A COMPARISON OF PLANE ROTOR AND ISING MODELS

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Let β_c^I be the critical inverse temperature in an Ising ferromagnet with pair interactions and let β_c^R be the critical temperatures in the plane rotor (classical *xy* model) with the same couplings. We prove that $\beta_c^R \geq 2\beta_c^I$. This yields the lower bound $\beta_c \geq 0.88$ for the Kosterlitz–Thouless transition.

Consider Ising spins $s_\alpha = \pm 1$ interacting under a hamiltonian $H = -\Sigma J_{\alpha\gamma} s_\alpha s_\gamma$ with $J_{\alpha\gamma} \geq 0$ let $\langle\ \rangle_{\beta,1}$ denote expectations in this model at inverse temperature β. Let $\boldsymbol{\sigma}_\alpha = (\sigma_\alpha^{(1)}, \sigma_\alpha^{(2)}) = (\cos\theta_\alpha, \sin\theta_\alpha)$ be two-component uniformly distributed unit vectors and let $\langle\ \rangle_{\beta,2}$ be the expectation for this model with $H = -\Sigma J_{\alpha\gamma} \boldsymbol{\sigma}_\alpha \cdot \boldsymbol{\sigma}_\gamma$ (same J's). In this note we will prove that

$$\langle \boldsymbol{\sigma}_\alpha \cdot \boldsymbol{\sigma}_\gamma \rangle_{2\beta,2} \leq \langle s_\alpha s_\gamma \rangle_{\beta,1} \,. \tag{1}$$

If β_c^I (respectively β_c^R) is the critical temperature, defined in terms of loss of exponential falloff, in the model with one (respectively two) component(s), (1) immediately implies that

$$\beta_c^R \geq 2\beta_c^I \,. \tag{2}$$

By the ghost spin method [1], (1) implies a bound on the magnetization:

$$m_{2\beta,2}(h) \leq m_{\beta,1}(h) \,, \tag{3}$$

so that (2) holds also if critical β's are defined in terms of the spontaneous magnetization.

Before proving (2), we note that since for nearest neighbor interaction on a two-dimensional square lattice $\beta_c^I(\text{2-dim}) = \frac{1}{2}\ln(\sqrt{2}+1)$, we have an upper bound on the Kosterlitz–Thouless transition temperature

$$\beta_c^R(\text{2-dim}) \geq \ln(1+\sqrt{2}) = 0.88137 \ldots,$$

to be compared with the mean field value 0.5 (which was known [2] [‡1] to be a lower bound on β_c^R), the gaussian value $2/\pi = 0.637$ and the value 1.11 ± 0.04 obtained in ref. [3] on the basis of Monte Carlo simulation. We also note that in three dimensions, and nearest neighbor interactions, the ratio of the two sides of (2) is 1.04 and that as $d \to \infty$, the ratio goes to 1, since the mean field theory is known to be exact in that limit [4].

As a final preliminary, we remark that (1), with 2β replaced by β and thus (2) without the factor of 2, is well known.

The proof of (1) is in two steps, neither of which is new! Let $\langle\ \rangle_{\beta,\mathrm{P}}$ denote expectations in the Z_4 Potts model, i.e. the variables before coupling take the values $0, \pi/2, \pi, 3\pi/2$ with equal weights. Then (1) follows from the pair of statements:

$$\langle \boldsymbol{\sigma}_\alpha \cdot \boldsymbol{\sigma}_\gamma \rangle_{2\beta,2} \leq 2\langle \sigma_\alpha^{(1)} \sigma_\gamma^{(1)} \rangle_{2\beta,\mathrm{P}} \,, \tag{4}$$

$$2\langle \sigma_\alpha^{(1)} \sigma_\gamma^{(1)} \rangle_{2\beta,\mathrm{P}} = \langle s_\alpha s_\gamma \rangle_{\beta,1} \,. \tag{5}$$

(4) is a well-known statement which we first learned from Bricmont [5] (see also ref. [6]) and (5) follows from Suzuki's observation [7] that the Z_4 Potts model is isomorphic to two uncoupled Ising models. To be more explicit, (4) is proven by noting that $\langle \boldsymbol{\sigma}_\alpha \cdot \boldsymbol{\sigma}_\gamma \rangle =$

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[‡1] J. Fröhlich has informed us that his method implies $\beta_c^R(\text{2-dim}) > 0.67$.

$2\langle\sigma_\alpha^{(1)}\sigma_\gamma^{(1)}\rangle$ and considering the hamiltonian $H' = H + \lambda\Sigma_\alpha \cos(4\theta_\alpha)$. By correlation inequalities of Ginibre [8], increasing λ increases $2\langle\sigma_\alpha^{(1)}\sigma_\gamma^{(1)}\rangle$. But the left side of (4) is the value at $\lambda = 0$ and the right side the value at $\lambda = \infty$.

To prove (5), let $s_\alpha = \sigma_\alpha^{(1)} + \sigma_\alpha^{(2)}$, $t = \sigma_\alpha^{(1)} - \sigma_\alpha^{(2)}$. The four values $\theta = 0, \pi/2, \pi, 3\pi/2$ correspond to $(s_\alpha, t_\alpha) = (\pm 1, \pm 1)$ with equal weights. Since

$$2\beta \sum J_{\alpha\gamma} \boldsymbol{\sigma}_\alpha \cdot \boldsymbol{\sigma}_\gamma = \beta \sum J_{\alpha\gamma}(s_\alpha s_\gamma + t_\alpha t_\gamma),$$

the s and t are independently distributed according to $\langle\ \rangle_{\beta,1}$. Since $2\sigma_\alpha^{(1)}\sigma_\gamma^{(1)} = \frac{1}{2}(s_\alpha + t_\alpha)(s_\gamma + t_\gamma)$ and $\langle s_\alpha\rangle = 0$, (5) is proven.

To see the subtlety of (1), we note that if $\langle\ \rangle'_{\beta,2}$ is the expectation for plane rotors but with the hamiltonian $H' = -\Sigma J_{\alpha\gamma}\sigma_\alpha^{(1)}\sigma_\gamma^{(1)}$, then by results of Wells [9]

$$\langle s_\alpha s_\gamma\rangle_{\beta,1} \leqslant 2\langle\sigma_\alpha^{(1)}\sigma_\gamma^{(1)}\rangle'_{2\beta,1}\ . \tag{6}$$

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References

[1] R. Griffiths, J. Math. Phys. 8 (1967) 478.
[2] M. Aizenman and B. Simon, Local Ward identities and the mass gap in ferromagnets, to be submitted to Commun. Math. Phys;
B. Durhuus and J. Fröhlich, A connection between ν-dimensional Yang–Mills theory and $(\nu - 1)$-dimensional nonlinear σ-models, Commun. Math. Phys., to be published;
J. Fröhlich and T. Spencer, in preparation.
[3] W.J. Shugard, J.D. Weeks and G.H. Gilmer, Monte Carlo simulation of the planar model using the dual solid-on-solid representation, Bell. Lab. preprint.
[4] F. Dyson, E. Lieb and B. Simon, J. Stat. Phys. 18 (1978) 335;
B. Simon, Mean field upper bounds on the transition temperature in multicomponent ferromagnets, J. Stat. Phys., to be published.
[5] J. Bricmont, private communication.
[6] S. Elitzur, R.B. Pearson and J. Shigemitsu, IAS preprint.
[7] M. Suzuki, Prog. Theor. Phys. 37 (1967) 770.
[8] J. Ginibre, Commun. Math. Phys. 16 (1970) 310.
[9] D. Wells, Some moment inequalitites for general spin Ising ferromagnets, Indiana Univ. preprint.