Brownian Motion and Harnack Inequality for Schrödinger Operators

M. AIZENMAN
Princeton University

AND

B. SIMON
California Institute of Technology

1. Introduction

In this paper we shall discuss solutions of the time independent Schrödinger equation

$$Hu = 0, \tag{1.1}$$

$$H = -\tfrac{1}{2}\Delta + V. \tag{1.2}$$

Since V will not be required to go to zero at infinity, one can obtain results on solutions of $Hu = Eu$ by changing V to $V - E$. Two kinds of results will interest us. The first are often called *subsolution estimates* since they hold for functions with $Hu \leqq 0$, $u \geqq 0$:

$$u(x) \leqq D \int_{|x-y| \leqq 1} u(y)\, dy, \tag{1.3}$$

the constant D depending only on some local norms of V. In many cases (1.3) will hold with a constant D which is independent of x. The second kind are results of the type of the *Harnack inequality*. It states that if Ω, Ω' are bounded open sets with $\bar{\Omega} \subset \Omega'$, then there is a constant C, depending only on Ω, Ω' and norms of V, such that all solutions of (1.1) with $u \geqq 0$ on Ω' satisfy

$$u(x) \leqq Cu(y) \tag{1.4}$$

for all $x, y \in \Omega$. We remark that if $H\phi = 0$, then $H|\phi| \leqq 0$ so that (1.3) holds with $u = |\phi|$.

Both types of results were proven for a wide class of potentials V by Trudinger [36], following the approach of Stampacchia [34], which uses in part a set of ideas due to Moser [20]. Our interest in these results was stimulated by their use in work by M. and T. Hoffmann-Ostenhof and collaborators [1], [14], [15]; in particular, (1.3) is an ideal tool for passing from exponential fall-off in

Communications on Pure and Applied Mathematics, Vol. XXXV, 209–273 (1982)
 CCC 0010-3640/82/020209-64$06.40

average sense to pointwise exponential fall-off [1] (see [30], [9] for other methods), and (1.4) is useful in studying nodes [14].

Our main goal here is to provide a proof of these and related results by exploiting Brownian motion ideas. This approach is suggested by relating (1.1) with an integral equation, *via* the Feynman–Kac formalism. (This relation has already been used in deriving a simple proof (see [15]) of one of the consequences of (1.4).) The naturalness of these methods is shown by the fact that we succeed in proving necessary and sufficient conditions on the potential V so that a strong form of (1.4) holds. Explicitly we present the following definitions.

DEFINITION. We say that $H = -\frac{1}{2}\Delta + V$ obeys a *strong Harnack inequality* if and only if for each $R, d > 0$ there is a function $g_{R,d}(\cdot)$ on $(0, d)$ with $\lim_{a\downarrow 0} g_{R,d}(a) = 0$, such that if u is defined in $\{z \mid |z - x| < 2d\}$, for some $|x| \leqq R$, and satisfies there

(i) $Hu = 0$ (distributional sense),

(ii) $u \geqq 0$,

then

$$|u(y) - u(x)| \leqq g_{R,d}(|y - x|)u(x) \tag{1.5}$$

for all y with $|y - x| \leqq d$.

DEFINITION. We say that $V \in K_\nu^{\text{loc}}$, $\nu \geqq 3$, if and only if, for each R,

$$\lim_{\alpha\downarrow 0}\left[\sup_{|x|\leqq R}\int_{|x-y|\leqq\alpha}|x-y|^{-(\nu-2)}|V(y)|\,d^\nu y\right] = 0. \tag{1.6}$$

When $\nu = 2$, $|x - y|^{-(\nu-2)}$ is replaced by $\{-\ln|x - y|\}$, and when $\nu = 1$, by 1.

Among other things, we prove the following result:

THEOREM 1.1. *If $V \in K_\nu^{\text{loc}}$, then H obeys the strong Harnack inequality. Conversely, suppose that*

(i) $V \leqq 0$,

(ii) *for all R, there are $\epsilon_R > 0$ and $c_R < \infty$ such that*

$$\frac{1}{2}\int|\nabla\phi(x)|^2\,dx + (1 + \epsilon_R)\int V(x)|\phi(x)|^2\,dx \geqq -C_R\int|\phi(x)|^2\,dx$$

for all $\phi \in C_0^\infty$ supported in $\{x \mid |x| \leqq R\}$, and

(iii) *H obeys the strong Harnack inequality.*

Then $V \in K_\nu^{\text{loc}}$.

We remark that if V is allowed to have severe local oscillations, then it is quite likely that the strong Harnack inequality can hold even though $V \notin K_\nu^{\text{loc}}$; see Example 3 in Appendix 1.

While one can construct rather pathological examples in K_ν^{loc} for which Trudinger's methods will not work (see Example 2 in Appendix 1), the main point of our work is not in such borderline cases, although it is a nice bonus to be able to handle them, and in physical situations one wants to be able to deal with some unbounded V's (like sums of Coulomb potentials which certainly lie in K_ν; see below). Rather, we feel that our approach is very natural. Indeed, for $V = 0$ the standard method for proving the Harnack inequality is to get upper and lower bounds on the Poisson kernel, as one variable varies on the boundary and the other on a compact set away from the boundary. In essence, this is exactly how we shall prove the Harnack inequality here!

To introduce the formula we shall use for the Poisson kernel for H, let us recall (without giving the precise conditions for validity) three formulas from the theory of Brownian motion as applied to Schrödinger operators [31] and to the Dirichlet problem [22]. Let $H_0 = -\frac{1}{2}\Delta$, and let E_x (respectively P_x) denote the expectation (respectively probability) with respect to Brownian motion starting at x. If A is a set and f a function, $E_x(f; A) \equiv E_x(f\chi_A)$ with χ_A the indicator function for A. The first formula is a standard application of Brownian motion:

$$(1.7) \qquad (\exp\{-tH_0\}f)(x) = E_x(f(b(t))).$$

The second is the Feynman–Kac formula:

$$(1.8) \qquad (\exp\{-tH\}f)(x) = E_x\left(\exp\left\{-\int_0^t V(b(s))ds\right\}f(b(t))\right),$$

and the last is the formula for solving the Dirichlet problem. While one big advantage of the Brownian motion approach to solving the Dirichlet problem lies in its ability to effortlessly accommodate irregular regions, we describe it for an open bounded region Ω with a smooth boundary $\partial\Omega$. For a continuous path, b, let $T(b) \equiv \inf\{s \geqq 0 \mid b(s) \notin \Omega\}$, the *first exit time*. By continuity, $b(T(b)) \in \partial\Omega$. Then, for any measurable f on $\partial\Omega$ define

$$(1.9) \qquad (M_0 f)(x) \equiv E_x(f(b(T))),$$

writing T for $T(b)$. This solves the Dirichlet problem in the sense that $M_0 f$ is harmonic in Ω and for any $y \in \partial\Omega$, $\lim_{x\to y}(M_0 f)(x) = f(y)$ if f is continuous at y. A glance at the change from (1.7) to (1.8) suggests that the solution of the Dirichlet problem for $H = -\frac{1}{2}\Delta + V$ should be

$$(1.10) \qquad (M_V f)(x) \equiv E_x\left(\exp\left\{-\int_0^T V(b(s))\,ds\right\}f(b(T))\right).$$

In Appendix 4, we verify that this indeed solves the Dirichlet problem for any $V \in K_\nu^{\mathrm{loc}}$ so long as there is an $\alpha > 0$ for which

$$\frac{1}{2}\int|\nabla\phi(x)|^2\,dx + \int V(x)|\phi(x)|^2\,dx \geqq \alpha\int|\phi(x)|^2\,dx \tag{1.11}$$

for all $\phi \in C_0^\infty$ with $\operatorname{supp}\phi \subset \Omega$ (and a condition like (1.11) is needed for M_V even to be defined).

We should emphasize that there has been a prior work on the solution of the Dirichlet problem for Schrödinger operators. While (1.9) may seem quite natural, it is not valid without some assumptions on Ω in relation to V (see Remark 1 in Section 2). The relation (1.9) was studied by Chung–Rao [7] (see also [6], [42]) who required V to be bounded; however the key condition (1.11) is not mentioned in that work (and neither are the various L^p estimates which we prove). Closely related ideas appear in an earlier work by Khas'minskii [17]. Explicitly, he shows that $(M_V 1)(x) \equiv u(x)$ solves $Hu = 0$, and he emphasizes the importance of a condition which is closely related to (1.11), namely the existence of a strictly positive solution of $Hu = 0$.

It is not the general solution of the Dirichlet problem itself that is of importance to us in proving (1.3) and (1.4). Rather it is the closely related fact that if $Hu = 0$ in a neighborhood of Ω, then for $x \in \Omega$

$$u(x) = E_x\left(\exp\left\{-\int_0^T V(b(s))\,ds\right\}u(b(T))\right). \tag{1.12}$$

We shall give several proofs of (1.12) in Section 2 for Ω sufficiently small spheres about any fixed point x_0. The one that is most illuminating is simple under the extra restriction that u is a global solution of $Hu = 0$. Then $e^{-tH}u = u$, and thus (1.12) holds for T replaced by any fixed time, t. The Markov property of Brownian motion then implies that

$$\exp\left\{-\int_0^t V(b(s))\,dr\right\}u(b(t)) \equiv f(b,t)$$

is a martingale. Relation (1.12) results by applying the standard theorems on evaluating a martingale at a stopping time (there is a critical subtlety in that T is unbounded and it is here that (1.11) enters).

Now fix x_0. Let $\Omega' \subset \Omega$ be two spheres about x_0, so small that (1.12) holds. We shall obtain (1.3), (1.4) from a pair of estimates which hold if Ω' and Ω are shrunk sufficiently, namely

$$\sup_{x\in\Omega'}|(M_V f)(x)| \leqq C_1\int_{\partial\Omega}|f(y)|\,d\sigma(y) \tag{1.13}$$

and

$$\inf_{x \in \Omega'} (M_V f)(x) \geqq C_2 \int_{\partial\Omega} f(y)\, d\sigma(y) \tag{1.14}$$

if $f \geqq 0$. In these formulae, $d\sigma$ is the usual surface measure. Inequality (1.13) implies (1.3) by averaging over a small interval of values for the radius of Ω, and (1.4) is an immediate consequence of (1.13), (1.14) and a compactness argument.

One of the nicest features of the Brownian motion approach to these problems is that in a certain sense (1.13) implies (1.14)! For the Schwarz inequality immediately implies that, for $f \geqq 0$,

$$[(M_0 f)(x)]^2 \leqq (M_V f)(x)(M_{-V} f)(x) \tag{1.15}$$

where M_0 is the map defined by (1.10) when $V = 0$, i.e., the Poisson kernel. Relation (1.15) says that if we have the upper bound (1.13) for M_{-V} and the lower bound (1.14) for M_0 (the ordinary Harnack inequality), then we have (1.14) in general. We describe the details of this argument in Section 2.

This focuses attention on the bound (1.13). Given the similarity of (1.10) and (1.8), the bound (1.13) is very close to the bound

$$\|e^{-tH}\phi\|_\infty \leqq C_t \|\phi\|_1. \tag{1.16}$$

Let us recall the proof of (1.16) found by Carmona [5] and Simon [31] (a different proof is implicit in Kovalenko and Semenov [18]; earlier, under stronger hypotheses on V, (1.16) was proven by related methods by Herbst–Sloan [12]). An important input is the following basic result, which we shall use several times.

THEOREM 1.2 (Khas'minskii's lemma). *Let g be a non-negative function on $\mathbb{R}^n$ with*

$$\sup_x E_x\left(\int_0^t g(b(s))\, ds\right) \equiv \alpha < 1. \tag{1.17}$$

Then

$$\sup_x E_x\left(\exp\left\{\int_0^t g(b(s))\, ds\right\}\right) \leqq (1 - \alpha)^{-1}. \tag{1.18}$$

Proof: Obviously, it suffices to show that

$$A_n = \sup_x E_x\left(\int_{0 < s_1 < \cdots < s_n < t} g(b(s_1)) \cdots g(b(s_n))\, ds_1 \cdots ds_n\right) \leqq \alpha^n.$$

In this expectation, fix $0 < s_1 < \cdots < s_{n-1}$ and condition on the path $b(s)$ for $s \in [0, s_{n-1}]$. Since Brownian motion starts afresh at $b(s_{n-1})$, and $s_n - s_{n-1}$ runs from 0 to $t - s_{n-1} < t$, we can use (1.17) to bound this conditional expectation. The result is

$$A_n \leqq \alpha A_{n-1},$$

from which $A_n \leqq \alpha^n$ follows by induction.

The result appears to go back at least to the paper of Khas'minskii [17] after whom we name it. It was later rediscovered independently by Portenko [23] and by Berthier–Gaveau [3]. (In [31] it is dubbed "Portenko's lemma" since its author did not know of Khas'minskii's earlier work.)

For later purposes, we note that this result holds in much greater generality than stated. First, since the proof involves integration over the last time interval, it is applicable also if t is replaced by a (nonanticipatory) stopping time T, which "starts afresh". Secondly, the proof extends without change to an arbitrary Markov process, replacing the Brownian motion.

We can now describe the Carmona–Simon proof of (1.16) under suitable hypotheses on V:

Step 1. Use Khas'minskii's lemma to conclude that

$$\|\exp\{-t(H_0 + \lambda V)\}\phi\|_\infty \leqq C\|\phi\|_\infty \tag{1.19}$$

for any $\lambda \leqq 2$ and $t > 0$ sufficiently small, and then use the semigroup property to get (1.19) for all $t > 0$ with C replaced by Ce^{At}.

Step 2. Use the Schwarz inequality in (1.8) to show that

$$|(\exp\{-t(H_0 + V)\}f)(x)| \leqq \left[(\exp\{-t(H_0 + 2V)\}1)(x)\right]^{1/2} \times \left[(\exp\{-tH_0\}|f|^2)(x)\right]^{1/2},$$

which implies

$$\|\exp\{-t(H_0 + V)\}f\|_\infty \leqq \|\exp\{-t(H_0 + 2V)\}1\|_\infty \|\exp\{-tH_0\}|f|^2\|_\infty^{1/2}.$$

Using properties of $\exp\{-tH_0\}$ and (1.19), we have that

$$\|\exp\{-t(H_0 + V)\}f\|_\infty \leqq C_t\|f\|_2. \tag{1.20}$$

Step 3. Use duality (selfadjointness of e^{-tH}) and (1.20) to conclude that

$$\|\exp\{-t(H_0+V)\}f\|_2 \leqq C_t\|f\|_1. \tag{1.21}$$

Step 4. By the semigroup property, (1.20) and (1.21), we have that

$$\|\exp\{-t(H_0+V)\}f\|_\infty \leqq C_{t/2}^2\|f\|_1.$$

If one tries to mimic this argument to prove (1.13), there is no difficulty with Steps 1 and 2 (see Section 2). This yields an effortless proof of the analogues of (1.13) with $\int |f(y)d\sigma(y)$ replaced by $[\int |(f(y)|^2 d\sigma(y)]^{1/2}$, and thus of (1.3) with $\int_{|x-y|<1} u(y)dy$ replaced by $(\int_{|x-y|<1}|u(y)|^2 dy)^{1/2}$ (which suffices for many applications; see [1]). Although at first sight, M does not seem to have a semigroup property, it does: namely if $\Omega' \subset \Omega'' \subset \Omega$, and if we make the Ω dependence of M explicit, then $(M^{\Omega''}g)(x) = (M^{\Omega}f)(x)$ for $x \in \Omega'$ denoting $g = (M^{\Omega}f)\restriction \partial\Omega''$. This is a simple consequence of the strong Markov property of Brownian motion (see Proposition A.4.9). Thus Step 4 poses no problem. It turns out that the trivial Step 3 of the above proof is the most subtle in the analogous proof of (1.13), since M is not symmetric. The key will be to realize the adjoint of M in terms of a Markov process (Brownian motion reversed at an exit time), which would permit to mimic Steps 1 and 2 also for the adjoint of M.

In Section 2, we prove (1.12) and then use it to get (1.15) and to proceed through the analogue of the above Steps 1 and 2. In Section 3, we develop a convenient way of studying the dual of the map M and then complete the proofs of (1.13) and (1.14). Section 4, discussed below, is devoted to the study of spaces K_ν and K_ν^{loc}. In Section 5, we prove Theorem 1.1 and, in Section 6, (1.3) is proven for solutions of $Hu \leqq 0$. In Appendix 1, we present certain pathological examples which delimit K. Appendix 2 contains some estimates on Green's functions needed in Section 3. In Appendix 3, we discuss various facts about exit times, including the remark that for any increasing function f, and all x and Ω, $E_x(f(T))$ is maximized, for a fixed value of $|\Omega|$, by taking $x = 0$ and Ω a ball centered at 0. For "nice" Ω's, we also give an explicit formula for the joint distribution of T and $b(T)$. In Appendix 4, we discuss the solution of the Dirichlet problem under the hypothesis (1.11).

We close this introduction with a few words about the classes K_ν and K_ν^{loc}. This is partly because in Sections 2 and 3 we require some facts not proven until Section 4. We also wish to emphasize that these classes, which have already been introduced by Kato [16], are very natural for the problems discussed here. K_ν^{loc} is defined before Theorem 1.1, and K_ν (at least for $\nu \geqq 2$) is its analogue, with $\sup_{|x|\leqq R}$ replaced by $\sup_x$ (K_1 is defined in Section 4). Spurred by their relevance, which is manifested by Theorem 1.1, we prove in Section 4 a number of equivalences. Some of these require that $V \leqq 0$ or that V has compact support.

To be able to summarize the results we suppose both conditions in the theorem below. We emphasize, however, that most results do not require these restrictions ($H_0 \equiv -\frac{1}{2}\Delta$).

THEOREM 1.3. *Let $V \leqq 0$ be a measurable function of compact support on $\mathbb{R}^\nu$. The following are equivalent*:

(i) $V \in K_\nu$;

(ii) $\lim_{t\downarrow 0} \sup_x E_x(\int_0^t |V(b(s))|\, ds) = 0$;

(iii) $(-\Delta)^{-1}|V|$ *is a bounded map from L^∞ to the subspace of continuous functions (if $\nu = 1,2$ we replace $(-\Delta)^{-1}$ by $(-\Delta+1)^{-1}$)*;

(iv) $(-\Delta)^{-1}|V|$ *is a compact map from L^∞ to L^∞*;

(v) *as map on L^1, V is $-\Delta$-bounded, in the sense of Kato, with relative bound zero*;

(vi) $\exp\{-t(H_0+V)\}$ *is bounded from L^∞ to L^∞ and its norm satisfies* $\lim_{t\downarrow 0} \|\exp\{-t(H_0+V)\}\|_{\infty,\infty} = 1$;

(vii) *for some α and $\epsilon > 0$*, $(\phi, H_0\phi) + (1+\epsilon)(\phi, V\phi) \geqq -\alpha(\phi,\phi)$ *and $H = H_0 + V$ obeys the strong Harnack inequality.*

Moreover, if V is spherically symmetric and $\nu \geqq 3$, then (i)–(vii) *hold if and only if*

(viii) $\int_0^R r|V(r)|\, dr < \infty$, *where R is chosen so that V is supported in the sphere of radius R.*

There are also a number of simple conditions which imply that $V \in K_\nu$; note that (i) (below) is slightly weaker than the hypothesis that Trudinger uses in his discussion of Harnack's inequality (he requires $C(\epsilon) \leqq D\epsilon^{-m}$):

THEOREM 1.4. *Any of the following imply that V (no restriction on sign or support) lies in K_ν*:

(i) $(\phi|V|\phi) \leqq \epsilon(\phi, H_0\phi) + C(\epsilon)\|\phi\|^2$ *with* $C(\epsilon) \leqq D\exp\{B\epsilon^{-\alpha}\}$ *for some* $\alpha < 1$;

(ii) $V(x) = W(Tx)$ *with T a linear map from $\mathbb{R}^\nu$ onto $\mathbb{R}^\mu$ and $W \in k_\mu$*;

(iii) *for some $p > \frac{1}{2}\nu$ ($\geqq 1$, if $\nu = 1$)*,

$$\sup_y \int_{|x-y|\leqq 1} |V(x)|^p\, d^\nu x < \infty;$$

(iv) $\nu \geqq 3$ *and, for some* $a > 0$,

$$\int_a^\infty |\mu\{x \mid |V(x)| \geqq \lambda\}|^{2/\nu}\, d\lambda < \infty.$$

Notice that, by (ii) and (iii), if $\nu = \mu N$ and

$$V(x) = \Sigma a_{ij}|x_i - x_j|^{-\alpha}$$

for $x = (x_1, \cdots, x_n)$, $x_i \in \mathbb{R}^{\mu}$, with some $\alpha < 2$ (if $\mu = 1$, $\alpha < 1$), then $V \in K_{\nu}$. In particular, the Coulomb potentials which arise in atomic physics, can be accommodated.

In Sections 2 and 3, we require the following two results, which are proven at the end of Section 4.

THEOREM 1.5. *If u is a distributional solution of $(H_0 + V)u = 0$ in an open set Ω (in the sense that u, $Vu \in L^1_{\mathrm{loc}}$ and $\frac{1}{2}(-\Delta\phi, u) + (\phi, Vu) = 0$ for any $\phi \in C_0^{\infty}(\Omega)$), and if $V \in K_{\nu}^{\mathrm{loc}}$, then u is a continuous function.*

Remark. By an explicit example, it is easy to see that it need not be true that u is Hölder continuous of any order.

THEOREM 1.6. *Fix x_0 and $V \in K_{\nu}^{\mathrm{loc}}$. Then for any ϵ there is an R so that*

$$\sup_{x \in \Omega} E_x\left(\int_0^T |V(b(s))| ds \right) \leqq \epsilon \tag{1.22}$$

for all $\Omega \subset \{ y \mid |y - x_0| \leqq R \}$. R depends only on local K_{ν} norms of V.

In (1.22), T is the first exit time from Ω. If $\nu \geqq 3$, we can replace T by ∞, if we also replace $V(y)$ by $\tilde{V}(y) = V\chi_{\Omega}(y)$ with χ_{Ω} the indicator function of Ω.

We shall also need the following result which, given (1.9), is just the usual Harnack inequality. It can be proven, for example, by noting uniform upper and lower bounds on the Poisson kernel, as x varies in Ω' and y in $\partial\Omega$.

THEOREM 1.7. *Let Ω, Ω' be concentric open balls with $\overline{\Omega'} \subset \Omega$ and let $d\sigma(y)$ be the usual surface measure on $\partial\Omega$. Then, there exist constants $0 < C$, $D < \infty$ such that, for any $f \in L^1(\partial\Omega)$,*

$$\sup_{x \in \Omega'} |E_x(f(b(T)))| \leqq C \int |f(y)| d\sigma(y), \tag{1.23}$$

and, for $f \geqq 0$ in $L^1(\partial\Omega)$,

$$\inf_{x \in \Omega'} \left[E_x(f(b(T))) \right] \geqq D \int f(y) d\sigma(y). \tag{1.24}$$

In these expressions, T is the first exit time for Ω.

One final remark: in Section 3 it will be convenient (although not really necessary) to suppose that $\nu \geqq 3$. While this is a restriction on (1.13) and (1.14), it is no restriction on (1.3) and (1.4), since by Theorem 1.4 (ii) we can always add extra dimensions and take u and V independent of the extra coordinates!

2. The Poisson Kernel for the Schrödinger Equation

Our main goal in this section is to prove (1.12) for eigenfunctions. Since this justifies the study of the map M_V, we shall then develop the simplest properties of M_V, explicitly (1.15) and the bound

$$\sup_{x \in \Omega'} |M_V f(x)|^2 \leq \int_{\partial\Omega} |f(y)|^2 \, d\sigma(y). \tag{2.1}$$

The following result is so basic that we shall give it three proofs.

THEOREM 2.1. *Let $V \in K_\nu^{\text{loc}}$. Let u be a continuous function obeying*

$$Hu = 0 \tag{2.2}$$

in distributional sense. For any x_0 there exists $R > 0$ (depending on local norms of V) such that, for every ball Ω about x_0 of radius at most R, we have

$$\sup_{x \in \Omega} E_x\left(\exp\left\{-\int_0^T V(s)\, ds\right\}\right) < \infty \tag{2.3}$$

and, for any $x \in \Omega$,

$$u(x) = E_x\left(\exp\left\{-\int_0^T V(b(s))\, ds\right\} u(b(T))\right), \tag{2.4}$$

where T is the first exit time from Ω.

We shall first give a mixed, analytic-probabilistic, proof which does not require any extra hypotheses on u. Remarks about extensions to other Ω will be offered later. Recall that, by Theorem 1.5, any distributional solution of $Hu = 0$ is continuous, after a change on a set of measure zero.

Proof: By Theorem 1.6, we choose R so that

$$\sup_{x \in \Omega} E_x\left(\int_0^T |V(b(s))|\, ds\right) < 1, \tag{2.5}$$

and then use the proof of Khas'minskii's lemma (Theorem 1.2) to conclude (2.3). Moreover, this lemma also shows that

$$\lim_{n \to \infty} \sup_x E_x\left(\int_{0 < s_1 < \cdots < s_n < T} |V(b(s_1))| \cdots |V(b(s_n))|\, ds_1 \cdots ds_n\right) = 0. \tag{2.6}$$

Let H_0^Ω be $-\frac{1}{2}\Delta$ with Dirichlet boundary conditions on $\partial\Omega$. Then, it is easy to see (see Section 4) that $(H_0^\Omega)^{-1}Vu$ is continuous on $\bar{\Omega}$ and vanishes on $\partial\Omega$. Moreover, in distributional sense,

$$H_0\left[u + \left(H_0^\Omega\right)^{-1}(Vu)\right] = H_0u + Vu = 0.$$

Since $u + (H_0^\Omega)^{-1}(Vu)$ has boundary values $u \restriction \partial\Omega$, we see that, on $\bar{\Omega}$,

$$u = g - \left(H_0^\Omega\right)^{-1}(Vu),$$

where the function g is harmonic on Ω and continuous on $\partial\Omega$, with $g(y) = u(y)$ for $y \in \partial\Omega$. Using (1.9) for g and Lemma A.4.4 (in Appendix 4) for $(H_0^\Omega)^{-1}Vg$, we see that

$$u(x) = E_x(u(b(t))) - E_x\left(\int_0^T V(b(s))u(b(s))\,ds\right).$$

Iterating and using the Markov property (since u is bounded on $\bar{\Omega}$, (2.5) justifies taking conditional expectations), we obtain

$$u(x) = E_x\left(\sum_{n=0}^{N}(-1)^n\int_{0<s_1<\cdots<s_n<T} ds_1 \cdots ds_n \right.$$
$$\left. \times V(b(s_1)) \cdots V(b(s_n))u(b(T))\right) + R_N$$

with

$$R_N = (-1)^{N+1}E_x\left(\int_{0<s_1<\cdots<s_{N+1}<T} ds_1 \cdots ds_{N+1}\right.$$
$$\left. \times V(b(s_1)) \cdots V(b(s_{N+1}))u(b(s_{N+1}))\right).$$

By (2.6), $R_N \to 0$ as $N \to \infty$ and the proof of (2.6) shows that one can sum the exponential series to get (2.4).

First Alternate Proof of Theorem 2.1: For this proof, we suppose that u has an extension to all of R^ν obeying $Hu = 0$ with $u \in L^\infty$, and that $V \in K_\nu$. It then follows from the Feynman–Kac formula (1.8) and the Markov property of Brownian motion that

$$f(b,t) \equiv \exp\left\{-\int_0^t V(b(s))\,ds\right\} u(b(t))$$

is a martingale. Moreover, it is continuous in t for a.e. b since u is continuous (by assumption) and since $\int_0^t |V(b(s))|\,ds < \infty$ for a.e. b. $V \in K_\nu$ implies (see Section 4) that, for each fixed n,

$$E_x\left(\exp\left\{+\int_0^n |V(b(s))|\,ds\right\}\right) < \infty. \tag{2.7}$$

Therefore, if $T_n \equiv \min(n, T)$,

$$u(x) = E_x\left(\exp\left\{-\int_0^{T_n} V(b(s))\,ds\right\} u(b(T_n))\right) \tag{2.8}$$

by the standard result on evaluating martingales at a stopping time; (2.7) is needed to pass from discrete stopping times to a continuous stopping time *via* a dominated convenience theorem.

Equation (2.8) holds without any restriction on the size of Ω. Some restriction is needed to assure that we can replace T_n by T (see Remark 1 below); with the restriction (2.5) and Khas'minskii's lemma, we have that

$$E_x\left(\exp\left\{\int_0^T |V(b(s))|\,ds\right\}\right) < \infty,$$

so that the limit $T_n \to T$ can be justified by dominated convergence.

Remarks 1. The following could almost be a textbook example of the dangers of evaluating unbounded martingales at a stopping time: let $\nu = 1$, $\Omega = (0, 1)$ and $V = -\frac{1}{2}\pi^2$, so that $u(x) = \sin(\pi x)$ obeys $Hu = 0$. Then (2.8) holds, since $u(b(T)) = 0$ for a.e. b and $\exp\{-\int_0^T V(b(s))ds\} < \infty$ a.e. b (since $T < \infty$ a.e. b). Nevertheless (2.4) fails. The problem is, of course, that

$$E_x\left(\exp\left\{-\int_0^n V(b(s))\,ds\right\} u(b(n));\, T \geqq n\right) \tag{2.9}$$

does not go to zero as $T \to \infty$.

2. One can use the argument in the proof of Lemma A.4.2, in place of Khas'minskii's lemma, to control (2.9). Doing this, one sees that (1.11) is really

the only restriction needed on Ω for (2.4) to hold. As we shall mention shortly, (1.11) holds automatically if Ω is shrunk.

Second Alternate Proof of Theorem 2.1: This proof uses the fact that we have solved the Dirichlet problem in Appendix 4. Since $V \in K_\nu^{\text{loc}}$, $H - \frac{1}{2}H_0 = \frac{1}{2}H_0 + V\chi_1$ is bounded below as an operator if χ_1 is the indicator function of the unit ball, Ω', about x_0. Thus, for $\phi \in C_0^\infty(\Omega)$ and $\Omega \subset \Omega'$,

$$\int (\phi, H\phi)\, d^\nu x \geqq \frac{1}{4}\int |\nabla\phi|^2\, d^\nu x - a\int |\phi|^2\, d^\nu x.$$

When the ball Ω is shrunk, the lowest eigenvalue of H_0 behaves like $c/\text{diam}(\Omega)^2$. We see therefore that $H \geqq \alpha > 0$ for small enough balls. Hence, by Theorem A.4.1, the right-hand side of (2.4), call it $f(x)$, is well defined. Furthermore, $u - f$ obeys: (i) $u - f \in C(\bar{\Omega})$, (ii) $H(u - f) = 0$, and (iii) $u - f = 0$ on $\partial\Omega$.

Since $\Delta(u - f) = V(u - f) \in L^1(\bar{\Omega})$ (by $u - f \in L^\infty(\bar{\Omega})$, $K_\nu^{\text{loc}} \subset L^1_{\text{loc}}$), we have that $\nabla(u - f) \in L^2(\bar{\Omega})$ (see, e.g., [33]). An elementary argument shows that if $g \in C(\bar{\Omega})$, $\nabla g \in L^2(\Omega)$ and g vanishes on $\partial\Omega$, then g is in $Q(H_0)$, the form domain of H_0; and if further $Hg = h \in L^2$ (distributional sense), then $(g, Hg) = (g, h)$ as the Dirichlet form. Since $H \geqq \alpha > 0$, we have a contradiction unless $u - f = 0$. This proves (2.4).

The last proof requires only $H \geqq \alpha$ on $L^2(\bar{\Omega})$, so it shows that R may only depend on K_ν^{loc} norms of $V_- = \max(0, -V)$. This remark will be used below. We now define a map M_V^Ω from $L^\infty(\partial\Omega)$ to $L^\infty(\Omega)$ by:

$$(M_V^\Omega f)(x) \equiv E_x\left(\exp\left\{-\int_0^T V(b(s))\,ds\right\} f(b(T))\right), \tag{2.10}$$

defined if Ω is a sufficiently small ball. Writing

$$|f| \leqq \left[\exp\left\{-\frac{1}{2}\int V\right\}|f|^{1/2}\right]\left[\exp\left\{+\frac{1}{2}\int V\right\}|f|^{1/2}\right]$$

and using the Schwarz inequality, we immediately obtain

PROPOSITION 2.2. $|(M_{V=0}^\Omega f)(x)|^2 \leqq [(M_V^\Omega|f|)(x)]^{1/2}[(M_{-V}^\Omega|f|)(x)]^{1/2}$.

From this and (1.24), we conclude

THEOREM 2.3. *Let Ω', Ω be concentric open balls with $\bar{\Omega}' \subset \Omega$. Let $V \in K_\nu^{\text{loc}}$ and suppose that, for some constant C,*

$$\sup_{x \in \Omega'} |(M_{-V}^\Omega f)(x)| \leqq C\int_{\partial\Omega} |f(y)|\, d\sigma(y).$$

Then, for some constant D and all $f \geqq 0$,

$$\inf_{x \in \Omega'} (M_V^\Omega f)(x) \geqq D \int_{\partial\Omega} f(y)\, d\sigma(y).$$

The other results which are immediate for Mf are analogous to Step 2 in the proof of (1.16):

THEOREM 2.4. *Suppose that $a \equiv \sup_{x \in \Omega}[(M_{2V}^\Omega 1)(x)] < \infty$ (which by Khas'minskii's lemma and Theorem 1.6 is true if we shrink Ω enough); then for any concentric ball $\Omega' \subset \Omega$ with $\bar{\Omega}' \subset \Omega$ we have*

$$\sup_{x \in \Omega'} |(M_V^\Omega f)(x)| \leqq C\left[\int_{\partial\Omega} |f(y)|^2 d\sigma(y)\right]^{1/2}, \tag{2.11}$$

where C depends only on a, and the geometric relation of Ω' and Ω.

Proof: By the Schwarz inequality,

$$|(M_V^\Omega f)(x)| \leqq \left[(M_{2V}^\Omega 1)(x)\right]^{1/2}\left[(M_{V=0}^\Omega |f|^2)^{1/2}(x)\right].$$

Now use the assumption $a < \infty$ and (1.23).

Averaging over the radius of Ω and using the remark that the truth of (2.4) only depends on local norms of V_- we have

COROLLARY 2.5. *Let $V \in K_\nu^{\text{loc}}$. Then, for every x_0 and every continuous distributional solution of $Hu = 0$ in $\{y \,|\, |y - x_0| < 1\}$, we have*

$$|u(x_0)| \leqq C_{x_0}\left[\int_{|y - x_0| \leqq 1} |u(y)|^2 d^\nu y\right]^{1/2}. \tag{2.12}$$

The constant C_{x_0} is bounded as x_0 runs through compact sets. If $V_- \in K_\nu$, then C_{x_0} can be replaced by a constant which is independent of x_0.

It is (2.12) that is used in [1]. By using Hölder's inequality in place of the Schwarz inequality, we can obtain (2.12) with 2 replaced by any fixed $p > 1$ but C may *a priori* diverge for $p > 1$ (or worse, the radius of Ω in (2.11) may go to zero). The goal of getting (2.11) with $p = 1$ will require the considerations of the next section.

3. The Exit Reversal Process

In this section, we want to complete the proof of (1.13). The idea will be to prove that

$$\left[\int_{\Omega'} |M_V^\Omega f(x)|^2 \, dx\right]^{1/2} \leqq C \int_{\partial\Omega} |f(y)| \, d\sigma(y) \tag{3.1}$$

and combine this with (2.10) and a "semigroup" property to show that (1.13) holds. Relation (3.1) will be proven by studying the adjoint of M_V^Ω. Of course to study an adjoint, we need to choose a measure on Ω. The choice of Lebesque measure is not appropriate since we clearly want to give little weight to points near $\partial\Omega$.

Obviously, since M_V involves running paths up to a stopping time, the adjoint must involve running paths backwards from a stopping time. Exactly how they run backwards will depend on what initial distribution we give to the Brownian paths; this choice is essentially equivalent to that needed to define an adjoint.

Of course, the most natural choice of initial distribution would be an invariant one but the whole point of exiting is that there is no such distribution. Given this, the most natural choice is then one that will give an invariance for the distribution conditional on not having exited. This picks out the choice we shall take and as a bonus it will follow (essentially automatically) that with this choice the exit distribution and exit time become independent!

Let Ω be an arbitrary bounded open set and let H_0^Ω be one-half the Dirichlet Laplacian on Ω. We denote by α its lowest eigenvalue, and by $\psi(x)$, $x \in \Omega$, the corresponding eigenfunction

$$H_0^\Omega \psi = \alpha\psi, \tag{3.2}$$

normalized by

$$\int_\Omega \psi(x) \, dx = 1. \tag{3.3}$$

(Note: not ψ^2). The eigenfunction ψ is unique, and positive. We first point out

THEOREM 3.1. *Let E denote the probability distribution for Brownian motion with initial distribution $\psi(x)\,d^\nu x$. Let T be the first exit time from Ω. Then T and $b(T)$ are independent random variables and the distribution of T is*

$$\alpha e^{-\alpha s} \, ds. \tag{3.4}$$

Proof: We begin by computing

$$E(T \geqq t;\, b(t) \in A) = \int E_x(T \geqq t;\, b(t) \in A)\psi(x)\, d^\nu x.$$

This is precisely (see [31])

$$\left(\psi, \exp\{-tH_0^\Omega\}\chi_A\right) = \left(\exp\{-tH_0^\Omega\}\psi, \chi_A\right) = e^{-t\alpha}\int_A \psi(x)\, d^\nu x$$

with χ_A the indicator function of A; thus the conditional distribution of $b(t)$ subject to $T \geqq t$ is exactly $\psi(x)\, d^\nu x$. It follows by the Markov property that

$$E(f(b(T));\, T \geqq t)$$

is independent of t, so that the claimed independence is proven. Moreover, taking $A = \Omega$ in the above equation, we find that

$$P(T \geqq t) = e^{-\alpha t},$$

from which (3.4) immediately follows.

Remark. A similar argument shows that the probability distribution for $\{b(T-s)\}_{0 \leq s \leq t}$ conditional on $T \geqq t$ is independent of T. This fact should be remembered in thinking about the constructions below.

For the case where Ω is a ball, the E-distribution of $b(T)$ is obviously $d\sigma(y)$ by symmetry. The formula for more general Ω is discussed extensively in Appendix 3. If $y_0 \in \partial\Omega$ and, near y_0, $\partial\Omega$ is a smoothly embedded submanifold of $\mathbb{R}^\nu$, then the E-distribution of the exit place for $b(T)$ is

$$(3.5) \qquad (2\alpha)^{-1}\frac{\partial\psi}{\partial n}\, d\sigma(y)$$

for y near y_0. Here n is an *inward* pointing normal and $d\sigma$ the usual surface area. When $\partial\Omega$ is everywhere smooth, the normalization condition

$$(3.6) \qquad \int (2\alpha)^{-1}\frac{\partial\psi}{\partial n}\, d\sigma(y) = 1$$

follows from

$$(3.7) \qquad \int_{\partial\Omega}\frac{\partial\psi}{\partial n}\, d\sigma(y) = \int_\Omega (2H_0^\Omega)\psi\, d^\nu x.$$

One way of understanding these formulas is in terms of the distributional equation (implied by (3.2) and Green's equality)

$$H_0[\psi d^\nu x] = \alpha[\psi d^\nu x] - \frac{1}{2}\frac{\partial\psi}{\partial n}\, d\sigma(y).$$

From now on, we restrict ourselves to the case where Ω is a ball, since that is all we need.

Define, for $x, y \in \Omega$,

$$Q_s(x, y) = \psi(x)^{-1}\exp\{-sH_0^\Omega\}(x, y)\psi(y), \tag{3.8}$$

where $e^{-sH}(x, y)$ is the integral kernel for e^{-sH}.

We shall prove the following technical result in Appendix 2:

THEOREM 3.2 ($\equiv$ Theorems A.2.3, A.2.4, A.2.8). *Let Ω be an open ball in $\mathbb{R}^\nu$. Let Q_s be defined by* (3.8), *on $\Omega \times \Omega \times (0, \infty)$. Then Q_s extends to a continuous function on $\bar\Omega \times \bar\Omega \times (0, \infty)$* (*which we still denote by Q_s*) *obeying*

$$Q_s(x, y) = 0 \quad \textit{for all} \quad y \in \partial\Omega, x \in \bar\Omega, \tag{3.9}$$

$$\int_\Omega Q_s(x, y)\, dy = e^{-\alpha s} \quad \textit{for all} \quad x \in \bar\Omega. \tag{3.10}$$

Furthermore, for any $\epsilon > 0$,

$$Q_s(x, y) \leqq c_\epsilon P_{(1+\epsilon)s}(x - y) \tag{3.11}$$

with some $c_\epsilon < \infty$ which is independent of Ω: P_s being the integral kernel of $\exp\{\frac{1}{2}s\Delta\}$ *on all of $\mathbb{R}^\nu$, i.e.,* $P_s(z) = (2\pi s)^{-\nu/2}\exp\{-z^2/2s\}$.

Actually, in Appendix 2 we prove the key estimate (3.11) for fairly general Ω, we do not try to prove the more technical continuity result in such generality.

Let now $\tilde Q_s(x, y) = e^{\alpha s}Q_s(x, y)$. By (3.10) and the obvious semigroup property of $\tilde Q_s$, we can define, for each $y \in \bar\Omega$, a probability measure $\tilde E_y$ on paths $\{q(s)\}_{0 \leqq s \leqq t}$ so that $q(0) = y$ and, for $0 < s_1 < \cdots < s_n$, the joint probability distribution of $q(s_1) = y_1, \cdots, q(s_n) = y_n$ is

$$\tilde Q_{s_1}(y, y_1)\,\tilde Q_{s_2 - s_1}(y_1, y_2) \cdots \tilde Q_{s_n - s_{n-1}}(y_{n-1}, y_n)\, dy_1 \cdots dy_n.$$

From the estimate (3.11) and Kolmogorov's lemma (see [31], Theorem 5.1), it easily follows that paths can be realized as continuous functions and $\{\tilde E_y\}_{y \in \bar\Omega}$ is a Markov process. By (3.9), $q(s) \in \Omega$ for all $s > 0$.

Now introduce a random variable $\hat{T}$ with the distribution $\alpha e^{-\alpha s}\,ds$. $\hat{T}$ will be chosen independently of q. Notice that the joint probability distribution with respect to $\tilde{E}_y$ of $q(s_1), \cdots, q(s_n)$ *and* $\hat{T} \geqq s_n$ is

$$Q_{s_1}(y, y_1) \cdots Q_{s_n - s_{n-1}}(y_{n-1}, y_n)\,dy_1 \cdots dy_n.$$

By $\hat{E}$ we shall denote the expectation of the Markov process $\{\tilde{E}_y\}$ with the initial distribution $d\sigma(y)$—the normalized surface measure on $\partial\Omega$, and an independent variable $\hat{T}$. The basic duality result is:

THEOREM 3.3. *The E-probability distribution of T and $\{b(T-s)\}_{0 \leqq s \leqq T}$ is identical to the $\hat{E}$ distribution of $\hat{T}$ and $\{q(s)\}_{0 \leqq s \leqq \hat{T}}$.*

Before proving this theorem, we want to note several things. First, the fact that we took a ball for Ω obscures somewhat the general formula for the correct initial distribution for $q(0)$. Obviously, we must take the distribution of $b(T)$, i.e., $(2\alpha)^{-1}(\partial\psi(y)/\partial n_y)\,d\sigma(y)$ for general "nice" Ω.

Secondly, we note that there is a very close relation between $\hat{E}$ and h-processes of Doob [11]. In place of $Q = \psi^{-1}P\psi$, Doob considers $h^{-1}\,Ph$ with h an excessive function; ψ is not an excessive function but it is close enough for considerable formal connections. Indeed, various authors (see [19], [21]) have discussed reversal of processes at an exit time in terms of h-processes.

Proof of Theorem 3.3: Since T and $\hat{T}$ have identical distributions, it suffices to fix $0 \leqq s_1 \leqq \cdots \leqq s_n$ and prove the equality of the distributions of $\{q(s_1), \cdots, q(s_n)\}$, conditioned on $\hat{T} \geqq s_n$, and of $\{b(T-s_1), \cdots, b(T-s_n)\}$, conditioned on $T \geqq s_n$. Let F be a continuous function on Ω^n and fix $s \geqq s_n$ and $\gamma > 0$. We claim that

$$\begin{aligned} &E(F(b(s-s_n), \cdots, b(s-s_1)); s \leqq T \leqq s+\gamma) \\ &\quad = \int \tilde{E}_y(F(q(s_n), \cdots, q(s_1))(\gamma\alpha e^{-\alpha s}))\,d\mu_\gamma(y), \end{aligned} \tag{3.12}$$

where

$$d\mu_\gamma(y) = (\gamma\alpha)^{-1}P_y(0 \leqq T \leqq \gamma)\psi(y)\,d^\nu y, \tag{3.13}$$

since the left side of (3.13) is, by the Markov property for Brownian motion (P_s being the kernel of $\exp\{-sH_0^\Omega\}$),

$$\begin{aligned} &\int F(x_1, \cdots, x_n)\psi(x_0)P^\Omega_{(s-s_n)}(x_0, x_1)P^\Omega_{(s_n - s_{n-1})}(x_1, x_2) \cdots P^\Omega_{s_1}(x_n, y) \\ &\quad \times P_y(0 \leqq T \leqq \gamma)\,dx_0 \cdots dx_n\,dy. \end{aligned}$$

If we compute the x_0-integral and replace P by $\tilde{Q}$, we find this is equal to

$$\int F(x_1, \cdots, x_n)\tilde{Q}_{s_n - s_{n-1}}(x_1, x_2) \cdots \tilde{Q}_{s_1}(x_n, y)\psi(y)P_y(0 \leqq T \leqq \gamma)\, dx_1 \cdots dx_n\, dy$$

which is the right side of (3.12).

Note that $d\mu_\gamma$ is a measure of total weight

$$\int d\mu_\gamma(y) = (\gamma\alpha)^{-1}E(0 \leqq 2T \leqq \gamma) = (\gamma\alpha)^{-1}\int_0^\gamma \alpha e^{-\alpha s}\, ds$$

which goes to 1 as $\gamma \to 0$. Moreover, $d\mu_\gamma$ is clearly rotation invariant, and, if $f \in C_0^\infty(\Omega)$, $\int f(y)\, d\mu_\gamma(y) \to 0$ as $\gamma \to 0$. Thus,

$$w^* - \lim_{\gamma \downarrow 0} d\mu_\gamma = d\sigma(y). \tag{3.14}$$

Now fix $m = 1, \cdots,$ let $\gamma_m = s_n/m$ and define $T_m = \gamma_m[\gamma_m^{-1}T]$, $[x]$ being the integral part of x_0. Then,

$$\begin{aligned} &E(F(b(T_m - s_n), \cdots, b(T_m - s_1)); T \geqq s_n) \\ &= \sum_{j=0}^{\infty} E(F(b(s_n + j\gamma_m - s_n), \cdots, b(s_n + j\gamma_m - s_1)); \\ &\qquad s_n + j\gamma_m \leqq T \leqq s_n + (j+1)\gamma_m). \end{aligned} \tag{3.15}$$

Using (3.12), we see that the right-hand side of (3.15) is equal to

$$\exp\{-\alpha s_n\}a_m \int \tilde{E}_y(F(q(s_n), \cdots, q(s_1)))\, d\mu_{\gamma_m}(y), \tag{3.16}$$

where

$$a_m = \sum_{j=0}^{\infty} \gamma_m \alpha \exp\{-j\alpha\gamma_m\} = \gamma_m\alpha/(1 - \exp\{-\alpha\gamma_m\}) \underset{m\to\infty}{\to} 1.$$

Taking $m \to \infty$, (3.16) converges to

$$\hat{E}(F(q(s_n), \cdots, q(s_1)); \hat{T} \geqq s_n),$$

by (3.14) and the continuity of the q-paths. The left side of (3.15) converges to

$$E(F(b(T - s_n), \cdots, b(T - s_1)); T \geqq s_n).$$

This completes the proof.

Let N be a map from functions on Ω to functions on $\bar{\Omega}$ defined by

(3.17) $$(N_V^\Omega f)(y) = \tilde{E}_y\left(\exp\left\{-\int_0^{\hat{T}} V(q(s))\,ds\right\} f(q(\hat{T}))\right).$$

Below, we shall worry about when N is well defined. For the time being, we take f positive so that $(Nf)(y)$ is obviously defined although it might be infinite. Clearly the above theorem implies that N and M are adjoints:

COROLLARY 3.4. *Let Ω be a ball. For any positive g on $\partial\Omega$ and f on Ω,*

(3.18) $$\int_{\partial\Omega} d\sigma(y)(N_V^\Omega f)(y)g(y) = \int_\Omega f(x)(M_V^\Omega g)(x)\psi(x)\,d^\nu x.$$

Proof: One uses a change of variables from s to $T - s$.

Remark. We emphasize again that for general nice Ω, $d\sigma(y)$ should be replaced by $(2\alpha)^{-1}(\partial\psi/\partial n_y)\,d\sigma(y)$.

Next we make the first step in a Khas'minskii analysis for N:

PROPOSITION 3.5. *Let $\tilde{V}(y) = (V\chi_\Omega)(y)$, χ_Ω being the indicator function for Ω. Then for some constant D, independent of Ω and V,*

(3.19) $$\sup_{y\in\bar{\Omega}} \tilde{E}_y\left(\int_0^{\hat{T}} |V(q(s))|\,ds\right) \leqq D \sup_{x\in R^\nu} E_x\left(\int_0^\infty |\tilde{V}(b(s))|\,ds\right).$$

Proof: Clearly, since q and $\hat{T}$ are independent,

$$\begin{aligned}
\tilde{E}_y\left(\int_0^{\hat{T}} |V(q(s))|\,ds\right) &= \int_0^\infty \tilde{E}_y(|V(q(s))|)P(\hat{T} \geqq s)\,ds \\
&= \int_0^\infty \int_{x\in\Omega} e^{-\alpha s}\tilde{Q}_s(y,x)|V(x)|\,dx\,ds \\
&= \int_0^\infty \int_{x\in\Omega} Q_s(y,x)|V(x)|\,dx\,ds \\
&< c_\epsilon \int_0^\infty \int_{x\in\Omega} P_{(1+\epsilon)s}(y,x)|V(x)|\,dx \\
&= (1+\epsilon)^{-1}c_\epsilon E_y\left(\int_0^\infty |\tilde{V}(b(s))|\,ds\right),
\end{aligned}$$

where we used (3.11).

$\tilde{T}$ is in a rather trivial way a stopping time; therefore, as we noted in Section 1, Khas'minskii's lemma holds. Using this lemma, the last proposition, and Theorem 1.6, we obtain

PROPOSITION 3.6. *Let $\nu \geqq 3$ and $V \in K_\nu^{\mathrm{loc}}$. Then, for any x_0, there is a small ball Ω about x_0 with*

$$\sup_{x \in \bar{\Omega}} |(N_{2V}^{\Omega} f)(x)| \leqq C \sup_{x \in \Omega} |f(x)|. \tag{3.20}$$

Remark. With a little more work, one can prove this result for $\nu = 1, 2$ as follows: one actually proves that (3.11) is true with an extra factor of $e^{-\beta s}$ on the right side; β is dependent on Ω but behaves like $[\mathrm{diam}(\Omega)]^{-2}$, by scaling. With this change, we obtain (3.19) with an extra $e^{-\beta s}$ on the right side. This is small if Ω is shrunk even if $\nu = 1, 2$.

We are now ready to prove the analogue of (1.21).

THEOREM 3.7. *Let* (3.20) *hold. Then for any Ω', a ball which is concentric with Ω, $\bar{\Omega}' \subset \Omega$, we have*

$$\left[\int_{\Omega'} |(M_V^{\Omega} g)(x)|^2 d^\nu x \right]^{1/2} \leqq C_{\Omega', \Omega, V} \int_{\partial\Omega} |g(y)| \, d\sigma(y). \tag{3.21}$$

Proof: Since ψ is bounded below on $\bar{\Omega}'$ by a strictly positive constant, we need only prove that

$$\left[\int_{\Omega'} |(M_V^{\Omega} g)(x)|^2 \psi(x) \, d^\nu x \right]^{1/2} \leqq C \int_{\partial\Omega} |g(y)| \, d\sigma(y).$$

By (3.18) this would follow from a result of the form

$$\sup_{y \in \partial\Omega} |(N_V^{\Omega} f)(y)| \leqq C \left[\int_{\Omega'} |f(y)|^2 \psi(y) \, d^\nu y \right]^{1/2}$$

for all f supported in Ω'. Again, since ψ is bounded below, we need only prove that

$$\sup_{y \in \partial\Omega} |(N_V^{\Omega} f)(y)| \leqq C \left[\int_{\Omega'} |f(y)|^2 d^\nu y \right]^{1/2}.$$

However, the Schwarz inequality in $\tilde{E}_y$ implies that

$$|(N_V^{\Omega} f)(y)|^2 \leqq |(N_{2V}^{\Omega} 1)(y)|^{1/2} \left[(N_{V=0}^{\Omega} |f|^2)(y) \right]^{1/2}.$$

Combined with (3.20) this reduces the problem to showing that

$$\sup_{y \in \partial\Omega} |(N_{V=0}^{\Omega} h)(y)| \leqq C \int_{\Omega'} h(y)\, d^{\nu} y$$

for $h \geqq 0$. Since ψ is bounded from above, we only need to show that

$$\sup_{y \in \Omega} \left|(N_{V=0}^{\Omega} h)(y)\right| \leqq C \int_{\Omega'} h(y) \psi(y)\, d^{\nu} y.$$

Using the duality (3.18) again, we see that this is equivalent to

$$\sup_{x \in \Omega'} |(M_{V=0}^{\Omega} g)(x)| \leqq C \int_{\partial\Omega} |g(y)|\, d\sigma(y)$$

which is (1.23).

The following is in many ways the first of the two main results of this paper ($\nu = 1, 2$ can be accommodated by using the remark after Proposition 3.6):

THEOREM 3.8. *Let $\nu \geqq 3$, $V \in K_{\nu}^{\mathrm{loc}}$. Then for any x_0, there are balls $\bar{\Omega}' \subset \Omega$ about x_0 (depending only on local norms of V) such that, for $f \in L^1(\partial\Omega; d\sigma)$,*

$$\sup_{x \in \Omega'} |M_V^{\Omega} f(x)| \leqq C \int |f(y)|\, d\sigma(y).$$

Proof: Let A be a closed union of spheres about x_0 with $\bar{\Omega}' \cap A = \emptyset$, $A \subset \Omega$ (see Figure 1). Let S_r be a sphere about x_0 contained in A and let Ω_r'' be the ball whose boundary is S_r. By the strong Markov property of Brownian motion, for $x \in \Omega'$,

$$(M_V^{\Omega''} g)(x) = (M_V^{\Omega} f)(x)$$

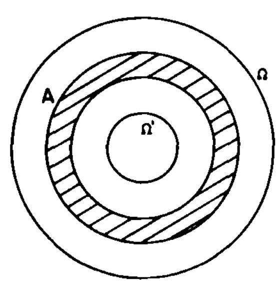

Figure 1. A reference for the proof of Theorem 3.8.

if $g = M_{V}^{\Omega}f \upharpoonright S_r$. Thus, averaging in r and using Theorem 2.4, we see that by shrinking Ω we can be sure that

$$\sup_{x \in \Omega'} |(M_{V}^{\Omega}f)(x)| \leqq C\left[\int_A |(M_{V}^{\Omega}f)(y)|^2 d^{\nu}y\right]^{1/2}.$$

By Theorem 3.7, the last quantity is bounded by $C\int_{\partial\Omega}|f(y)|\, d\sigma(y)$.

Given Theorem 2.1, we conclude:

COROLLARY 3.9. *Let $V \in K_{\nu}^{\text{loc}}$. If u is a continuous function obeying $Hu = 0$ in a neighborhood of $\{y \mid |y - x_0| \leqq 1\}$, then*

$$|u(x_0)| \leqq C\int_{|x_0 - y| \leqq 1} |u(y)|\, d^{\nu}y,$$

where C depends only on local norms of V_-. In particular if $V_- \in K_{\nu}$, C may be chosen independently of x_0.

Finally, given Theorem 2.3 and a very elementary covering argument, we have proven *Harnack's inequality*—the second main result of this paper.

THEOREM 3.10. *Let $V \in K_{\nu}^{\text{loc}}$. For each pair of a compact set K and an open set Ω, $K \subset \Omega$, there exists a constant C (depending only on K, Ω and local norms of V) with the property that, for any function $u \in C(\Omega)$, which satisfies $u \geqq 0$ and $Hu = 0$ on Ω, one has*

$$\sup_{x \in K} u(x) \leqq C \inf_{x \in K} u(x).$$

4. The Spaces K_{ν} and K_{ν}^{loc}

In this section we study the classes K_{ν} and K_{ν}^{loc} defined as follows.

DEFINITION. Let $\nu \geqq 2$. $V \in K_{\nu}$ if and only if

$$\lim_{\alpha \downarrow 0} \sup_{x} \int_{|x-y| \leqq \alpha} [g(x-y)]\,|V(y)|\, d^{\nu}y = 0, \tag{4.1}$$

where

$$g(z) = |z|^{-(\nu-2)} \quad \text{if} \quad \nu \geqq 3, \tag{4.2a}$$

$$g(z) = -\ln(|z|) \quad \text{if} \quad \nu = 2. \tag{4.2b}$$

We say $V \in K_\nu^{\text{loc}}$ if and only if $V\phi \in K_\nu^{\text{loc}}$ for all $\phi \in C_0^\infty$. Obviously this is equivalent to requiring that (4.1) holds when $\sup_x$ is replaced by $\sup_{|x| \leqq R}$ for each fixed R. For $\nu = 1$, we say that $V \in K_1$ if and only if

$$\sup_x \int_{|x-y| \leqq 1} |V(y)|\, d^\nu y < \infty, \tag{4.3}$$

i.e., $K_1 = L^1_{\text{unif}}(\mathbb{R})$, and $K_1^{\text{loc}} = L^1_{\text{loc}}(\mathbb{R})$.

By the monotone convergence theorem, if

$$\int_{|x-y| \leqq \alpha_0} [g(x-y)]|V(y)|\, d^\nu y < \infty,$$

then

$$\lim_{\alpha \downarrow 0} \int_{|x-y| \leqq \alpha} [g(x-y)]|V(y)|\, d^\nu y = 0.$$

Nevertheless, there are pathological examples (see Appendix 1) where $V \notin K_\nu$ even though V has compact support and

$$\sup_x \int_{|x-y| \leqq 1} g(x-y)|V(y)|\, d^\nu y < \infty.$$

As we shall try to demonstrate, these classes are exceedingly natural for the problems studied here; even more so than in the context in which they were introduced by Kato [16]. Related classes were studied by Schechter [26], who defines the class $M_{\beta,p}$ by the requirement

$$\sup_x \int_{|x-y| \leqq 1} |x-y|^{\beta-\nu} |V(y)|^p\, d^\nu y < \infty,$$

(for $p = 2$ the condition was introduced by Stummel [35]). If one examines his definitions, one finds that Schechter on page 155 of [26] defines $-\Delta + q$ as a form-sum when $\min(q, 0) \in K_\nu$ but, as he subsequently realized (see [27]), this is not a natural condition for the problem of form-sums. Obviously we have

PROPOSITION 4.1. *If $\beta > 2$ and $\nu \geqq 3$, then $M_{\beta,1} \subset K_\nu$.*

It is a direct consequence of Hölder's inequalities that $M_{\alpha,p} \subset M_{\beta,1}$ so long as $\alpha < p\beta$, and thus one can state

PROPOSITION 4.2. *If $\alpha > 2p$ and $\nu \geqq 3$, then $M_{\alpha,p} \subset K_\nu$. In particular, $M_{\alpha,2} \subset K_\nu$ if $\alpha > 4$.*

The classes $M_{\alpha,2}$ with $\alpha > 4$ are precisely the classes introduced by Stummel [35]. By another application of Hölder's inequality, we have

$$\int_{|x-y|\leqq\alpha} g(x-y)V(y)\,d^\nu y \leqq \left[\int_{|x-y|\leqq\alpha} |V(y)|^{1/p}\,d^\nu y\right]^{1/p} f(\alpha)$$

with

$$f(\alpha) = \left[\int_{|z|<\alpha} |g(z)|^q\,dz\right]^{1/q},$$

where q is the dual index to p. If $g \in L^q$, $\lim_{\alpha\downarrow 0} f(\alpha) = 0$. Thus we obtain the following condition:

PROPOSITION 4.3. *If $p > \frac{1}{2}\nu$, $\nu \geqq 2$, and if*

$$\sup_x \int_{|x-y|\leqq 1} |V(y)|^p\,d^\nu y < \infty,$$

then $V \in K_\nu$. If $V \in L^p_{\text{loc}}$, then $V \in K_\nu^{\text{loc}}$.

The following elementary fact will occasionally be useful:

LEMMA 4.4. *If $V \in K_\nu$, then*

$$\sup_x \left[\int_{|x-y|\leqq 1} |V(y)|\,d^\nu y\right] < \infty.$$

Proof: The bound is an immediate consequence of the fact that $\inf_{|z|\leqq 1/2} g(z) > 0$.

Next we want to link K_ν to the kind of condition needed for Khas'minskii's lemma:

THEOREM 4.5. *$V \in K_\nu$ if and only if*

$$\lim_{t\downarrow 0} \sup_x E_x\left(\int_0^t |V(b(s))|\,ds\right) = 0. \tag{4.4}$$

Proof: Suppose first that $V \in K_\nu$ and $\nu \geqq 2$. Fix α and note that, for $t \leqq \alpha^2/\nu$,

$$\int_0^t E_x(|V(b(s))|; |b(s)-x| \geqq \alpha)\,ds$$

$$\leqq t\int_{|y|>\alpha} (2\pi t)^{-\nu/2} \exp\{-y^2/2t\}|V(x-y)|\,dy$$

which tends to zero as $t\downarrow 0$, by Lemma 4.4. On the other hand,

$$\begin{aligned}\int_0^t E_x(V(b(s)); |b(s)-x| \leqq \alpha)\, ds &\leqq e^t \int_0^\infty E_x(e^{-s}V(b(s)); |b(s)-x| \leqq \alpha)\, ds \\ &\leqq e^t \int_{|x-y|\leqq\alpha} f(x-y)|V(y)|\, dy,\end{aligned}$$

where f is the integral kernel of $(-\frac{1}{2}\Delta + 1)^{-1}$. Since $|f(z)| \leqq (\text{const})\, g(z)$ in the region $|z| \leqq \frac{1}{2}$ (a restriction needed if $\nu = 2$), this term goes to zero as $\alpha\downarrow 0$. These two estimates imply (4.4).

For $\nu = 1$, let $V \in K_1$,

$$\lim_{t\downarrow 0} \sup_x \int_0^t E_x(|V(b(s))|; |b(s)-x| \geqq 1)\, ds = 0.$$

Moreover,

$$\sup_x \int_0^t E_x(|V(b(s))|; |b(s)-x| \leqq 1)\, ds \leqq \left[\int_0^t (2\pi s)^{-1/2}\, ds\right] \sup_x \int_{|x-y|\leqq 1} |V(y)|\, dy$$

which vanishes as $t\downarrow 0$.

To prove the converse, let $P_s(x-y)$ be the integral kernel of $\exp\{-sH_0\}$ and suppose that $\nu \geqq 3$. We claim that for $|x-y| \leqq t^{1/2}$ we have

$$\int_0^t P_s(x-y)\, ds \geqq (\text{const})\, g(x-y). \tag{4.5}$$

Postponing the proof of (4.5), we note that it implies that, for α sufficiently small,

$$\sup_x \int_{|x-y|\leqq\alpha} g(x-y)|V(y)|\, dy \leqq (\text{const}) \sup_x E_x\left(\int_0^{\alpha^2} |V(b(s))|\, ds\right),$$

thus (4.4) implies $V \in K_\nu$.

Since $P_s(x-y) = (2\pi s)^{-\nu/2} \exp\{-|x-y|^2/2s\}$, we see that

$$\begin{aligned}\int_0^t P_s(x-y)\, ds &= |x-y|^{-(\nu-2)} \int_0^{t/|x-y|^2} \exp\{-y^{-1/2}\}(2\pi y)^{-\nu/2}\, dy \\ &\geqq |x-y|^{-(\nu-2)} \int_0^1 \exp\{-y^{-1/2}\}(2\pi y)^{-\nu/2}\, dy\end{aligned}$$

if $|x-y| \leqq t^{1/2}$. This proves the required result (4.5).

As for $\nu = 2$, the above steps show that

$$\int_0^1 P_s(x-y)\,ds \geqq \int_0^{t/(x-y)^2} \exp\{-y^{-1/2}\}(2\pi y)^{-1}\,dy \geqq c\ln(2t/|x-y|^2),$$

if $|x-y| \leqq t^{1/2}$. Thus, if (4.4) holds, then

$$\lim_{\alpha\downarrow 0} \sup_x \int_{|x-y|\leqq\alpha} \ln(2\alpha^2/|x-y|^2)|V(y)|\,dy = 0. \tag{4.6}$$

However, for $|x-y| \leqq \alpha^2$

$$\ln(2\alpha^2/|x-y|^2) \geqq \ln(1/|x-y|) + \ln 2 \geqq \ln(1/|x-y|);$$

thus (4.6) implies (4.1) if $\nu = 2$.

Finally for $\nu = 1$, we note that if

$$\sup_x \int_0^t E(|V(b(s))|\,ds)\,ds < \infty,$$

hence $P_s(x-y) \geqq (\text{const}) > 0$ for $\frac{1}{2}t < s < t$ and $|x-y| \leqq 1$; thus $V \in K_1$.

As Example 2 in Appendix 1 shows, it can happen that

$$\sup_x E_x\left(\int_0^t |V(b(s))|\,ds\right) \to 0 \qquad \text{as} \quad t\downarrow 0$$

even though $\int_0^t[\sup_x E_x(|V(b(s))|)]\,ds = \infty$.

COROLLARY 4.6. *Let T be a linear map of $\mathbb{R}^\nu$ onto $\mathbb{R}^\mu$. Let $V(x) = W(Tx)$. Then $V \in K_\nu$ if and only if $W \in K_\mu$.*

Proof: Suppose first $\mu = \nu$. Then a simple change of variables, and $c_1|T(z)| \leqq |z| \leqq c_2|T(z)|$, show that the result is true. For this reason, there is no loss in supposing that T is an orthogonal projection onto a subspace of $\mathbb{R}^\nu$. But then

$$E_x(V(b(s))) = E_{Tx}(W(\tilde{b}(s))),$$

$\tilde{b}$ being a μ-dimensional Brownian motion. Now use the above Theorem 4.5.

DEFINITION. We say that a self adjoint operator A on $L^2(R^\nu)$ generates a regular L^∞-semigroup if and only if for each t there exists a constant c_t with which

$$\|e^{-tA}\phi\|_\infty \leqq c_t\|\phi\|_\infty$$

for all $\phi \in L^2 \cap L^\infty$ and, moreover,

$$\lim_{t\downarrow 0} c_t = 1. \tag{4.7}$$

THEOREM 4.7. *If $V \in K_\nu$, then V is $-\Delta$-form bounded with relative bound zero and the operator $H = -\frac{1}{2}\Delta + V$ generates a regular L^∞-semigroup. Conversely, if $V \leqq 0$, and if*

$$H = s\text{-}\underset{n\to\infty}{\text{resolvent limit}}\left(-\tfrac{1}{2}\Delta + \max(V, -n)\right)$$

defines a selfadjoint operator which generates a regular L^∞-semigroup, then $V \in K_\nu$.

Remark. As Example 3 in Appendix 1 shows, the condition $V \leqq 0$ is needed for the converse.

Proof: From the Khas'minskii lemma and Theorem 4.5, we conclude that if $V \in K_\nu$, then, for any λ,

$$\varlimsup_{t\downarrow 0} \sup_x E_x\left(\exp\left\{-\lambda\int_0^t V(b(s))\,ds\right\}\right) = 1.$$

From Jensen's inequality we obtain

$$E_x\left(\exp\left\{-\lambda\int_0^t V(b(s))\,ds\right\}\right) \geqq \exp\left\{-E_x\int_0^t \lambda V(b(s))\,ds\right\};$$

thus by Theorem 4.5 we see that, for any λ,

$$\varliminf_{t\downarrow 0} \sup_x E_x\left(\exp\left\{-\lambda\int_0^t V(b(s))\,ds\right\}\right) = 1.$$

These facts and the ideas in [31] immediately prove the first half of the theorem.

For the converse, we note that, if $V \leqq 0$,

$$E_x\left(\exp\left\{-\int_0^t V(b(s))\,ds\right\}\right) - 1 \geqq E_x\left(-\int_0^t V(b(s))\,ds\right),$$

since $e^x \geqq 1 + x$ for $x \geqq 0$.

COROLLARY 4.8. *If $V \leqq 0$ and $-\frac{1}{2}\Delta + V$ generates a regular L^∞-semigroup, then so does $-\frac{1}{2}\Delta + \lambda V$ for all $\lambda > 0$.*

Remark. This is interesting since $-r^{-2}$ is a counterexample for the analogous L^2-result. If we drop the $\lim_{t\downarrow 0}\|e^{-tH}\|_{\infty,\infty} = 1$ hypothesis, then the result is false as Example 1 in Appendix 1 shows.

Theorem 4.5 allows us to show that K_ν contains the class considered by Trudinger:

THEOREM 4.9. *Let V be a function on $\mathbb{R}^\nu$ such that for all $\epsilon > 0$, there is a $c(\epsilon)$ with which*

$$\langle \phi, |V|\phi \rangle \leqq \tfrac{1}{2}\epsilon\langle \phi, (-\Delta)\phi\rangle + c(\epsilon)\langle \phi, \phi\rangle \tag{4.8}$$

and suppose that, for some $\alpha < 1$, and some A, B,

$$c(\epsilon) \leqq A\exp\{B\epsilon^{-\alpha}\}. \tag{4.9}$$

Then $V \in K_\nu$.

Remarks 1. Trudinger [31] requires (4.8) with (4.9) replaced by the stronger $c(\epsilon) \leqq A\epsilon^{-N}$, but his method actually works under the hypothesis (4.9) (see [13]).

2. As we shall see, these kinds of hypotheses lead to

$$\int_0^1 \Big[\sup_x E_x(|V(b(s))|)\Big]\,ds < \infty$$

which is strictly stronger than $V \in K_\nu$; see Example 2 in Appendix 1.

3. It is a remarkable fact that L^2 estimates like (4.8) lead to L^∞ bounds on e^{-tH}.

Proof: Let $P_s(x, y)$ be the integral kernel of $\exp\{-sH_0\}$ and let $\phi_{s,y}(x) = (P_s(x, y))^{1/2}$. Then, noting that $\langle \phi, \phi\rangle = 1$ and $\langle \nabla\phi, \nabla\phi\rangle = \nu/4s$, we have

$$E_y(|V(b(s))|) \leqq \nu\epsilon/4s + c(\epsilon)$$

for any y, s and $\epsilon > 0$. Choosing $\epsilon = |\ln s|^{-\gamma}$ with $\gamma = 2/(1+\alpha)$ and using (4.9) we see that

$$\lim_{t\downarrow 0}\int_0^t \sup_y\big[E_y(|V(s)|)\big]\,ds = 0.$$

Theorem 4.5 completes the argument.

For radially symmetric potentials, K_ν is a rather familiar class, at least with regard to the question of singularity at the origin. Let us first consider the case of V having compact support.

PROPOSITION 4.10. *Let $\nu \geqq 3$. Let V be radially symmetric with support in $\{x \mid |x| \leqq 1\}$. Then $V \in K_\nu$ if and only if*

$$\int_0^1 r|V(r)|\,dr < \infty. \tag{4.10}$$

Proof: Let $V \in K_\nu$. By Lemma 4.4, for any α,

$$\sup_x \int_{\alpha < |x-y| < 1} |x-y|^{-(\nu-2)} |V(y)|\, dy < \infty.$$

Thus, for $V \in K_\nu$ we have

$$\int_{|y|<1} |y|^{-(\nu-2)} |V(y)|\, d^\nu y < \infty$$

which is (4.10).

Conversely, suppose (4.10) holds. For each ϵ, we can write $V = V_1 + V_2$ with V_1 bounded with compact support and $\int_0^1 r|V_2(r)|\,dr < \epsilon$. Then

$$\sup_x \int |x-y|^{-(\nu-2)} |V_2(y)|\, d^\nu y$$

$$= \sup_x \int \left[\max(|x|, |y|)\right]^{-(\nu-2)} |V_2(y)|\, d^\nu y$$

$$= \int_0^1 r|V_2(r)|\, dr < \epsilon.$$

Since $V_1 \in L^p$ with $p > \frac{1}{2}\nu$, $V_1 \in K_\nu$. Thus $V \in K_\nu$.

THEOREM 4.11. *Let $\nu \geqq 3$ and let V be spherically symmetric. Then V is in K_ν if and only if* (4.10) *holds and*

$$\sup_{|x| \geqq 2} \left[\int_{||x|-r| \leqq 1} |V(r)|\, dr \right] < \infty. \tag{4.11}$$

Proof: If $V \in K_\nu$, then (4.10) follows from Proposition 4.10 and (4.11) from Lemma 4.4. Conversely, let (4.10) and (4.11) hold. Write $V_1 + V_2 = V$ with $V_1(r) = V(r)$ (respectively 0) if $|r| \leqq 1$ (respectively $|r| > 1$). By Proposition 4.10, $V_1 \in K_\nu$. As for V_2, we note that, for any $r_0 \geqq 1$, $\alpha \leqq 1$, any x and any unit vector, $\hat{e}$,

$$\int_{\substack{|y| \leqq r_0 \\ |y - |y|\hat{e}| \leqq \alpha}} |x-y|^{-(\nu-2)}\, d\tilde{\Omega}(y) \leqq \int_{\substack{|y| = r_0 \\ |y - r_0\hat{e}| \leqq \alpha}} |y - r_0\hat{e}|^{-(\nu-2)}\, d\tilde{\Omega}(y) \leqq C\alpha$$

for some constant C. Here $d\tilde{\Omega}(y)$ is the surface measure normalized so that the total surface area is $\int_{|y|=r} d\tilde{\Omega}(y) \equiv Dy^{\nu-1}$. From this fact, we see that, for $\alpha \leqq 1$,

$$\int_{|x-y| \leqq \alpha} |x-y|^{-\nu-2} |V_2(y)|\, d^\nu y$$

is, at most, $c\alpha$ multiplied by the right-hand side of (4.11); thus $V_2 \in K_\nu$.

Remarks 1. These arguments show that the phenomena given in Example 1 of Appendix 1 cannot take place for centrally symmetric potentials, i.e., if V is central and $\sup_x E_x(\int_0^t |V(b(s))|\,ds) < \infty$ for some t, then $V \in K_\nu$. Since this holds for $\nu = 1$, it is not surprising that it is also true for central potentials.

2. As far as total singularities are concerned, these results nicely delimit K_ν: e.g. $r^{-2}[\ln r]^{-1-\epsilon}$ or $r^{-2}[\ln r]^{-1}[\ln(\ln r^{-1})]^{-1-\epsilon}$ are in K_ν but $r^{-2}[\ln r]^{-1}$ or $r^{-2}[\ln r][\ln(\ln r^{-1})]^{-1}$ are not.

THEOREM 4.12. *Suppose that* $\nu \geqq 3$ *and*

$$\int_a^\infty |\mu\{x \,|\, |V(x)| \geqq \lambda\}|^{2/\nu}\, d\lambda < \infty$$

with $\mu(\cdot) \equiv$ *Lebesque measure. Then* $V \in K_\nu$.

Proof: Without loss of generality, we can suppose that on the support of V, $V(x) \geqq 2a$, since $\min(2a, V(s)) \in L^\infty \subset K_\nu$. Let V^* be the spherically symmetric decreasing rearrangement of $|V|$. By definition,

$$\int_0^\infty |\mu\{x \,|\, V^*(x) \geqq \lambda\}|^{2/\nu}\, d\lambda = \int_0^\infty |\mu\{x \,|\, |V(x)| \geqq \lambda\}|^{2/\nu}\, d\lambda < \infty.$$

Suppose that V^* is strictly monotone and continuous on its support. Define $r(\lambda)$ by $V^*(r(\lambda)) = \lambda$ for $2a \leqq \lambda < \infty$ and $r(\lambda) = r(2a)$ for $\lambda < 2a$. Then

$$\int_0^\infty r|V^*(r)|\, dr = \int_0^\infty d\lambda\, \tfrac{1}{2} r(\lambda)^2 = c\int_0^\infty |\mu\{x \,|\, V^*(x) \geqq \lambda\}|^{2/\nu}\, d\lambda.$$

Therefore, $V^* \in K_\nu$.

However, by general principles (see [4]),

$$\sup_x \int_{|x-y| \leqq \alpha} |x-y|^{-(\nu-2)} |V(y)|\, d^\nu y \leqq \int_{|y| \leqq \alpha} |y|^{-(\nu-2)} |V^*(y)|\, d^\nu y.$$

Thus $V \in K_\nu$. The general case follows by an elementary limiting argument.

COROLLARY 4.13. *Let* $\nu \geqq 3$. *Let* $G(y)$ *be positive and monotone nondecreasing in s with*

$$\int_a^\infty \left[G'(s)\right]^{-2/(\nu-2)} ds < \infty.$$

Suppose that

$$\int G(|V(y)|)\, d^\nu y < \infty.$$

Then $V \in K_\nu$.

Proof: We know that

$$\int G(|V(y)|)\, d^\nu y \geqq \int_{V(y)\geqq a}\left[\int_a^{|V(y)|} d\lambda G'(\lambda)\right] d^\nu y \tag{4.12}$$

$$= \int_a^\infty G'(\lambda)|\mu\{y \,|\, |V(y)| \geqq \lambda\}|\, d\lambda.$$

However, by Hölder's inequality,

$$\int_a^\infty |\mu\{y \,|\, |V(y)| \geqq \lambda\}|^{2/\nu} \leqq (4.12) \times \int_a^\infty (G'(\lambda))^{-2/(\nu-2)}\, d\lambda.$$

EXAMPLE. $G(s) = s^{\nu/2}[\log(s+2)]^\alpha$ with $\alpha > \frac{1}{2}(\nu - 2)$.

The next result shows the connection between K_ν and the class of Kovalenko–Semenov [18].

THEOREM 4.14. *$V \in K_\nu$ if and only if, for all ϵ, there is a $C(\epsilon)$ with which*

$$\|Vu\|_1 \leqq \tfrac{1}{2}\epsilon\|\Delta u\|_1 + C(\epsilon)\|u\|_1 \tag{4.13}$$

for all $u \in C_0^\infty$, where $\|\cdot\|_1 \equiv L^1$-norm.

Proof: We first claim that (4.13) holds if and only if $(H_0 \equiv -\frac{1}{2}\Delta)$

$$\lim_{a\to\infty} \|V(H_0 + a)^{-1}\|_{1,1} = 0, \tag{4.14}$$

where $\|\cdot\|_{1,1}$ is the norm as an operator from L^1 to L^1. Relation (4.14) implies (4.13), using

$$\|Vu\|_1 \leqq \|V(H_0 + a)^{-1}\|_{1,1}\left[\tfrac{1}{2}\|\Delta u\|_1 + a\|u\|_1\right],$$

and (4.13) implies (4.14) since

$$\|H_0(H_0 + a)^{-1}\|_{1,1} \leqq 2, \qquad \|(H_0 + a)^{-1}\|_{1,1} \leqq a^{-1}.$$

By duality,

$$\|V(H_0 + a)^{-1}\|_{1,1} = \|(H_0 + a)^{-1}V\|_{\infty,\infty}$$

and, since $(H_0 + a)^{-1}$ is positivity preserving,

$$\|(H_0 + a)^{-1}V\|_{\infty,\infty} = \|(H_0 + a)^{-1}|V|\,\|_\infty$$

(on the right side of this equation, we apply $(H_0 + a)^{-1}$ to the function $|V|$ and compute its L^∞-norm). Thus, (4.14) is equivalent to

$$\lim_{a\to\infty} \sup_x E_x\left(\int_0^\infty e^{-at}|V(b(s))|\,ds\right) = 0$$

and this is easily seen to be equivalent to (4.4).

Remark. Kovalenko–Semenov [18] consider the set of all V for which (4.13) holds for some $\epsilon < 1$. Thus, K_ν is the set of potentials all of whose multiples obey the condition of [18]. The difference is only in the pathologies of Example 1 of Appendix 1.

The next set of ideas involving K_ν that we want to present involves the study of $(-\Delta)^{-1}V$ as a map on L^∞. We discuss in detail the case $\nu \geqq 3$. All the arguments carry easily over to $\nu = 1, 2$ after replacing $(-\Delta)^{-1}$, and its kernel, by what corresponds to $(-\Delta + 1)^{-1}$.

THEOREM 4.15. *Let $\nu \geqq 3$. Let V have compact support. Then $V \in K_\nu$ if and only if* (i) *the integral defining*

$$f(x) \equiv \int |x-y|^{-(\nu-2)}|V(y)|\,d^\nu y \tag{4.15}$$

converges for all x, and (ii) *f is a continuous function.*

Proof: Suppose first that $V \in K_\nu$. Let f_α be the function defined by adding the condition $|x - y| \geqq \alpha$ in (4.15). Then f_α is trivially continuous and $V \in K_\nu$ implies that $\lim_{\alpha\downarrow 0}\|f - f_\alpha\|_\infty = 0$, therefore f is continuous, being a uniform limit of continuous functions.

Conversely, suppose that f is a continuous function. Let $|V| = V_n + W_n$, where $V_n = \min(|V|, n)$. Let $f = f_n + g_n$, where f_n, g_n are the functions obtained by replacing $|V|$ by V_n and W_n. Now, for fixed n,

$$\lim_{\alpha\downarrow 0} \sup_x \int_{|x-y|\leqq\alpha} |x-y|^{-(\nu-2)}|V_n(y)|\,d^\nu y = 0,$$

so that

$$\overline{\lim_{\alpha\downarrow 0}} \sup_x \int_{|x-y|\leqq\alpha} |x-y|^{-(\nu-2)}|V(y)|\,d^\nu y \leqq \lim_{n\to\infty} \|g_n\|_\infty.$$

But f_n converges monotonically upwards to f, which by assumption is continuous. Therefore, $g_n \to 0$ uniformly on compacts (by Dini's theorem). Control of g_n far from the support of V is trivial; hence $\lim_{n\to\infty}\|g_n\|_\infty = 0$.

The above proof, especially the Dini's theorem argument, can be summarized as follows:

THEOREM 4.16. *Let $\nu \geqq 3$. Let V have compact support. Then $V \in K_\nu$ if and only if, for all ϵ, there is a function W_ϵ such that $V - W_\epsilon$ is bounded and such that*

$$\sup_x \int |x-y|^{-(\nu-2)} |W_\epsilon(y)| \, d^\nu y \leqq \epsilon.$$

THEOREM 4.17. *Let $\nu \geqq 3$ and let V lie in L^1 and have compact support. Then $V \in K_\nu$ if and only if $(-\Delta)^{-1}V$ defines a bounded map of L^∞ into $C_\infty(\mathbb{R}^\nu)$, the continuous functions vanishing at infinity.*

Proof: Let $V \in K_\nu$. Write $V = W_\epsilon + (V - W_\epsilon)$ in accordance with Theorem 4.16. Since $\|(-\Delta)^{-1}W_\epsilon\|_{\infty,\infty} \leqq C\epsilon$, we see that $(-\Delta)^{-1}V$ is a norm limit of bounded maps from L^∞ to C_∞ and therefore it itself is bounded.

Conversely, if $(-\Delta)^{-1}V$ is such a map, let $g(x) = \overline{V(x)}|V(x)|^{-1}$ (respectively $= 0$) if $V(x) \neq 0$ (respectively $= 0$). Then $(-\Delta)^{-1}Vg = (-\Delta)^{-1}|V|$. If this is continuous, then $V \in K_\nu$ by Theorem 4.15.

The following result is primarily of academic interest:

THEOREM 4.18. *Let $\nu \geqq 3$ and let V be an L^1 function of compact support. Then $V \in K_\nu$ if and only if $(-\Delta)^{-1}V$ is a compact map of L^∞, i.e., $[(-\Delta)^{-1}V] \times [\{f \mid \|f\|_\infty \leqq 1\}]$ is precompact in L^∞.*

Proof: If $V \in K_\nu$, as above, $(-\Delta)^{-1}V$ is a norm limit of maps $(-\Delta)^{-1}V_n$ with $V_n \in L^\infty$ and $\operatorname{supp} V_n$ compact. Such maps are compact by the Arzela–Ascoli theorem; hence $(-\Delta)^{-1}V$ is compact.

For the converse, suppose that $V \notin K_\nu$ but that $(-\Delta)^{-1}V$ is a bounded map on L^∞. Let

$$g_{\alpha,z}(x) \equiv c \int_{|z-y| \leqq \alpha} |x-y|^{-(\nu-2)} |V(y)| \, d^\nu y,$$

where c is chosen so that $c|x-y|^{-(\nu-2)}$ is the integral kernel of $(-\Delta)^{-1}$. Notice that if $h \in L^1$, we can always, given ϵ, pick α so that

$$\sup_z \int_{|z-y| \leqq \alpha} |h(y)| \, d^\nu y < \epsilon.$$

(We write h as a sum of two functions, one bounded and the other with a very small L^1 norm). Thus, since we are supposing $(-\Delta)^{-1}|V| \in L^\infty$, for any *fixed* x,

$$\lim_{\alpha \downarrow 0} \sup_z |g_{\alpha,z}(x)| = 0. \tag{4.16}$$

Since $V \notin K_\nu$, we have

$$\lim_{\alpha \downarrow 0} \sup_x g_{\alpha,x}(x) = \epsilon > 0. \tag{4.17}$$

Next let us pick α_n, z_n inductively by letting $\alpha_1 = 1$ and picking z_1 so that $g_{\alpha_1, z_1}(z_1) \geqq \frac{1}{2}\epsilon$. Supposing $\alpha_1, \cdots, \alpha_{n-1}$ and $z_1, \cdots, z_{n-1}$ are picked, we first select α_n so that

$$\sup_z |g_{\alpha_n, z}(z_i)| \leqq \tfrac{1}{4}\epsilon, \qquad i = 1, \cdots, n-1,$$

which is possible by (4.16), and then z_n so that

$$|g_{\alpha_n, z_n}(z_n)| \geqq \tfrac{1}{2}\epsilon.$$

Let $f_n \equiv g_{\alpha_n, z_n}$. Clearly, $f_n = (-\Delta)^{-1} V\rho_n$ with $\|\rho_n\|_\infty = 1$. Moreover, if $n > m$,

$$|f_n(z_m) - f_m(z_m)| \geqq \tfrac{1}{4}\epsilon;$$

hence $\|f_n - f_m\|_\infty \geqq \frac{1}{4}\epsilon$ for all n, m. It follows that $(-\Delta)^{-1}V$ is not compact.

Finally, we want to prove Theorems 1.5, 1.6. We emphasize that all that is really used in their proof is Theorem 4.16. Again we only give the proof for $\nu \geqq 3$.

Proof of Theorem 1.5: Here we give the proof assuming that $u \in L^\infty_{\text{loc}}$. Later (Theorem 4.18) we show that this is automatic. We only need to prove continuity near a fixed point, say near $x = 0$. Let

$$f(x) \equiv \int_{|y| \leqq 1} c|x-y|^{-(\nu-2)} V(y) u(y)\, dy \tag{4.18}$$

with c chosen so that $c|x-y|^{-(\nu-2)}$ is the integral kernel of $(-\Delta)^{-1}$. Then, for ϕ in $C_0^\infty(\{x \mid |x| \leqq 1\})$

$$(\Delta\phi, u + f) = (\phi, Vu) + \big(\Delta\phi, (-\Delta)^{-1}\chi V u\big) = 0,$$

where χ is the indicator function of the unit ball. Thus $u + f$ is harmonic in that ball and so continuous there. Since $V \in K_\nu^{\text{loc}}$, f is continuous by Theorem 4.17.

Proof of Theorem 1.6: Consider first the case $\nu \geqq 3$, without the stopping time, where we want to show that

$$\sup_{x \in \Omega} E_x\left(\int_0^\infty |(\chi_\Omega V)(b(s))|\, ds\right) < \epsilon. \tag{4.19}$$

Since $(-\Delta)^{-1}$ has an integral kernel $c|x-y|^{-(\nu-2)}$ and

$$\int_0^\infty (e^{-t\Delta}f)(x) = \left[(-\Delta)^{-1}f\right](x) \quad \text{if} \quad f \geqq 0,$$

we see that the left-hand side of (4.11) is at most

$$c \sup_x \int_{|x-y| \leqq 2R} |x-y|^{-(\nu-2)} |V(y)| \, d^\nu y,$$

because of the location of Ω. Thus (4.19) follows directly from the definition of K_ν.

The result for $\nu \geqq 3$ with stopping time follows from what we have just proven. If $\nu = 1, 2$, we proceed as follows. Let T_x be the first exit time from $|y-x| \leqq 2R$. Then $T_x \geqq T$ and

$$E_x\left(\int_0^T |V(b(s))| \, ds\right) \leqq E_x\left(\int_0^{T_x} |V(b(s))| \, ds\right)$$
$$\leqq \int_{|x-y| \leqq 2R} G(x, y; R) |V(y)| \, dy,$$

where $G(x, y; R)$ is the Green's function for $\{y \,|\, |y-x| \leqq 2R\}$ with Dirichlet boundary conditions. Explicitly,

$$G(x, y; R) = \begin{cases} -(\pi)^{-1} \ln[|x-y|/2R] & \text{for} \quad \nu = 2, \\ 2R - |x-y| & \text{for} \quad \nu = 1. \end{cases}$$

Noting that, for $2R \leqq 1$, $G(x, y; R) \leqq -(\pi)^{-1} \ln[|x-y|]$ in two dimensions and $G(x, y; R) \leqq 2R$ in one dimension, the result follows.

To get the strongest claim of Theorem 1.5, we need

THEOREM 4.18. *Let $V \in K_\nu^{\text{loc}}$, let u, $Vu \in L_{\text{loc}}^1$ with*

$$-\tfrac{1}{2}\Delta u + Vu = 0$$

in distributional sense inside some set Ω. Then u is locally bounded.

Proof: We suppose without loss of generality that $\nu \geqq 3$, since we can always add on extra dimensions. We shall prove boundedness of $u(x)$ for x near 0, assuming $\Omega \ni 0$. As in the proof of Theorem 1.5, we can write

$$u = f + g$$

with g harmonic near zero, and f given by (4.18), $|y| \leqq 1$ being replaced by $|y| \leqq \delta$ for suitable δ. By hypothesis, $Vu \in L_{\text{loc}}^1$, so by Young's inequality f is in

L^p_{loc} for any $p < \nu/(\nu-2)$. Since g is harmonic near 0, we conclude that, inside Ω, u is in L^p_{loc} for $p < \nu/(\nu-2)$. Similarly, since

$$(\nabla f)(x) = \int_{|y| \leqq \delta} c(2-\nu)|x-y|^{-(\nu-1)} V(y) u(y)\, d^\nu y$$

(distributional gradient) we see that $\nabla u \in L^p_{\text{loc}}$ if $p < \nu/(\nu-1)$.

Now fix some p with $1 < p < \nu/(\nu-1)$ and let q be its dual index. Choose δ so small that $S \equiv \{y \mid |y| \leqq \delta\} \subset \Omega$ and so that

$$\sup_{|x| \leqq \delta} E_x\left(\int_0^\infty |Q(b(s))|\, ds\right) < q^{-1}, \tag{4.20}$$

where

$$Q(y) = \chi_S(y)[V(y) - 1]$$

with χ_S the indicator function of S; (4.20) can be arranged since $V \in K_\nu^{\text{loc}}$ and $\nu \geqq 3$.

Pick $\eta \in C_0^\infty(\mathbb{R}^\nu)$ with $\operatorname{supp}\eta \subset S^{\text{int}}$ and with $\eta \equiv 1$ in a neighborhood of $\frac{1}{2}S \equiv \{y \mid |y| \leqq \frac{1}{2}\delta\}$. Let $w = \eta u$. We shall show that $w \in L^\infty$ in $\frac{1}{2}S$; hence u is bounded in $\frac{1}{2}S$ proving the theorem.

Note that w obeys, in distributional sense,

$$-\tfrac{1}{2}\Delta w + Qw + w = h, \tag{4.21}$$

where $h = \frac{1}{2}(\Delta\eta)u + (\nabla\eta)\cdot\nabla u \in L^p$ and $\operatorname{supp} h \subset \{x \mid \frac{1}{2}\delta < |x| < \delta\}$. Now $w \in L^1$ and, by (4.21), $\Delta w \in L^1$ and moreover, $\nabla w \in L^1$. It follows that w lies in the domain of the generator of the semigroup $e^{t\Delta}$ on L^1. Since Q is a Kato bounded perturbation of the generator (see Theorem 4.14), w is in the domain of the generator of the semigroup e^{-tH} with $H = -\frac{1}{2}\Delta + Q$. Thus (4.21) which was proven in distributional sense is true in L^1-operator sense. Therefore,

$$w = (H+1)^{-1}h = \int_0^\infty (e^{-t}e^{-tH}h)\, dt,$$

where the second formula is true by (4.20) and Khas'minskii's lemma, which tell us that $\sup_t \|e^{-tH}\|_{1,1} < \infty$. By Hölder's inequality, first in path space and then in dt, we find that

$$|w(x)| \leqq \left\{\left[\left(-\tfrac{1}{2}\Delta + qQ + 1\right)^{-1}1\right](x)\right\}^{1/2}\left\{\left[\left(-\tfrac{1}{2}\Delta + 1\right)^{-1}|h|^p\right](x)\right\}^{1/q}.$$

By (4.20) and Khas'minskii's lemma, the first factor is bounded. Since $|h|^p \in L^1$ and $\operatorname{supp} h \subset \{y \mid |y| > \frac{1}{2}\delta\}$, the second factor is bounded.

5. The Strong Harnack Inequality

In this section, we shall prove Theorem 1.1. The first half of it is

THEOREM 5.1. *Let $V \in K_\nu^{\mathrm{loc}}$. Then H obeys the strong Harnack inequality.*

Proof: Without loss of generality (by adding extra dimensions) we may suppose that $\nu \geqq 3$. For fixed R, d, assume that there is a function $u \geqq 0$ which satisfies $Hu = 0$ in a neighborhood of $\Omega' = \{z \mid |z - y| \leqq 2d\}$ with $|x| < R$. Let us denote $\Omega = \{z \mid |z - x| \leqq \frac{3}{2}d\}$. By Harnack's inequality (see Section 3), we know that

$$\sup_{z \in \Omega} u(z) \leqq Cu(x) \tag{5.1}$$

with C depending only V, R and d. As in Section 2, we know that in Ω,

$$u(y) = g(y) - \left[\left(H_0^\Omega\right)^{-1} Vu\right](y) \equiv g(y) + f(y),$$

H_0^Ω being the Dirichlet Laplacian in Ω and g the function which is harmonic on Ω^{int}, continuous on Ω, and equal to u on $\partial\Omega$. By (5.1) and the explicit formula for g which yields a bound on ∇g, we see that

$$|g(y) - g(x)| \leqq A|x - y| u(x) \quad \text{if} \quad |x - y| \leqq d \tag{5.2}$$

for some A depending only on V, R and d. Now write $V = V_n + W_n$ with

$$V_n(y) = \begin{cases} V(y) & \text{if } |V(y)| \leqq n, \\ -n & \text{if } V(y) \leqq -n, \\ n & \text{if } V(y) \geqq n, \end{cases}$$

and write $f = f_n + g_n$ correspondingly. Since

$$\lim_{n \to \infty} \sup_{|y| \leqq d + R} \int_{|z-y| \leqq d} |z - y|^{-(\nu-2)} |W_n(z)|\, dz = 0$$

(see Theorem 4.16) and $(H_0^\Omega)^{-1}$ has an integral kernel dominated by a multiple of $|z - y|^{-(\nu-2)}$, we see (using (5.1)) that

$$|g_n(y)| \leqq B_n u(x),$$

where

$$\lim_{n \to \infty} B_n = 0. \tag{5.3}$$

Thus, if $|x-y| \leqq d$,

$$(5.4) \qquad |g_n(x) - g_n(y)| \leqq 2B_n u(x).$$

Finally, since $|V_n| \leqq n$, the explicit form of $(H_0^\Omega)^{-1}$ gives us control of ∇f_n, leading to

$$(5.5) \qquad |f_n(x) - f_n(y)| \leqq C_n |u(x)|\,|x-y| \quad \text{for} \quad |x-y| \leqq d.$$

As a result,

$$(5.6) \qquad |u(x) - u(y)| \leqq u(x) f(|x-y|),$$

where

$$f(a) = \inf_n \left[aA + 2B_n + C_n a \right]$$

depends only on R, d and V. From (5.3), we see that $\lim_{a \downarrow 0} f(a) = 0$.

As usual, we give the proof of the converse just for $\nu \geqq 3$, only to avoid notational complexities.

THEOREM 5.2. *Let $V \leqq 0$ be a measurable function on $\mathbb{R}^\nu$. Suppose that for any R there are $\epsilon_R > 0$ and c_R such that*

$$(5.7) \qquad \frac{1}{2}\int |\nabla\phi|^2 dx + (1+\epsilon_R)\int V(x)|\phi(x)|^2 dx \geqq -c_R \int |\phi(x)|^2 dx$$

for all $\phi \in C_0^\infty(|x| < R)$. Let H be a corresponding form sum, and let H obey the strong Harnack inequality. Then $V \in K_\nu^{\text{loc}}$.

Proof: We shall show that, given ϵ, we can find α for which

$$\int_{|y| \leqq \alpha} |x-y|^{-(\nu-2)} |V(y)|\, d^\nu y \leqq \epsilon$$

for all x with $|x| < R$. Obviously, by compactness, we need only do this for x very near 0. What we shall do is prove it for $x = 0$ but in our proof α will only depend on c_R and on the functions $f_{d,R}$ in the strong Harnack inequality, so the bound will hold uniformly in x near $x = 0$.

Inequality (5.7) implies that

$$(\phi, H\phi) \geqq \tfrac{1}{2}\epsilon_R(1+\epsilon_R)^{-1}\int |\nabla\phi|^2 - c_R \int |\phi|^2$$

if $\phi \in C_0^\infty(|x| \leqq R)$. It follows, that we can take δ so small that $H^\Omega \geqq 1$, where $\Omega = \{|x| \mid |x| \leqq 2\delta\}$ and H^Ω is the "Friedrich's extension" of H on $C_0^\infty(\Omega)$ (δ

only depends on c_R, ϵ_R and the fact that $\Omega \subset \{|x| \subset R\}$ and not on the fact that the center of the sphere is at zero). Pick any function f supported in $\{x|\delta < x < 2\delta\}$ which is C^∞ and positive. Let

$$u = (H^\Omega)^{-1} f.$$

Then $u \geqq (H_0^\Omega)^{-1} f > 0$ since $(-V) \geqq 0$. In a neighborhood of $\{x||x| < \delta\}$, u obeys $Hu = 0$, thus by the strong Harnack inequality, given $\tilde{\epsilon}$ we can find r such that

$$\inf_{|y| \leqq r} \frac{u(y)}{u(0)} \geqq 1 - \tfrac{1}{2}\tilde{\epsilon}. \tag{5.8}$$

We claim that, for any t,

$$u(0) \geqq E_0\left(\exp\left\{-\int_0^t V(b(s))\right\} u(b(t)); |b(t)| \leqq r, \sup_{0 \leqq s \leqq t} |b(s)| \leqq \delta\right). \tag{5.9}$$

Inequality (5.9) would hold by considerations of the type found in Section 2 if we knew that $V \in K_\nu$. To get it in general, let $V_n = \max(V, -n)$, $H_n^\Omega = H_0^\Omega + V_n$ and $u_n = (H_n^\Omega)^{-1} f$. Since $V_n \in K_\nu$, (5.9) holds if u is replaced by u_n and V by V_n. Now, u_n is monotone in n since this is also true for V_n. By the monotone convergence theorem for forms, $(H_n^\Omega)^{-1} \to (H^\Omega)^{-1}$ strongly, therefore $u_n \to u$ in L^2. Since u is continuous, the convergence is uniform. Applying the monotone convergence theorem, we obtain (5.9).

Since (5.8) and (5.9) hold, we see that

$$E_0\left(\exp\left\{-\int_0^t V(b(s))\right\}; |b(t)| \leqq r, \sup_{0 \leqq s \leqq t} |b(s)| \leqq \delta\right) \leqq (1 - \tfrac{1}{2}\tilde{\epsilon})^{-1}. \tag{5.10}$$

Since δ and r are now fixed, we have

$$\lim_{t \downarrow 0} E_0\left(|b(t)| \leqq r \quad \text{and} \quad \sup_{0 \leqq s \leqq t} |b(s)| \leqq \delta\right) = 1. \tag{5.11}$$

The last two formulas imply, using $e^a \geqq 1 + a$ for $a \geqq 0$, that we can pick T so small that, if $t < T$,

$$E_0\left(\int_0^t |V(b(s))|\, ds; |b(t)| \leqq r; \sup_{0 \leqq s \leqq t} |b(s)| \leqq \delta\right) \leqq \tilde{\epsilon}. \tag{5.12}$$

Let $P_s'(x, y)$ be the integral kernel of the semigroup generated by the Dirichlet Laplacian in $\{z||z| < \delta\}$. Equation (5.12) says that

$$\int_0^t \int_{|z| \leqq r} P_s'(0, y)|V(y)|P_{t-s}'(y, z)\, d^\nu y\, dz \leqq \tilde{\epsilon}.$$

If $|y| \leqq \frac{1}{2}r$, we can be sure that $\int_{|z|<r} P_u'(y,z)\,dz \geqq \frac{1}{2}$ if u is small. Thus, shrinking t if necessary, we know that

$$\int_{|y|\leqq r/2} d^\nu y\,|V(y)| \int_0^t P_s'(x,y)\,ds \leqq 2\tilde{\epsilon}.$$

But if $|y| \leqq t^{1/2} < \frac{1}{2}\delta$, we have, by scaling, that

$$\int_0^t P_s'(0,y)\,ds \geqq c|y|^{-(\nu-2)}.$$

Thus, for $\alpha = \min(t^{1/2}, \frac{1}{2}r)$,

$$\int_{|y|\leqq\alpha} |y|^{-(\nu-2)}|V(y)|\,d^\nu y \leqq 2c^{-1}\tilde{\epsilon}.$$

Since c is independent of $\tilde{\epsilon}$, we can make the *a priori* choice $\tilde{\epsilon} = \frac{1}{2}c\epsilon$.

6. Subsolution Estimates

DEFINITION. Let $V \in L^1_{\text{loc}}$. We say that u is a subsolution for $H = \frac{1}{2}\Delta + V$ if and only if

(i) $u \geqq 0$,
(ii) *u is locally bounded,*
(iii) *u is upper semicontinuous,*
(iv) $Hu \leqq 0$ *(in distributional sense).*

Remark. By Kato's inequality (see [16]), if $H\phi = 0$ with ϕ locally bounded and continuous, then $u = |\phi|$ is a subsolution.

In this section we want to prove the following:

THEOREM 6.1. *Suppose $V_- \equiv \min(V,0)$ is in K_ν^{loc} and let u be a subsolution. Then*

$$u(x) \leqq C\int_{|x-y|=\delta} u(y)\,d\sigma(y)$$

so long as δ is sufficiently small. How small δ must be and how large C is depends only on local norms of V near x. In particular, if $V_- \in K_\nu$, these may be chosen to be independent of x.

As a preliminary remark, we note that

$$(-\tfrac{1}{2}\Delta + V_-)u = Hu - V_+u \leqq 0;$$

thus without loss of generality we can suppose that $V \leqq 0$ and $V \in K_\nu^{\text{loc}}$. Given this, the theorem clearly follows from the results in Section 3 and

LEMMA 6.2. *Let u be a subsolution for $H = -\frac{1}{2}\Delta + V$ with $V \in K_\nu^{\text{loc}}$ and $V \leqq 0$. Then, for any small enough ball Ω, T – the first exit time from Ω, and all $x \in \Omega$,*

$$u(x) \leqq E_x\left(\exp\left\{-\int_0^T V(b(s))\right\} u(b(T))\right). \tag{6.1}$$

Remark. Formal considerations suggest that (6.1) is true even if V is not negative but since the proof is more direct in case $V \leqq 0$ and that is all we need, we only consider that case.

Proof: Let $f = (H_0^\Omega)^{-1} V u$. Then f is continuous and vanishes on $\partial\Omega$ by Lemma A.4.4. Let $\eta = u + f$. Then

$$\Delta\eta = -2Hu \geqq 0$$

and η is upper semicontinuous; hence η is subharmonic. Thus, since $f = 0$ on $\partial\Omega$,

$$\eta(x) \leqq E_x(u(b(T))).$$

Using Lemma A.4.4 again to write $(H_0^\Omega)^{-1} V u$, we see that

$$u(x) \leqq E_x(u(b(T))) + E_x\left(\int_0^T [-V(b(s))] u(b(s))\, ds\right).$$

Since $-V \geqq 0$, we can iterate controlling the remainder by supposing Ω to be small and so obtain (6.1).

Appendix 1. Three Examples in Search of a Paper

In this appendix, we present some pathological potentials which illustrate various points made in the paper.

EXAMPLE 1. Take $\nu \geqq 3$. Let x_n be points $(2^{-n}, 0, \cdots, 0)$ in $\mathbb{R}^\nu$, let S_n be the ball of radius 8^{-n} about x_n and let $V_n(x)$ be the function at x which is -8^{2n} on S_n and zero off S_n. Let

$$V(x) = \sum_{n=2}^{\infty} V_n(x). \tag{A.1.1}$$

We claim that

(A.1.2) $$\sup_x \int |x-y|^{-(\nu-2)} |V(y)|\, dy < \infty$$

and that, for each fixed x,

(A.1.3) $$\lim_{\alpha \downarrow 0} \int_{|x-y| \leqq \alpha} |x-y|^{-(\nu-2)} |V(y)|\, dy = 0$$

but that nevertheless

(A.1.4) $$\lim_{\alpha \downarrow 0} \sup_x \int_{|x-y| \leqq \alpha} |x-y|^{-(\nu-2)} |V(y)|\, dy \neq 0.$$

After proving these facts, we shall discuss the significance of such an example.

Given (A.1.2), (A.1.3) is an immediate consequence of the monotone convergence theorem. To prove (A.1.2), let

$$\rho_n(x) = \int |x-y|^{-(\nu-2)} |V_n(y)|\, dy.$$

By scaling, $\sup_x \rho_n(x)$ is independent of n; thus

(A.1.5) $$\rho_n(x) \leqq C$$

for all n and x and some C (indeed, the best C is $\frac{1}{2}\nu\tau_\nu$, τ_ν being the volume of the unit ball). Let T_n be the sphere of radius 4^{-n} about x_n. Since $n, m \geqq 2$,

(A.1.6) $$T_n \cap T_m = \emptyset \quad \text{for} \quad n \neq m.$$

Outside S_n, ρ_n equals $d_n |x - x_n|^{-(\nu-2)}$ by Newton's law, and by scaling we have $d_n = d 8^{-n(\nu-2)}$. Thus

(A.1.7) $$\sup_{y \notin T_n} \rho_n(y) = d\left(\tfrac{4}{8}\right)^{n(\nu-2)} = d 2^{-n(\nu-2)}.$$

By (A.1.5–7),

$$\sum_n \rho_n(y) \leqq C + d \sum_{n=2}^{\infty} 2^{-n(\nu-2)} < \infty,$$

proving (A.1.2). To prove (A.1.4), we note that, so long as $\alpha \geqq 8^{-n}$,

$$\int_{|x_n-y| \leqq \alpha} |x-y|^{-(\nu-2)} |V(y)|\, d^\nu y \geqq \rho_n(x_n) = \tfrac{1}{2}\nu\tau_\nu.$$

Thus (A.1.4) holds. By (A.1.2),

$$\sup_x E_x\left(a\int_0^\infty |V(b(s))|\,ds\right) < 1$$

for a small. Therefore,

$$\exp\{-t(-\tfrac{1}{2}\Delta + aV)\} \equiv F_t(a)$$

defines an exponentially bounded semigroup on L^∞ (see Theorem 1.2), for which

$$\lim_{t\downarrow 0} \|F_t(a)\|_{\infty,\infty} \neq 1,$$

by our basic result on K_ν. Moreover, while $F_t(a)$ is bounded from L^∞ to L^∞ for a small, it is not even bounded from L^2 to L^2 if a is sufficiently large. Indeed, let $\psi(x)$ be the lowest eigenfunction of the Dirichlet Laplacian in the unit sphere and let $2e_0$ be the corresponding eigenvalue. Let

$$\psi_n(x) = \psi(8^n(x - x_n)).$$

Then

$$(\psi_n, (H_0 + aV)\psi_n)/(\psi_n, \psi_n) = (e_0 - a)8^{2n},$$

by scaling. Thus, if $a > e_0$, $F_t(a)$ is not even bounded from L^2 to L^2.

Notice that the distribution function of V is essentially the same as that for r^{-2}. From this point of view, the surprise is that $F_t(a)$ is even bounded from L^∞ to L^∞.

EXAMPLE 2. Let V_n be the potentials of example 1 and let

$$W = \sum_{n=2}^{\infty} n^{-1} V_n(x).$$

Then clearly,

$$\sup_x \int_{|x-y|\leqq\alpha} |x-y|^{-(\nu-2)} |W(y)|\,dy \leqq cn^{-1} + b_{\alpha,n},$$

where

$$c = \left\|\sum_j \rho_j\right\|_\infty < \infty \qquad \text{(by A.1.2)}$$

and

$$b_{\alpha,n} = \sup_x \int_{|x-y|\leqq \alpha} |x-y|^{-(\nu-2)} \Big[\sum_{j\leqq n} j^{-1} |V_j(y)| \Big] d^\nu y.$$

Since $\sum_{j\leqq n} j^{-1}|V_j(y)| \in L^\infty \cap L^1$, we see that

$$\lim_{\alpha\downarrow 0} b_{\alpha,n} = 0$$

for each n. It follows that $W \in K_\nu$. Thus (see Theorem 4.5)

$$\lim_{t\downarrow 0} \Big[\sup_x \int_0^t E_x(|W(b(s))|)\, ds \Big] = 0. \tag{A.1.8}$$

However, we claim that

$$\int_0^t \sup_x [E_x(|W(b(s))|)]\, ds = \infty \tag{A.1.9}$$

for any $t > 0$. Using the scaling we have

$$\begin{aligned} \sup_x E_x(|W(b(s))|) &\geqq \sup_x E_x\big(n^{-1}|V_n(b(s))|\big) \\ &\geqq E_{x_n}\big(n^{-1}|V_n(b(s))|\big) \geqq (2n)^{-1} 8^{2n} \end{aligned}$$

if $0 \leqq s \leqq \alpha 8^{-2n}$ for some α. Thus

$$\int_{\alpha 8^{-2(n+1)}}^{\alpha 8^{-2n}} \sup_x E_x(|W(b(s))|)\, ds \geqq (2n)^{-1}\alpha\Big(\frac{63}{64}\Big).$$

Summing over n, we obtain (A.1.9).

The point is that while Trudinger type estimates can be used to prove that a potential is in K_ν, they always imply that $\int_0^t \sup_x[\cdots] < \infty$; thus W will not obey Trudinger type estimates with a $c(\epsilon)$ which can be turned around to give us $W \in K_\nu$. Indeed, the $c(\epsilon)$ for W will be essentially identical to the $c(\epsilon)$ for $r^{-2}(\log r)^{-1}$ for which Harnack's inequality can fail.

EXAMPLE 3. It has been known for some years (see, e.g., Schechter [28], [29], Combescure–Ginibre [8] or Baetman–Chadan [2]) that there exist potentials V with severe oscillations which allow $-\Delta + V$ to be bounded from below on L^2 even though $-\Delta + \min(V,0)$ has no lower bound. We want to show here that in such a case it still may happen that $\exp\{-t(H_0 + V)\}$ is bounded on L^∞. It is probable that one could prove a Harnack type inequality here also. The point of this example is that if V is allowed to oscillate, one cannot hope for any simple K_ν-type condition to be necessary and sufficient for properties like L^∞ boundedness. Let $V(\mathbf{x}) = V(|\mathbf{x}|)$ on $\mathbb{R}^3$ with

$$V(r) = \begin{cases} 0 & \text{for } 1 \leqq r \text{ or } r \geqq 2, \\ -(2m)^{-1}\epsilon(r-1)^{-m}\cos\left((r-1)^{-2m}\right) & \text{for } 1 < r < 2, \end{cases}$$

where m is fixed, but arbitrarily large, and ϵ is a parameter which will be adjusted.

Of course, V is so singluar that one cannot directly define $H = \frac{1}{2}\Delta + V$ by just integrating. There are several equivalent definitions, see [8]:

(a) Define

$$\int |\phi(x)|^2 V(x)\,dx \text{ to be } \lim_{\delta \downarrow 0} \int_{|r-1| \geqq \delta} |\phi(x)|^2 V(x)\,dx$$

which exists for $\phi \in C_0^\infty$. H is then form-bounded from below on C_0^∞, with a well-behaved Friedrichs extension.

(b) Let V_δ be the potential obtained by replacing $r-1$ by $(r-1+\delta)$ and note that $H_0 + V_\delta$ has a nice limit in norm-resolvent sense.

(c) Note that

$$V(r) = \epsilon \frac{d}{dr}(r-1)^{m+1}\sin(r-1)^{-2m}\big) + W \tag{A.1.10}$$

with W bounded, and interpret

$$(\phi, V\phi) = (\phi, W\phi) - 2\epsilon\,\mathrm{Re}\left(\phi, (r-1)^{m+1} \cdots d\phi/dr\right).$$

These equivalent definitions yield an operator H, with e^{-tH} a bounded semigroup on L^2. We shall show that for ϵ small, the semigroup is bounded from L^∞ to L^∞. If $H\phi = 0$ has a solution ϕ which is in L^∞ and $\inf_x \phi(x) > 0$, then, by standard ideas (see [32]) the semigroup is bounded. (Actually, it is only for nice V's that one can easily see this; hence one uses the ideas to get a δ-independent bound on $\|\exp\{-t(H_0 + V_\delta)\}\|_{\infty,\infty}$ with V_δ given in construction (b) above.)

Our construction uses the following result:

THEOREM A.1.1. *Let M, N, K be smooth, finite matrix-valued, functions on (a, b) with*

$$K(x) = \frac{dM}{dx} + N(x).$$

Suppose that $\lim_{x\downarrow a} M(x)$ *exists, and that*

$$\alpha = \int_a^b \|N(x)\|\,dx < \infty, \quad \beta = \int_a^b \|M(x)K(x)\|\,dx < \infty, \quad \gamma = \sup_{a<x<b} \|M(x)\| < 1.$$

Then, for any vector u_0, *there is a unique* C^1 *function*, u, *on* (a,b) *obeying* $\dot{u}(x) = K(x)u(x)$ *and* $\lim_{x\to a} u(x) = u_0$. *Moreover, if* u_0 *is fixed but* M, N *are varied in such a way that* α, $\beta, \gamma \to 0$, *then* $u(x) \to u_0$ *uniformly on* (a,b).

Proof: Let $v(x) = (1 - M(x))u(x)$. Since $(1 - M)$ is invertible for all x $(\gamma < 1)$, it is easy to see that $\dot{u} = Ku$ if and only if $\dot{v} = (N - MK)(1 - M)^{-1}v$, and $\lim_{x\to a} u(x) = u_0$ if and only if $\lim_{x\to a} v = (1 - M(a))u_0 = v_0$. Since, furthermore, $\|(N - MK)(1 - M)^{-1}\| \in L^1([a,b])$ we can solve for v by standard methods.

Remark. This theorem is a result of Wintner [38], [39]. In [24] it is attributed to Dollard–Friedman [10] who rediscovered it. We learned of Wintner's work from Devinatz.

Now, to solve $H\phi = 0$, $\phi \geqq 0$, we try $\phi(x) = |x|^{-1}u(|x|)$ (as usual) with $u''(r) = V(r)u(r)$. Letting $u(x) = \mu(x)x + \nu(x)$ and $u'(x) = \mu(x)$, we see that the differential equation is equivalent to

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix}' = K(x)\begin{pmatrix} \mu \\ \nu \end{pmatrix} \tag{A.1.11}$$

with

$$K(x) = V(x)\begin{pmatrix} x & 1 \\ x^2 & x \end{pmatrix}.$$

Because of the form of V, K obeys the hypothesis of Theorem A.1.1, and as $\epsilon \to 0$ the parameter α, β, γ (with $[a,b] = [1,2]$) all converge to zero. We now solve (A.1.11) on $[1,2]$ with $(\mu(1), \nu(1)) = (1,0)$ and define

$$u(x) = \begin{cases} x & \text{for } 0 < x < 1, \\ \mu(x)x + \nu(x) & \text{for } 1 < x < 2, \\ \mu(2)x + \nu(2) & \text{for } 2 < x < \infty. \end{cases}$$

As $\epsilon \downarrow 0$, $\inf u(x)/x \to 1$; thus we can find a suitable solution for ϵ small. This shows that e^{-tH} defines a bounded semigroup on L^∞.

Appendix 2. Estimates on Green's Functions

Let $Q_s(x, y)$ be the kernel given in (3.8). Our main goal is to prove Theorem 3.2 for the case where Ω is a ball, but we shall prove (3.11) for fairly general Ω.

LEMMA A.2.1. *Let Ω be the unit ball about zero. Then*

$$c(1-r) \leqq \psi(\mathbf{r}) \leqq d(1-r) \tag{A.2.1}$$

for some $0 < c, d < \infty$.

Proof: This is an immediate consequence of the fact that ψ vanishes only at $|\mathbf{r}| = 1$, ψ is C^∞ up to the boundary, ψ is radial and $\partial\psi/\partial r(r = 1) \neq 0$.

LEMMA A.2.2. *Let Ω be the unit ball about zero in $\mathbb{R}^\nu$. Then, there are constants $m, C, D < \infty$ such that for an orthonormal basis of eigenfunctions ϕ of H_0^Ω we have*

$$|\phi(\mathbf{r})| \leqq C(E+1)^m \tag{A.2.2}$$

and

$$|\phi(\mathbf{r})| \leqq D(E+1)^{m+1}(1-|\mathbf{r}|), \tag{A.2.3}$$

where E is defined by $H\phi = E\phi$.

Proof: By separation of variables, we can find a basis of eigenfunctions of the form

$$\phi(\mathbf{r}) = r^{-(\nu-1)/2} u(r) Y_l^m(\hat{r}),$$

where Y_l^m is a spherical harmonic and u obeys

$$-u'' + L(L+1)r^{-2}u = 2Eu$$

with $L = l + \frac{1}{2}(\nu - 3)$. Now, $(H_0^\Omega + 1)^{-m}$ maps L^2 to L^∞, if m is sufficiently large, so that (A.2.2) is immediate. This yields (A.2.3) in the region $|\mathbf{r}| < \frac{1}{2}$. Noticing that $-\frac{1}{4} \leqq L(L+1) \leqq 2E$ (the latter since $\int_{|r| \leqq 1} r^{-2} L(L+1) u^2 \, dr \leqq 2E$) we see that, by (A.2.2),

$$|u''(x)| \leqq C'(E+1)^{m+1}, \qquad \tfrac{1}{2} \leqq x \leqq 1, \tag{A.2.4}$$

where, without loss of generality, we can assume that $C' > 96$. We claim that

$$|u'(1)| \leqq C'(E+1)^{m+1}. \tag{A.2.5}$$

Suppose the contrary and assume, without loss of generality, that $u'(1) > 0$. Then, by (A.2.4),

$$u'(x) \geqq \tfrac{1}{2} C'(E+1)^{m+1}, \qquad \tfrac{1}{2} < x < 1,$$

which since $u(1) = 0$ implies that

$$u(x) \geqq \tfrac{1}{2} C'(E+1)^{m+1}(1-x), \qquad \tfrac{1}{2} < x < 1,$$

so that

$$\int_{\frac{1}{2}}^{1} |u(x)|^2 \, dx \geqq \frac{1}{96} C'(E+1)^{m+1} > 1,$$

which violates the normalization condition. Thus (A.2.5) holds. Using (A.2.4) again we get

$$|u'(x)| \leqq \tfrac{3}{2} C'(E+1)^m,$$

which yields (A.2.3).

THEOREM A.2.3. *Let ϕ_n, E_n be an orthonormal basis of eigenfunctions for H_0^Ω, with Ω a unit sphere in $\mathbb{R}^\nu$. Then, for $s > 0$,*

$$Q_s(x, y) \equiv \lim_{E \to \infty} \sum_{E_n \leqq E} e^{-sE_n} \big[\phi_n(x)\psi(x)^{-1}\big]\big[\phi_n(y)\psi(y)\big]$$

converges. The convergence is uniform on set of the form $\bar{\Omega} \times \bar{\Omega} \times [s_0, \infty)$, with $s_0 > 0$, and Q_s obeys

$$|Q_s(x, y)| \leqq C_s(1 - |y|)^2. \tag{A.2.6}$$

Proof: Given the above bounds and the fact that $\#\{E_n \leqq E\}$ is bounded by $C \cdot E^{\nu/2}$ (see e.g., [25]) the convergence is easily seen, as is the bound (A.2.6).

THEOREM A.2.4. *Q_s is continuous on $\bar{\Omega} \times \bar{\Omega} \times (0, \infty)$ and obeys* (3.9), (3.10).

Proof: Continuity follows from the obvious continuity of the sums $\sum_{E_n \leqq E}$ and the uniform convergence. (A.2.6) implies (3.9), and (3.10) follows from the orthonormality of the ϕ_n.

We shall prove (3.11) for the following class of sets:

DEFINITION. We say that Ω is a rounded convex set if and only if Ω is a bounded open convex set, and there is an $r > 0$ with which for every $x \in \partial\Omega$ there exists an open ball B_x of radius r, with $x \in \partial B_x$ and $B_x \subset \Omega$.

Remarks 1. In some sense r is just an upper bound on the radius of curvature of $\partial\Omega$.

2. Except in $\nu = 1$, cubes are obviously not rounded but as products of interva s (which are rounded) they clearly obey (3.11) once we know the bound for rounded sets.

LEMMA A.2.5. *Let Ω be a rounded convex set. Then*

$$c\,\mathrm{dist}(x, \partial\Omega) \leqq \psi(x) \leqq d\,\mathrm{dist}(x, \partial\Omega) \tag{A.2.7}$$

for suitable $0 < c, d < \infty$.

Proof: Since $H_0^{\Omega}\psi = \alpha\psi$,

$$\psi(x) = \alpha\Big[\big(H_0^{\Omega}\big)^{-1}\psi\Big](x). \tag{A.2.8}$$

For any $x \in \Omega$, let $y \in \partial\Omega$ be chosen so that $\mathrm{dist}(x, \partial\Omega) = \mathrm{dist}(x, y)$ and let π_x be the half-space with $\Omega \subset \pi_x$ and $y \in \partial\pi_x$. Then by (A.2.8) and the monotonicity of the integral kernel of $(H_0^R)^{-1}$ in the region R, we obtain

$$\psi(x) \leqq \alpha\big[(H_0^{\pi_x})^{-1}\psi\big](x). \tag{A.2.9}$$

Let $G(y,z)$ be the integral kernel of $(H_0^{\pi_x})^{-1}$. Then, by the method of images, we have

$$\frac{\partial G}{\partial(y\hat{e})}(y,z) \leqq c|y - z|^{-(\nu-1)}$$

if $\hat{e}$ is the direction perpendicular to $\partial\pi_x$. This implies that, with $\eta(y) = (H_0^{\pi_x}\psi)(y)$, $\eta(y) \leqq c\|\psi\|_\infty(y \cdot \hat{e})$ which proves the upper bound in (A.2.7).

For the lower bound, we first note that since ψ is continuous and non-vanishing on Ω, we can find a non-zero lower bound, b, on ψ in the region $R_1 = \{y | \mathrm{dist}(y, \partial\Omega) \geqq \frac{1}{2}r\}$. Now, let $x \in \Omega$, with $\mathrm{dist}(x, \partial\Omega) \leqq r$. Let $y \in \partial\Omega$ be chosen with $\mathrm{dist}(y, x) = \mathrm{dist}(x, \partial\Omega)$ and let B_y be a ball of radius r as in the definition of roundedness. Then, by (A.2.8),

$$\psi(x) \geqq \alpha\Big[\big(H_0^{B_y}\big)^{-1}\psi\Big](x).$$

Using the contribution of all points, z, in the ball B_y with $|z - z_0| \leqq \frac{1}{2}r$, where z_0 is the center of B_y, we obtain the lower bound in (A.2.7).

The penultimate preparatory step to proving (3.11) is to verify it for a half-line:

LEMMA A.2.6. *For $x, y, \in [-\infty, \infty)$, let*

$$P_s(x, y) = (2\pi s)^{-1/2}\exp\{-(x - y)^2/4s\}$$

and, for $x, y > 0$, let

$$Q_s(x, y) = x^{-1}y[P_s(x, y) - P_s(x, -y)].$$

Then, for any ϵ, there is a C_ϵ with

$$Q_s(x, y) \leqq C_\epsilon P_{(1+\epsilon)s}(x, y) \tag{A.2.10}$$

for all $s > 0$, $x, y > 0$.

Proof: By scaling we can suppose, without loss of generality that $s = 1$. We consider separately the two cases: $y \geqq 2x$ and $y \leqq 2x$. If $y \leqq 2x$, then

$$Q_1(x, y) \leqq 2P_1(x, y) \leqq 2(1 + \epsilon)^{1/2} P_{(1+\epsilon)}(x, y),$$

since $s^{1/2}P_s(x, y)$ is monotone increasing in s.

If $y \geqq 2x$, we proceed as follows:

$$\begin{aligned} &Q_1(x, y)[P_{(1+\epsilon)}(x, y)]^{-1} \\ &\quad = (1 + \epsilon)^{1/2}\left[\exp\{-\tfrac{1}{2}(1 + \epsilon)^{-1}\epsilon(x - y)^2\}\right]\left[x^{-1}y(1 - e^{-2xy})\right] \\ &\quad \leqq (1 + \epsilon)^{1/2} 2y^2 \exp\{-(8 + 8\epsilon)^{-1}\epsilon y^2\} \end{aligned}$$

which is bounded.

LEMMA A.2.7. *For $x, y \in \mathbb{R}^\nu$, let*

$$P_s(x, y) = (2\pi s)^{-\nu/2} \exp\{-(x - y)^2/2s\} \tag{A.2.11}$$

and, for x, y with $x_1, y_1 > 0$, let

$$Q_s(x, y) = x_1^{-1}y_1[P_s(x, y) - P_s(x, (-y_1, y_2, \cdots, y_3))].$$

Then, for every ϵ, there is a C_ϵ such that (A.2.10) *holds.*

Proof: Make the ν-dependence explicit by writing $P_s^{(\nu)}$, $Q_s^{(\upsilon)}$. Then

$$P_s^{(\nu)}(x, y) = \prod_{i=1}^{\nu} P_s^{(1)}(x_i, y_i)$$

and

$$Q_s^{(\nu)}(x, y) = Q_s^{(1)}(x_1, y_1) \prod_{i=2}^{\nu} P_s^{(1)}(x_i, y_i).$$

Thus, (A.2.10) follows from the one-dimensional case (Lemma A.2.7) and the bound

$$P_s^{(1)}(x, y) \leqq (1+\epsilon)^{1/2} P_{(1+\epsilon)s}^{(1)}(x, y)$$

which follows from the monotonicity of $s^{1/2} P_s^{(1)}(x, y)$ in s.

THEOREM A.2.8. *Let Ω be a rounded convex set in $\mathbb{R}^\nu$. Let P_s be given by* (A.2.11) *and let*

$$Q_s(x, y) = \psi(x)^{-1}\psi(y) P_s^{\Omega}(x, y),$$

P^{Ω} being the integral kernel of the semigroup generated by the Dirichlet Laplacian in Ω. Then (A.2.10) *holds for any $\epsilon > 0$. Moreover, C_ϵ depends only on ν and on the ratio*

$$\sup[\operatorname{dist}(x, \partial\Omega)\psi(x)] / \inf[\operatorname{dist}(x, \partial\Omega)\psi(x)].$$

In particular, the same C_ϵ will work for all balls.

Proof: We bound instead

$$\operatorname{dist}(x, \partial\Omega)^{-1} P_s^{\Omega}(x, y) \operatorname{dist}(y, \partial\Omega) \tag{A.2.12}$$

by $C_\epsilon P_{(1+\epsilon)s}(x, y)$ independently of Ω and then use Lemma A.2.5. Given x, pick $z \in \partial\Omega$ so that $\operatorname{dist}(z, x) = \operatorname{dist}(x, \partial\Omega)$ and let π be the half-space with $z \in \partial\pi$, and $\Omega \subset \pi$. By translation and rotation, we can suppose without loss of generality that $\pi = \{z \mid z_1 > 0\}$. Then $\operatorname{dist}(x, \partial\Omega) = x_1$ and $\operatorname{dist}(y, \partial\Omega) \leqq y_1$. Since P^{Ω} is monotone in Ω, we see that (A.2.12) is bounded by

$$y_1 x_1^{-1}[P_s(x, y) - P_s(x, (-y_1, y_2, \cdots, y_\nu))]$$

so Lemma A.2.7 yields the required bound.

Appendix 3. Some Remarks on First Exit Times

Let Ω be an open subset of $\mathbb{R}^\nu$ and let T be the first hitting time for $\mathbb{R}^\nu \backslash \Omega$. The first fact that we want to note helps explain why Chung–Rau [7] obtain results which depend only on $|\Omega|$.

THEOREM A.3.1. *Let f be a monotone non-decreasing function on* $[0, \infty)$. *Then*

$$E_x(f(T)) \leqq E_0(f(\tilde{T})),$$

where $\tilde{T}$ is the first exit time for $\tilde{\Omega}$—the ball about zero with $|\tilde{\Omega}| = |\Omega|$ ($|A| \equiv$ Lebesgue measure of A).

Proof: By standard arguments, it suffices to consider the case $f(s) = 0$ (respectively 1) for $s \leqq a$ (respectively $s > a$), i.e.,

$$P_x(T > a) \leqq P_0(\tilde{T} > a). \tag{A.3.1}$$

Let Ω_n be an increasing sequence of open sets with $\overline{\Omega}_n \subset \Omega_{n+1}$ and $\cup \Omega_n = \Omega$. Then

$$P_x(T > a) = \lim_{n\to\infty}\lim_{m\to\infty} P_x\big(b(ja/m) \in \overline{\Omega}_n;\, j = 0, 1, \cdots, m\big),$$

where

$$\begin{aligned} &P_x\big(b(ja/m) \in \overline{\Omega}_n;\, j = 0, \cdots, m\big) \\ &\quad = \int G_m(x - x_1)\chi_n(x_1)G_m(x_1 - x_2)\chi_n(x_2) \cdots \chi_n(x_m)\, d^\nu x_1 \ldots d^\nu x \end{aligned} \tag{A.3.2}$$

with G_m a Gaussian function, and χ_n the characteristic function of Ω_n. A general result of Brascamp et. al. [4] asserts that any integral like (A.3.2) increases if every function is replaced by its spherically decreasing rearrangement. For $G_m(x - \cdot)$ this is just $G_m(\cdot)$, for $G_m(\cdot)$ it is $G_m(\cdot)$ and for χ_n it is dominated by $\tilde{\chi}$, the characteristic function of $\tilde{\Omega}$. Thus

$$P_x\big(b(ja/m) \in \overline{\Omega}_n;\, j = 0, \cdots, m\big) = P_0\big(b(ja/m) \in \tilde{\Omega};\, j = 0, \cdots, m\big)$$

which yields (A.3.1) upon taking limits.

The second result involves an explicit formula for the joint distribution of T and $b(T)$:

THEOREM A.3.2. *Let Ω be a bounded open set with smooth boundary. Suppose that:*

(i) *the integral kernel $P_s(x, y)$ of $\exp\{-sH_0^\Omega\}$ has, for each fixed $s > 0$ and $x \in \Omega$, an extension to $\overline{\Omega}$ which is C^1 up to the boundary,*

(ii) *the restrictions to $\partial\Omega$ of functions which are harmonic in Ω, and C^1 up to the boundary, is dense in the set of continuous functions on $\partial\Omega$.*

Then for fixed x the joint probability distribution of $T = s$ and $b(T) = y$ is

$$\frac{1}{2}\frac{\partial}{\partial n_y} P_s(x, y)\, d\sigma(y)\, ds, \tag{A.3.3}$$

where n_y is an inward pointing normal and $d\sigma$ is the conventional surface measure.

Remarks. 1. We shall not worry about when suppositions (i), (ii) hold. They should be valid under rather minimal conditions. It is easy to verify them for balls and annuli.

2. For the case of balls and annuli, $P_s(x, y)$ has an explicit spherical harmonic expansion from which one can easily read off $E_x(e^{-sT}Y(b(t)))$ (where Y is a spherical harmonic) in terms of Bessel functions. Thus we obtain, again, recent results of Wendel [37].

3. There is an analogous formula when Ω is the complement of a bounded set.

4. Smoothness of $\partial\Omega$ everywhere is not really essential. For example, one can prove this formula for Ω when it is a cube.

5. Both E. Dynkin and S.R.S. Varadhan have emphasized to us that this formula is just a "parabolic" analogue of the fact that exit distributions yield harmonic measures which are normal derivatives of fundamental solutions, and that therefore this formula is "obvious".

We shall need the following lemma, which is of independent interest.

LEMMA A.3.3. *Let Ω be an arbitrary open set and h an arbitrary bounded function on $\partial\Omega$. Define f on $\bar{\Omega}$ by*

$$f(y) = \begin{cases} h(y) & \text{if } y \in \partial\Omega, \\ E_y(h(b(T))) & \text{if } y \in \Omega. \end{cases}$$

Then, for any $s > 0$ and any $x \in \Omega$,

$$f(x) = E_x(f(b(T \vee s))),$$

where $T \vee s = \min(s, T)$.

Proof: By the Markov property, the E_x-distribution of $b(t + s)$ conditioned on $T > s$ and $b(s) = y$ is just the E_y-distribution of $b(t)$. Thus

$$E_x(h(b(T)); T > s) = \int P_x(b(s) \in dy; T > s) E_y(h(b(T)))$$

$$= E_x(f(b(s)); T > s),$$

which proves the result.

Proof of Theorem A.3.2: Let a function h be harmonic in Ω and C^1 up to $\partial\Omega$. Then, by Green's formula, for any s, x:

$$\frac{1}{2} \int \frac{\partial P_s}{\partial n_y}(x, y) h(y)\, d\sigma(y) = \int -\frac{1}{2} \Delta_y P_s(x, y) h(y)\, d^\nu y$$

$$= -\frac{d}{ds} \int P_s(x, y) h(y)\, d^\nu y,$$

where we used $P_s(x, y) = 0$ for $y \in \partial\Omega$ and $\Delta h = 0$. Integrating over s from 0 to t, we find

$$\int_0^t ds \int \left(\frac{1}{2} \frac{\partial P_s}{\partial n_y}(x, y) \right) h(y)\, d\sigma(y) = h(x) - \int P_s(x, y) h(y)\, d^\nu y.$$

But, by the path integral formula for P_s,

$$\int P_s(x, y) h(y)\, d^\nu y = E_x(h(b(s)); T > s).$$

Since

$$h(x) = E_x(h(b(T))),$$

the lemma tells us that

$$\int_0^t ds \int \left[\frac{1}{2} \frac{\partial}{\partial n_y} P_s(x, y) \right] h(y) d\sigma(y) = E_x(h(b(T)); T \leqq s).$$

In view of the assumed density of these trial h's we have identified the required joint distribution.

If $\psi(x)$ is the ground state of H_0^Ω and $H_0^\Omega \psi = \alpha\psi$, then, at least formally,

$$\int \psi(x) E_x(T \in d\sigma(y), T \in ds)\, d^\nu x$$

$$= \frac{1}{2} \frac{\partial}{\partial n_y} \left[\int P_s(x, y) \psi(x)\, d^\nu x \right] d\sigma(y)\, ds$$

$$= \left[\left(\frac{1}{2\alpha} \right) \frac{\partial \psi}{\partial n_y}(y)\, d\sigma(y) \right] \left[e^{-\alpha s} \alpha\, ds \right]$$

in line with our discussion in Section 3.

Let us sketch the ideas that should lead to a proof of (3.5). Since it is peripherial to our concerns, we do not try to give technical details. Because of the independence of $b(T)$ and T and the known $\alpha e^{-\alpha s} ds$ distribution of T, we need only to show that

$$\frac{d}{ds} P(T \leqq s; b(T) \in A) \Big|_{s=0} = \frac{1}{2} \int_A \frac{\partial \psi}{\partial n}\, d\sigma$$

for A a small set near y_0. But for s near zero all paths must come from very near A. Therefore one should be able to reduce this to a statement about the one-dimensional Brownian motion:

$$\lim_{s\downarrow 0} \int_{|x|\leqq 1} \frac{ax}{s} P_x(T \leqq s)\, dx = \frac{1}{2} a$$

which, given the explicit formula for T in one dimension, is equivalent to

$$\lim_{s\downarrow 0} \int_0^1 dx \left(\frac{x}{s} \right) \int_0^s (2\pi t)^{-1/2} \left(\frac{x}{t} \right) \exp\{-x^2/2t\}\, dt = \frac{1}{2}$$

verifiable by an elementary calculation.

Appendix 4. The Dirichlet Problem for Schrödinger Operators

Let Ω be a fixed but arbitrary bounded open region in $\mathbb{R}^\nu$. We let H_0^Ω denote the Dirichlet Laplacian on Ω times $\frac{1}{2}$, i.e., the Friedrichs extension of $-\frac{1}{2}\Delta$ on $C_0^\infty(\Omega)$. For any continuous path b on $[0, \infty)$, we define $T(b) = \inf\{s > 0 \mid b(s) \notin \Omega\}$. By continuity, $b(T(b)) \in \partial\Omega$. We recall (see, e.g., Port and Stone [22]) that a point $y \in \partial\Omega$ is called *regular* if and only if $P_y(T = 0) = 1$, where P_y is the probability for Brownian motion starting at y. If there is an open cone, K, with vertex y so that $K \cap \{x \mid |x - y| < R\} \cap \Omega = \emptyset$, then y is a regular point (see [22], Proposition 2.3.3). In particular, if Ω is convex, every $y \in \partial\Omega$ is regular.

In this section we want to study the Dirichlet problem for the Schrödinger equation. We emphasize that we know of no direct application of the solution of this problem to quantum mechanics (although, as we have seen, the explicit form of solutions of $Hu = 0$ in terms of their boundary values is significant, at least for nice Ω). Nevertheless, it is a natural mathematical problem with a solution which is easy to describe in terms of the ideas in this paper. We shall prove the following:

THEOREM A.4.1. *Let Ω be a bounded open subset of $\mathbb{R}^\nu$ and let $V \in K_\nu^{\rm loc}$. If $\inf \operatorname{spec}(H) > 0$, for $H = H_0^\Omega + V$ as an operator on $L^2(\Omega)$, then for any function f in $L^\infty(\partial\Omega)$, the expectation (T is the stopping time described above)*

$$(Mf)(x) = E_x\left(\exp\left\{-\int_0^T V(b(s))\, ds\right\} f(b(T))\right) \tag{A.4.1}$$

defines a bounded continuous function on Ω which is a distributional solution of $Hu = 0$ (i.e., $(H\phi, Mf) = 0$ for all $\phi \in C_0^\infty(\Omega)$). Moreover, if $y \in \partial\Omega$ is a regular point and if f is continuous at y, then

$$\lim_{\substack{x\to y\\ x\in\Omega}} (Mf)(x) = f(y). \tag{A.4.2}$$

For bounded V a closely related result has been obtained previously by Chung and Rao [7].

Before turning to the proof of this theorem, we want to note that the condition $\inf \operatorname{spec}(H) > 0$ is more or less necessary for Mf even to be finite. For example, let us show that if $V \leqq 0$ and $f = 1$, then $(Mf)(x) < \infty$ for a single $x \in \Omega$ implies that $E_0 = \inf \operatorname{spec}(H) > 0$. For H has compact resolvent since V is relatively form-bounded on $L^2(\Omega)$ with respect to H_0^Ω with relative bound zero on account of $V \in K_\nu^{\text{loc}}$. Thus, there exists a non-zero $u \in L^\infty$ with $Hu = E_0 u$ and, by general principles, $u \geqq 0$. By Harnack's inequality, $u(x) > 0$ for all x. Let

$$a_n = E_x\left(\exp\left\{-\int_0^n V(b(s))\,ds\right\}; T \geqq n\right).$$

Then, by the Feynman–Kac formula,

$$a_n = \left[\exp(-nH)1\right](x) \geqq \|u\|^{-1}(e^{-nH}u)(x) \geqq Ce^{-nE}$$

with $C \neq 0$. But $(Mf)(x) < \infty$ and $V \leqq 0$, implies that $a_n \to 0$ as $n \to \infty$ by a simple use of the dominated convergence theorem. Thus $E_0 > 0$.

The defect in the theorem is that if $0 \notin \operatorname{spec}(H)$, then one would expect to be able to solve the Dirichlet problem; for nice Ω's and V's one can do this by analytic methods (in terms of the normal derivatives of the integral kernel of H^{-1}).

We also note that if there is a function u which is strictly positive in a neighborhood of Ω and $Hu = 0$, then it is not hard to show that $E_0 > 0$; see Theorem A.4.9.

We prove Theorem A.4.1 through a sequence of lemmas:

LEMMA A.4.2. *Let* $V \in K_\nu^{\text{loc}}$. *If* $E_0 = \inf \operatorname{spec}(H) > 0$, *then* Mf *given by* (A.4.1) *defines a bounded map from* $L^\infty(\partial\Omega)$ *to* $L^\infty(\Omega)$.

Proof: Clearly, $|(Mf)(x)| \leqq |(M\mathbf{1})(x)|\,\|f\|_\infty$, so we need only show that $|(M\mathbf{1})(x)| < \infty$. Let

$$a_n(x) = E_x\left(\exp\left\{-\int_0^T V(b(s))\,ds\right\}; n \leqq T < n+1\right).$$

Let χ be the characteristic function of Ω. Then

$$a_1(x) \leqq E_x\left(\exp\left\{+\int_0^1 |(\chi V)|(b(s))\,ds\right\}\right);$$

hence

$$\sup_x |a_1(x)| < \infty$$

in view of the fact that $\chi V \in K_\nu$ (see Theorem 4.7). But, clearly,

$$a_n(x) = (e^{-nH}a_1)(x),$$

and thus, for $n \geqq 2$,

$$\|a_n\|_\infty \leqq \|e^{-H}\|_{2;\infty}\|\exp\{-(n-1)H\}a_1\|_2$$

$$\leqq \|e^{-H}\|_{2;\infty}\exp\{-(n-1)E_0\}\|a_1\|_\infty\left[\int_\Omega dx\right]^{1/2}.$$

Since $\chi V \in K_\nu$, e^{-H} is bounded from L^1 to L^∞; consequently,

$$\|a_n\|_\infty < D\exp\{-nE_0\}.$$

Since $E_0 > 0$,

$$\sum_n \|a_n\|_\infty < \infty.$$

LEMMA A.4.3. *If $E_0 > 0$, $V \in K_\nu^{\mathrm{loc}}$, then, for some $\epsilon > 0$,*

$$\sup_x E_x\left(\exp\left\{-(1+\epsilon)\int_0^T V(b(s))\,ds\right\}\right) < \infty. \tag{A.4.3}$$

Proof: In view of the previous lemma, it is sufficient to show that $E_\epsilon = \inf\operatorname{spec}(H_0^\Omega + (1+\epsilon)V) > 0$ for some $\epsilon > 0$. But, since V is H_0^Ω-form bounded with relative bound zero, E_ϵ is continuous in ϵ.

LEMMA A.4.4. *Let $V \in K_\nu^{\mathrm{loc}}$, $g \in L^\infty(\Omega)$. Then*

$$E_x\left(\int_0^T V(b(s))g(b(s))\,ds\right) = \left((H_0^\Omega)^{-1}Vg\right)(x). \tag{A.4.4}$$

Proof: Since $\chi V \in K_\nu$, the right side is the limit as $n, m \to \infty$ of the same thing with V replaced by $\min[\max(-n, V(x)), m]$ (see Theorem 4.16). By using monotone convergence theorems, one concludes that it suffices to consider the case $V \in L^\infty$ and thus, replacing Vg by g, to suppose that $V = 1$. Also we can assume $g \geqq 0$. Clearly,

$$E_x\left(\int_0^T g(b(s))\,ds\right) = \int_0^\infty E_x(g(b(s)); s < T)\,ds$$

$$= \int_0^\infty \left(\exp\{-sH_0^\Omega\}\,g\right)(x)\,ds$$

$$= \left[(H_0^\Omega)^{-1}g\right](x).$$

The last two equalities are only intended to be true in the L^2 sense. However (using the fact that the distributional Laplacian of $(H_0^{\Omega})^{-1}h$ is h) it is easy to see that both sides of (A.4.4) are continuous in x (when $V = 1$); hence equality in the L^2 sense implies equality pointwise.

LEMMA A.4.5. *Suppose the hypothesis of Lemma* A.4.2. *For any $f \in L^\infty$ write $g(x) = (Mf)(x)$. Then*

$$g(x) = E_x(f(b(T))) - E_x\left(\int_0^T V(b(s))g(b(s))\,ds\right). \tag{A.4.5}$$

Proof: We begin by noting that, for any function $q(s)$ and T,

$$\exp\left\{-\int_0^T q(s)\,ds\right\} = 1 - \int_0^T ds\, q(s)\exp\left\{-\int_s^T q(u)\,du\right\}.$$

(A.4.5) follows once we justify the fact that one can take a conditional expectation on $\{b(u)\}_{t \geqq s}$ inside

$$E_x\left(\int_0^T V(b(s))\right)\exp\left\{-\int_s^T V(b(u)\,du)f(b(T))\right\}.$$

To do this, we need only prove finiteness of the expectation when we replace f by $|f|$ and V by $|V|$ in the first place it appears (but *not* in the exponential). However, for this positive integral, we can take the conditional expectation and note that, by the previous lemma, the result

$$\left[(H_0^{\Omega})^{-1}|V|(Mf)\right](x)$$

is finite.

LEMMA A.4.6. *Suppose that the hypotheses of Lemma A.4.2 hold. Then, for any $f \in L^\infty(\partial\Omega)$, $g \equiv Mf$ is continuous on Ω and $(H\phi, g) = 0$ for any $\phi \in C_0^\infty(\Omega)$.*

Proof: As is well known (see, e.g., [22] or note the mean-value property), the first term in (A.4.5) is harmonic. We have already noted the continuity of the second term. Moreover, if $\phi \in C_0^\infty(\Omega)$,

$$(H_0^{\Omega}\phi, g) = (H_0^{\Omega}\phi, h) - \left(H_0^{\Omega}\phi, (H_0^{\Omega})^{-1}Vg\right),$$

h being the first term in (A.4.5). We have used Lemmas A.4.4 and A.4.5. Since h is harmonic we see that

$$(H_0^{\Omega}\phi, g) = -(\phi, Vg).$$

The following is standard (see, e.g., [22], Proposition 2.3.4). Since the proof is so easy, we give it to keep this paper relatively selfcontained.

LEMMA A.4.7. *Let $y \in \partial\Omega$ be a regular point. Fix δ, $t > 0$. Then*

$$\text{(i)}\quad \lim_{\substack{x \to y \\ x \in \Omega}} P_x(T \leqq t) = 1,$$

$$\text{(ii)}\quad \lim_{\substack{x \to y \\ x \in \Omega}} P_x\Big(\sup_{0 \leqq s \leqq T} |b(s) - y| \leqq \delta\Big) = 1,$$

$$\text{(iii)}\quad \lim_{\substack{x \to y \\ x \in \Omega}} P_x(f(b(T))) = f(y),$$

for any bounded function f on $\partial\Omega$ which is continuous at y.

Proof: (iii) follows trivially from (ii). (ii) follows from (i) and the fact that $\lim_{t \downarrow 0} P_x(|b(s) - x| \leqq \frac{1}{2}\delta;\ \text{all}\ 0 \leqq s \leqq t) = 1$ uniformly in x. Thus, we need only prove (i). Note that

$$\begin{aligned} P_z(T \geqq t) &= P_z(b(s) \in \Omega; 0 \leqq s < t) \\ &= \lim_{\epsilon \downarrow 0} P_z(b(s) \in \Omega; \epsilon \leqq s < t) \equiv \lim_{\epsilon \downarrow 0} f_\epsilon(z). \end{aligned}$$

Notice that $f_\epsilon(z) = (\exp\{-\epsilon H_0\} f_0)(z)$, with $H_0 = -\frac{1}{2}\Delta$, implying that $f_\epsilon(z)$ is continuous. Since y is regular, $\lim_{\epsilon \downarrow 0} f_\epsilon(y) = 0$, so given δ, find $\epsilon > 0$ with $f_\epsilon(y) \leqq \frac{1}{2}\delta$. Then find γ such that $|x - y| < \gamma$ implies $f_\epsilon(x) \leqq \delta$. But $f_0(x) < f_\epsilon(x)$ and thus we conclude that

$$\lim_{x \to y} P_x(T \geqq t) = 0.$$

LEMMA A.4.8. *Let y be a regular point in $\partial\Omega$ and let the hypothesis of Lemma A.4.2 hold. Then*

$$\lim_{\substack{x \to y \\ x \in \Omega}} E_x\left(\left|\left[\exp\left\{-\int_0^T V(b(s))\,ds\right\} - 1\right]\right|\right) = 0.$$

Proof: For each $\delta > 0$, write the expectation as a sum of two terms $f_\delta(x)$ and $g_\delta(x)$ corresponding to the expectation over those paths for which $\sup_{0 \leqq s \leqq T} |b(s) - y| \leqq \delta$ and, correspondingly, over those where this fails. We shall prove that $\lim_{\delta \to 0} \sup_x |f_\delta(x)| = 0$ and that $\lim_{x \to y} |g_\delta(x)| = 0$ for any fixed $\delta > 0$, from which the result follows.

Let q be the dual index of $p = (1 + \epsilon)$. Then, by Hölder's inequality,

$$|g_\delta(x)| \leqq P_x\Big(\sup_{0 \leqq s \leqq T} |b(s) - y| > \delta\Big)^{1/q} \alpha$$

with

$$\alpha = 1 + \sup_x \left[E_x\left(\exp\left\{ -\int_0^T (1 + \epsilon) V(b(s))\, ds \right\}\right)\right]^{1/(1+\epsilon)}.$$

By Lemma A.4.3, $\alpha < \infty$; hence by Lemma A.4.7 (ii),

$$\lim_{x \to y} |g_\delta(y)| = 0.$$

Now let χ_δ be the characteristic function of $\{x \,|\, |x - y| \leqq \delta\}$. Then, by Khas'minskii's argument (see Section 1),

$$\sup_x |f_\delta(x)| \leqq \beta(1 - \beta)^{-1},$$

where $\beta = \sup_x E_x(\int_0^T (\chi_\delta V)(b(s))\, ds)$, provided $\beta < 1$. However $\beta \to 0$ as $\delta \to 0$, since $V \in K_\nu^{\text{loc}}$.

Proof of Theorem A.4.1: We have already shown (Lemma A.4.6) that Mf is continuous and a distributional solution of $Hu = 0$. Thus, we need only prove (A.4.2). But

$$|(Mf)(x) - E_x(f(b(T)))| \leqq \|f\|_\infty E_x\left(\left|\exp\left(-\int_0^T V(b(s))\, ds\right) - 1\right|\right).$$

Therefore, the desired result follows from Lemma A.4.7 (iii) and Lemma A.4.8.

The following result is of interest since it tells us when $\inf \operatorname{spec}(H) > 0$.

THEOREM A.4.9. *Let $V \in K_\nu^{\text{loc}}$, and assume Ω is a bounded open set in $\mathbb{R}^\nu$. Suppose that there exists a function u on Ω such that $Hu = 0$ (distributional sense) and*

$$\inf_{x \in \Omega} u(x) > 0. \tag{A.4.6}$$

Then $\inf \operatorname{spec}(H) > 0$.

Remarks 1. For the case $V \leqq 0$, Khas'minskii essentially proved this result in [17]; his proof exploits $V < 0$ heavily.

2. Of course, if $\inf\operatorname{spec}(H) > 0$, there exists a u obeying (A.4.6); for, by Theorem A.4.1, the function

$$u(x) = E_x\left(\exp\left\{-\int_0^T V(b(s))\,ds\right\}\right)$$

obeys $Hu = 0$ and,

$$\inf_{x\in\Omega} u(x) \geqq \inf E_x\left(\exp\left\{-\int_0^T V_+(b(s))\,ds\right\}\right)$$

$$\geqq \inf\left[E_x\left(\exp\left\{\epsilon\int_0^T V_+(b(s))\,ds\right\}\right)\right]^{-1/\epsilon} > 0,$$

with $V_+(x) = \max(V(x), 0)$. The second inequality is Hölder's inequality and the last follows from Khasmin'skii's lemma and the hypothesis $V_+ \in K_\nu^{\mathrm{loc}}$.

Proof: Let $\tilde{u}$ be any strictly positive function on $\bar{\Omega}$ which is C^∞ on Ω. Let $\tilde{V} \equiv \frac{1}{2}\Delta\tilde{u}/\tilde{u}$ and let $f \in C_0^\infty(\Omega)$. Integrating by parts, one finds that

$$\left(f, \left(-\frac{1}{2}\Delta + \tilde{V}\right)f\right) = \frac{1}{2}\int |\tilde{u}(x)|^2\left[\nabla(f\tilde{u}^{-1})(x)\right]^2 d^\nu x.$$

Letting $\tilde{u}_n$ be a smooth set of approximates to u, we see that

$$(f, Hf) = \frac{1}{2}\int |u(x)|^2\left[\nabla(fu^{-1})(x)\right] \tag{A.4.7}$$

at first for smooth f's and then, by a limiting argument, for all $f \in Q(H) = Q(H_0^\Omega)$. (A.4.7) shows that $\inf\operatorname{spec}(H) \geqq 0$. Since Ω is bounded, H has compact resolvent so that if $\inf\operatorname{spec}(H) = 0$, there is an f with $(f, Hf) = 0$. By (A.4.7), this can only happen if $f = u$; i.e., it can only happen if $u \in Q(H_0^\Omega)$. Of course, since u does not vanish on $\partial\Omega$ and functions in $Q(H_0^\Omega)$ vanish on $\partial\Omega$ in some sense, one expects to be able to show that $u \notin Q(H_0^\Omega)$.

Following we give a proof that $u \notin Q(H_0^\Omega)$. The semigroup generated by H_0^Ω is a contraction on all L^p. The ideas of Beurling–Deny (see Theorem XIII.51 of [25] and its proof) imply that, for any constant c, the map $f \to \min(f, c)$ maps $Q(H_0^\Omega)$ into itself. Thus, if $u \in Q(H_0^\Omega)$ so is $(\inf u)\mathbf{1}$. Since $(f, H_0 f) = \frac{1}{2}\int |\nabla f|^2\, d^\nu x$ for *any* $f \in Q(H_0^\Omega)$, we see that if $u \in Q(H_0^\Omega)$, then $\inf\operatorname{spec}(H_0^\Omega) = 0$ which is impossible for a bounded Ω. This contradiction shows that $u \notin Q(H_0)$, and concludes the proof of the fact that $\inf\operatorname{spec}(H) > 0$.

Finally, we want to note an immediate consequence of the strong Markov property of Brownian motion and our definition of M; making the Ω dependence of M explicit by writing M^{Ω}:

PROPOSITION A.4.10. *Let Ω obey the hypothesis of Theorem* A.4.1 *and let Ω' be an open set with $\overline{\Omega}' \subset \Omega$. If $g = M^{\Omega}f \restriction \partial\Omega'$, then $M^{\Omega'}g = M^{\Omega}f$ on Ω'.*

Proof: Without loss of generality we can suppose that $f \geqq 0$. Let $x \in \Omega'$ and let T', T be the first exit times from Ω', respectively Ω. Clearly, $T' < T$ and

$$(M^{\Omega}f)(x) = E_x\left(\exp\left\{-\int_0^{T'} V(b(s))\,ds\right\}\exp\left\{-\int_{T'}^{T} V(b(s))\,ds\right\} f(b(T))\right).$$

Now use the strong Markov property (since t is positive we are justified in taking conditional expectation).

Note. R. Getoor has kindly pointed out to us that by using the general theory of [21] (especially in the form given in M. Sharpe, Ann. Prob. 8, 1980, pp. 1157–1162), some of the technical results of Section 3 (e.g., Theorem 3.3) can be proven. This is because our process q is a special case of Nagasawa's theory.

Acknowledgment. It is a pleasure to thank a number of people for valuable discussions or correspondence: R. Carmona, K. Chung, M. Donsker, J. Doob, E. Dynkin, M. and T. Hoffman-Ostenhof, E. H. Lieb, and S. R. S. Varadhan. In addition we thank R. Carmona, E. Cinlar, and A. Devinatz for telling us about references [17], [19], [21] and [38], [39], respectively. The research for this paper was supported in part by the National Science Foundation under Grant PHY-78-25390-A01 (M. A.) and under Grant MCS-78-01885 (B. S.).

Bibliography

[1] Ahlrichs, R., Hoffman-Ostenhof, M., Hoffman-Ostenhof, T., and Morgan, J., *Bounds on the Decay of Electron Densities with Screening*, Phys. Rev. 23A, 1981, pp. 2106–2117.

[2] Baetman, M. B., and Chadan, K., *Scattering theory with highly singular oscillating potentials*, Ann. Inst. H. Poincaré, A24, 1976, pp. 1–16.

[3] Berthier, A., and Gaveau, B., *Critère de convergence des fonctionelles de Kac et applications en mechanique et en géometrie*, J. Functional Anal. 29, 1978, pp. 416–424.

[4] Brascamp, H., Lieb, E. H., and Luttinger, J. M., *A general rearrangement inequality for multiple integrals*, J. Functional Anal. 17, 1974, pp. 227–237.

[5] Carmona, R., *Regularity properties of Schrödinger and Dirichlet semigroups*, J. Functional Anal. 33, 1979, pp. 259–296.

[6] Chung, K. L., *On stopped Feynman–Kac functionals*, Sem. de Prob. XIV, 1978–79, Lect. Notes in Math. No. 784, Springer-Verlag, New York, Heildelberg, Berlin, 1980.

[7] Chung, K. L., and Rao, K. M., *Sur la théorie du potentiel avec la fonctionelle de Feynman–Kac*, C. R. Acad. Sci. Paris 290A, 1980, pp. 629–631.

[8] Combescure, M., and Ginibre, J., *Spectral and scattering theory for the Schrödinger operator with strongly oscillating potentials*, Ann. Inst. H. Poincaré, A24, 1976, pp. 17–29.

[9] Deift, P., Hunziker, W., Simon, B., and Vock, E., *Pointwise bounds on eigenfunctions and wave packets in N-body quantum systems*, IV, Comm. Math. Phys. 64, 1978, pp. 1–34.

[10] Dollard, J., and Friedman, C., *On strong product integration*, J. Func. Anal. 28, 1978, pp. 309–354.
[11] Doob, J. L., *Conditional Brownian motion and the boundary limits of harmonic functions*, Bull. Soc. Math. France, 85, 1957, p. 431.
[12] Herbst, I., and Sloan, A., *Perturbation of translation invariant positivity preserving semigroups in* $L^2(R)$, Trans. Amer. Math. Soc. 236, 1978, pp. 325–360.
[13] Hoffman-Ostenhof, M., and Hoffman-Ostenhof, T., private communication.
[14] Hoffman-Ostenhof, M., Hoffman-Ostenhof, T., and Simon, B., *On the nodal structure of atomic eigenfunctions*, J. Phys. A13, 1980, pp. 1131–1133.
[15] Hoffman-Ostenhof, M., Hoffman-Ostenhof, T., and Simon, B., *Brownian motion and a consequence of Harnack's inequality, nodes of quantum wave functions*, Proc. A.M.S. 80, 1980, pp. 301–305.
[16] Kato, T., *Schrödinger, operators with singular potentials*, Israel J. Math. 13, 1973, pp. 135–148.
[17] Khas'miniskii, R. Z., *On positive solutions of the equation* $Uu + Vu = 0$, Theoret. Probability Appl. 4, 1959, pp. 309–318.
[18] Kovalenko, V. F., and Semenov, Yu., *Some problems on expansion in generalized eigenfunctions of the Schrödinger operator with strongly singular potentials*, Russian Math. Surveys, 33, 1978, pp. 119–157.
[19] Kunita, H., and Watanabe, T., *Notes on transformations of Markov processes connected with multiplicative functionals*, Mem. Fac. Sci. Kyusha Univ. Ser. A. Math. 17, 1963, pp. 181–191.
[20] Moser, J., *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. 14, 1961, pp. 577–591.
[21] Ngarawa, M., *Time reversal of Markov processes*, Nagoya Math. J. 24, 1964, pp. 177–205.
[22] Port, S. C., and Stone, C. J., *Brownian Motion and Classical Potential Theory*, Academic Press, New York, 1978.
[23] Portenko, N. I., *Diffusion processes with unbounded drift coefficient*, Theoret. Probability Appl. 20, 1976, pp. 27–37.
[24] Reed, M., and Simon, B., *Methods of Modern Mathematical Physics, III. Scattering Theory*, Academic Press, New York, 1979.
[25] Reed, M., and Simon, B., *Methods of Modern Mathematical Physics, IV. Analysis of Operators*, Academic Press, New York, 1978.
[26] Schechter, M., *Spectra of Partial Differential Operators*, North Holland, New York, 1971.
[27] Schechter, M., *Hamiltonian for singular potentials*, Indiana Univ. Math. J. 22, 1972, pp. 483–603.
[28] Schechter, M., *Spectral and scattering theory for elliptic operators of arbitrary order*, Comment. Math. Helv. 49, 1974, pp. 84–113.
[29] Schechter, M., *Scattering theory for the Schrödinger equation with potentials not of short range*, Vekua Jubilee Volume, 1977.
[30] Simon, B., *Pointwise bounds on eigenfunctions and wave packets in N-body quantum systems*, I, Proc. Amer. Math. Soc. 42, 1974, pp. 395–401.
[31] Simon, B., *Functional Integration and Quantum Physics*, Academic Press, New York, 1979.
[32] Simon, B., *Brownian motion,* L^p *properties of Schrödinger operators and the localization of binding*, J. Functional Anal. 35, 1980, pp. 215–229.
[33] Simon, B., *Large time behavior of the* L^p *norm of Schrödinger semigroups*, J. Functional Anal., 40, 1981, pp. 66–83.
[34] Stampacchia, G., *Le probleme de Dirichlet par les equations elliptiques du second ordre à coefficients discontinue*, Ann. Inst. Four. 15, 1965, pp. 189–258.
[35] Stummel, F., *Singuläre elliptische Differentialoperatoren in Hilbertschen Räumen*, Math. Ann. 132, 1956, pp. 150–176.
[36] Trudinger, N., *Linear elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa 27, 1973, pp. 265–308.
[37] Wendel, J. G., *Hitting spheres with Brownian motion*, Ann. Probability 8, 1980, pp. 164–169.

[38] Wintner, A., *On a theorem of Bochner in the theory of ordinary linear differential equations*, Amer. J. Math. 76, 1954, pp. 183–190.

[39] Wintner, A., *Addenda to* [38], Amer. J. Math. 78, 1956, pp. 893–897.

[40] R. Jensen, *Uniqueness of solutions to* $-\Delta u - qu = 0$, Commun. Partial Differential Equations 3, 1978, pp. 1053–1076.

[41] A. Devinatz, *Schrödinger operators with singular potentials*, J. Op. Th. 4, 1980, pp. 25–35.

[42] Chung, K. L., and Varaakan, S. R. S., *Kac function and Schrödinger's equation*, Studia Math. 68, 1980, pp. 249–260.

Received April, 1981.