new inequality which implies such a property for the two point correlation function. Thus, this inequality leads to an upper bound on the corresponding critical exponent. Furthermore, the inequality was used in [2] to provide upper bounds for the critical temperature of the mass gap. In fact, incorporating an improvement due to Lieb [3], one obtains a sequence of upper bounds, calculable by finite algorithms, which converge to the exact value. The derivation of the mass gap from the above mentioned inequality of [2] is related to its derivation from Griffith's third inequality [4], see [5, 6]. The latter is a particular case of the new inequality, in its improved version of [3].

The main purpose of this note is to prove inequalities similar to those of [2] for multicomponent, ferromagnetic, spin models with $O(N)$ symmetry. We use in the

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derivation local Ward identities [7, 8] and certain correlation inequalities. For the case $N=2$ (plane rotor), a somewhat weaker result, is already contained in [2].

In particular, our results imply a mass gap above the mean field transition temperature for any $N \geqq 2$, an upper bound on suitable critical exponents for $N=2,3,4$, and a lower bound, in terms of m, on the susceptibility, which diverges as $m \to 0$.

2. Local Ward Identities

Local Ward Identities were developed by Driessler et al. [7]; similar ideas were discovered by Fröhlich and Spencer [8]. Certain special cases go back to Mermin [9].

We consider a classical system of spins with an a-priori (uncoupled) measure, for which the expectation value is denoted by $\langle \cdot \rangle_0$, and an interacting expectation

$$\langle A \rangle = \langle A e^{-\beta H} \rangle / \langle e^{-\beta H} \rangle_0 . \tag{2.1}$$

Let γ_t be a family of automorphisms [i.e. $\gamma_t(AB) = \gamma_t(A)\gamma_t(B)$] which preserve the a-priori expectation values of the functions, i.e.

$$\langle \gamma_t(A) \rangle_0 = \langle A \rangle_0 , \tag{2.2}$$

and let

$$\dot{A} = \frac{d}{dt} \gamma_t(A)|_{t=0} . \tag{2.3}$$

Then, the local Ward Identities assert that

$$\langle \dot{A} \rangle = \beta \langle A \dot{H} \rangle . \tag{2.4}$$

This comes from

$$\frac{d}{dt} \langle \gamma_t(A) e^{-\gamma_t(\beta H)} \rangle_0 = 0$$

which follows from (2.2).

Driessler et al. applied the local Ward Identities to prove an inequality on the local magnetization function (somewhat similar to Griffith's third inequality), which was then used to prove the vanishing of the magnetization above, approximately, the mean field transition temperature (the full mean field bound requires for $N \geqq 5$, an improvement, made in [10]). In the next section we apply a similar procedure to prove an inequality for the correlation function. For temperatures above the mean field transition, the inequality implies a mass gap, by a general argument which, in one way or another, was used in [5, 6, 2] (and which we reformulate in yet another way here).

3. Mass Gap Above the Mean Field Transition Temperature

The systems we consider consist of N-component spin variables, $\boldsymbol{\sigma}_i = (\sigma_{i,\lambda})$ $\lambda = 1, \ldots, N$, which are associated with lattice sites $i \in \mathbb{L} = \mathbb{Z}^d$, where d is the space dimension. The spins have as the a-priori measure the isotropic distribution on $(S^{n-1})^{\mathbb{L}}$, and interact through the Hamiltonian

$$H=-\sum_{ij} J_{ij}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j,$$

with some $J_{ij}\geqq 0$. We consider states which are limits of the finite volume states (2.1), with either free or periodic boundary conditions, for sequences of domains which increase to $\mathbb{L}$.

Theorem 3.1. *Under the above hypotheses:*

$$\langle\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\rangle\leqq\beta/N\sum_{k\in\mathbb{L}} J_{ik}\langle\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}_j\rangle \tag{3.1}$$

for any $i\neq j$.

Proof. Let $\gamma_t^{(i)}$ be the family of maps on $(S^{n-1})^{\mathbb{L}}$ with

$$\gamma_t^{(i)}(\sigma_{j,\lambda})=\begin{cases}\sigma_{j,\lambda} & j\neq i \quad\text{or}\quad \lambda\neq 1,2\\ (\cos t)\sigma_{i,1}-(\sin t)\sigma_{i,2} & j=i,\quad \lambda=1\\ (\sin t)\sigma_{i,1}+(\cos t)\sigma_{i,2} & j=i,\quad \lambda=2.\end{cases}$$

Then (2.4), with $A=\sigma_{i,2}\sigma_{j,1}$ and $\gamma_t=\gamma_t^{(i)}$, becomes:

$$\begin{aligned}\langle\sigma_{i,1}\sigma_{j,1}\rangle&=\beta\sum_{k\in\mathbb{L}} J_{ik}\langle\sigma_{i,2}^2\sigma_{j,1}\sigma_{k,1}-\sigma_{i,2}\sigma_{i,1}\sigma_{j,1}\sigma_{k,2}\rangle\\ &\leqq\beta\sum J_{ik}\langle\sigma_{i,2}^2\sigma_{j,1}\sigma_{k,1}\rangle &(3.2)\\ &\leqq\beta/N\sum_k J_{ik}\langle\sigma_{j,1}\sigma_{k,1}\rangle. &(3.3)\end{aligned}$$

In (3.2), we use the elementary $\langle\prod\sigma_{k,\lambda}\rangle\geqq 0$, obtained by expanding $e^{-\beta H}$. (3.3) follows by the symmetry:

$$\langle\sigma_{i,\lambda}^2\sigma_{j,1}\sigma_{k,1}\rangle=\langle\sigma_{i,2}^2\sigma_{j,1}\sigma_{k,1}\rangle,\quad\text{for}\quad\lambda\geqq 2,$$

an inequality of Simon [10]:

$$\langle\sigma_{i,1}^2\sigma_{j,1}\sigma_{k,1}\rangle\geqq\langle\sigma_{i,2}^2\sigma_{j,1}\sigma_{k,1}\rangle, \tag{3.4}$$

and the constraint

$$\sum_{\lambda=1}^{N}\langle\sigma_{i,\lambda}^2\rangle=1.$$

By the isotropy, (3.3) is the same as (3.1). □

As it was shown in [2], (3.1) implies a mass gap. Thus we may conclude:

Theorem 3.2. *If for some* $\varepsilon>0$

$$\sup_{i\in\mathbb{L}}\sum_{k\in\mathbb{L}} J_{ik}e^{\varepsilon|i-k|}<\infty, \tag{3.5}$$

then for any β *such that*

$$\beta^{-1}>\sup_{i\in\mathbb{L}} N^{-1}\sum_{k\in\mathbb{L}} J_{ik} \tag{3.6}$$

there is some $m>0$ with which

$$\langle \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \rangle \leqq e^{-m|i-j|}. \tag{3.7}$$

Of course, for translation invariant finite range interactions, (3.5) is superflous.

While we could give a number of references, and in particular [2], for the deduction of Theorem 3.2 from Theorem 3.1, we prove it here both for completeness and to offer still another point of view on the derivation.

Proof. By a continuity (or, alternatively, interpolation) argument, the assumptions imply that for some $\mu>0$

$$\sup_{i\in\mathbb{L}} \beta/N \sum_k J_{ik} e^{\mu|i-k|} \equiv e^{-\alpha} < 1. \tag{3.8}$$

Let us define now an auxiliary Markov random walk process on the lattice, with discrete time. Denoting its paths by $x(t)$, $(t\in\mathbb{Z},\ x\in\mathbb{L})$ we choose for its transition probabilities from i to k:

$$q_{ki} = \beta/N J_{ik} e^{\mu|i-k|} e^{\alpha_i}, \tag{3.9}$$

with

$$e^{-\alpha_i} \equiv \beta/N \sum_{k\in\mathbb{L}} J_{ik} e^{\mu|i-k|} \leqq e^{-\alpha} < 1.$$

Thus, if it is only known that $x(t)=i$, then the conditional distribution of $x(t+1)$ is given by $q_{.i}$.

For a fixed pair $i\neq j$ we shall consider the walk starting at $x(0)=i$, and for each $N\geqq 0$ define the stopping time, τ_N, as either the earliest time $t\in[1,2,\dots,N]$ for which $x(t)=j$, or N, if $x(t)\neq j$ on $[1,N]$. Equation (3.1) can now be rewritten as follows:

$$\begin{aligned}\langle \boldsymbol{\sigma}_{x(t)} \cdot \boldsymbol{\sigma}_j \rangle &\equiv E_t(\langle \boldsymbol{\sigma}_{x(t)} \cdot \boldsymbol{\sigma}_j \rangle) \\ &\leqq \begin{cases} E_t(\langle \boldsymbol{\sigma}_{x(t+1)} \cdot \boldsymbol{\sigma}_j \rangle \exp[-\alpha_{x(t)} - \mu|x(t+1)-x(t)|]) & x(t)\neq j \\ 1 & x(t)=j \end{cases}\end{aligned} \tag{3.10}$$

denoting by $E(\cdot)$ the expectation over the random walk process $x(\cdot)$, and using the subscript t to indicate the conditioning on $x(t)$.

Replacing α_k by its lower bound α and iterating (3.10), we obtain:

$$\begin{aligned}\langle \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \rangle &\equiv E(\langle \boldsymbol{\sigma}_{x(\tau_0)} \cdot \boldsymbol{\sigma}_j \rangle) \\ &\leqq E(\langle \boldsymbol{\sigma}_{x(\tau_1)} \cdot \boldsymbol{\sigma}_j \rangle e^{-\mu|x(\tau_1)-x(\tau_0)|-\alpha} \\ &\leqq \dots \leqq E(\langle \boldsymbol{\sigma}_{x(\tau_N)} \cdot \boldsymbol{\sigma}_j \rangle \exp[-\mu|X(\tau_N)-X(\tau_0)| - \alpha\tau_N]) \qquad (3.11) \\ &\xrightarrow[N\to\infty]{} e^{-\mu|i-j|} \langle \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_j \rangle \lim_{N\to\infty} E(e^{-\alpha\tau_N}) \\ &\leqq e^{-\mu|i-j|}, \qquad (3.12)\end{aligned}$$

where we have also used the triangle inequality in (3.11) and the boundedness of $\langle \boldsymbol{\sigma}_k \boldsymbol{\sigma}_j \rangle$ in (3.12). □

Remarks. We would like to add the following comments, in which, for simplicity, we refer to translation invariant, ferromagnetic, interactions.

1. (3.6) is the condition for the lack of spontaneous magnetization in the mean field approximation. For any $\varepsilon > 0$, there are models with $\beta/N \sum_k J_{0k} = 1 + \varepsilon$ without a mass gap [11].

2. (3.6) is less restrictive than the known conditions for the applicability of Dobrushin uniqueness theorem [12]. The condition of [13] is

$$\beta \sum_k |J_{0k}| < \sqrt{N}$$

and that of [14]:

$$\beta \sum_k |J_{0k}| < N/\sqrt{5}.$$

However, these results require neither positivity nor isotropy and are not restricted to pair interactions. Moreover, the conclusions of the Dobrushin uniqueness theorem are stronger (although for $N=2$, known correlation inequalities [15, 16] imply a unique *translation invariant* state when there is no spontaneous magnetization).

3. For $N=2$ the above mean field bound is not as good as our other bound [17]:

$$\beta_{c,2} \geqq 2\beta_{c,1}, \tag{3.13}$$

where $\beta_{c,1}$ refers to an Ising spin system with the same interaction J, and both critical β are either defined via the mass gap or the magnetization. The latter result is better because the mean field bounds ($\beta_{c,N} \geqq \beta_{c,N}^{\mathrm{M.F.}}$) hold also for Ising models ($N=1$) [21, 22, also in 2], and as function of N they obey the equality in (3.13).

4. Incorporating a basic observation of Gross [18] let us point out that the only assumption about the metric $|i-j|$ used in the above proof was the triangle inequality. Any function with this property could be substituted instead of $|i-j|$ in the assumption (3.5) and the conclusion (3.7) of Theorem 3.2. This may be useful if $\sum_{k \in \mathbb{L}} J_{0k} e^{\varepsilon |k|} = \infty$ for any $\varepsilon > 0$, but $\sum_{k \in \mathbb{L}} J_{0k} e^{d(0,h)} < = 0$ for some other metric on $\mathbb{L}$ (e.g. $d(i,j) = \ln[|i-j|+1]$, in which case the theorem would just lead to a power fall-off).

4. Further Inequalities for $N=2,3,4$-Component Spins

We shall now prove the following inequality which is similar to one proven in [2] for Ising spins.

Theorem 4.1. *Let Λ be a finite subset of $\mathbb{L}$. Then under the assumptions of Sect. 3 (and in particular $J_{ij} \geqq 0$), for any N component system with $N=2,3,4$ and $i \in \Lambda$, $j \in \Lambda^c$*

$$\langle \sigma_{i,1} \sigma_{j,1} \rangle \leqq \beta \sum_{\substack{k \in \Lambda^c \\ l \in \Lambda}} J_{lk} \langle \sigma_{i,1} \sigma_{l,1} \rangle \langle \sigma_{k,1} \sigma_{j,1} \rangle . \tag{4.1}$$

Proof. Rotating all the spins in Λ by

$$\gamma_t^\Lambda = \prod_{k \in \Lambda} \gamma_t^{(k)},$$

and applying (2.4) with $A=\sigma_{i,2}\sigma_{j,1}$, we obtain

$$\langle\sigma_{i,1}\sigma_{j,1}\rangle \leqq \beta \sum_{\substack{k\in\Lambda^c \\ l\in\Lambda}} J_{lk}\langle\sigma_{i,2}\sigma_{l,2}\sigma_{k,1}\sigma_{j,1}\rangle, \tag{4.2}$$

by the same argument as (3.2).

Since $N=2,3,4$, we have that [19, 20]

$$\langle\sigma_{i,2}\sigma_{l,2}\sigma_{k,1}\sigma_{j,1}\rangle \leqq \langle\sigma_{i,2}\sigma_{l,2}\rangle\langle\sigma_{k,1}\sigma_{j,1}\rangle. \tag{4.3}$$

(4.1) follows now by the isotropy. □

Remarks

1. (4.1) can be rewritten as

$$\langle\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\rangle \leqq \beta/N \sum_{\substack{k\in\Lambda^c \\ l\in\Lambda}} J_{lk}\langle\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_l\rangle\langle\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}_j\rangle, \tag{4.4}$$

to make it look more like (3.1).

2. Once we have (4.1), most of its applications made in [2] carry through, for example:

Corollary 4.2. *Suppose J is translation invariant and that, for some $\varepsilon>0$, $\sum J_{0l}e^{\varepsilon|l|}<\infty$. Under the hypotheses of Theorem 4.1, if*

$$\langle\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\rangle \leqq C|i-j|^{-x} \tag{4.5}$$

with $x+1>d$, the dimension of the lattice, then

$$\langle\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\rangle \leqq C' e^{-m|i-j|}$$

for some $m>0$.

The corollary follows from (4.4) by the proof given to Proposition 3.2 from (3.1), after choosing Λ large enough so that

$$\beta/N\left(\sum_k J_{0k}e^{\varepsilon|k|}\right)\sum_{l\in\Lambda} e^{-\varepsilon\,\mathrm{dist.}(l,\Lambda^c)}\langle\sigma_0\sigma_l\rangle<1. \tag{4.6}$$

The stopping times τ_N are to be defined by the first instance that $j-x(t)\in\Lambda$. This leads to the extra coefficient $C'\left(=\exp\left[\mu\max_{l\in\Lambda}|l|\right]\right)$.

The last result clearly implies an upper bound for the possible values of the critical exponent x in such models.

3. Most of the other applications of [2] which do not use Lieb's improvement [3] depend on estimating

$$F(n)=\sum_{i_1=n}\langle\sigma_{0,1}\sigma_{i,1}\rangle$$

(i.e. the sum is over a hyperplane). Assuming $\langle\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\rangle \geqq e^{-m|i-j|}$, and finite susceptability, (4.1) implies that, for example, in the nearest neighbor use

$$F(j+k+1)\leqq F(j)F(k) \quad \text{for} \quad j,k>0$$

By induction,

$$F(j)^{2n}\geqq F(2^n(j+1)-1)$$

and thus, for $j \geqq 0$,

$$F(j) \geqq e^{-m[j+1]}.$$

Summing over j

$$X \equiv \sum_{j=-\infty}^{\infty} F(j) \geqq e^{-m} \coth\left(\frac{m}{2}\right).$$

Thus, the susceptibility diverges a $m \to 0$.

4. The main improvement made in [3] of the inequality of [2] suggests that it might be possible to replace the factor $\langle \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_l \rangle$ in (4.4) by $\langle \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_l \rangle'$, with the prime indicating that all the interactions between Λ and Λ^c were set to zero. We do not have a proof of such an inequality for the multicomponent system. For nearest neighbor model this would imply that the mass gap goes to zero at the critical point, by an argument given in [2]. This argument is applicable for $N=1,2$, as a consequence of the results of [3] and [23].

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