JOURNAL OF FUNCTIONAL ANALYSIS **40**, 66–83 (1981)

Large Time Behavior of the L^p Norm of Schrödinger Semigroups

BARRY SIMON*

Departments of Mathematics and Physics, Princeton University, Princeton, New Jersey 08544

Communicated by L. Gross

Received April 28, 1980

Let $V \leqslant 0$, $V \in C_0^\infty(R^\nu)$ with $\nu \geqslant 3$ be such that $H = -\frac{1}{2}\Delta + V \geqslant 0$ but for any $\varepsilon > 0$, $-\frac{1}{2}\Delta + (1+\varepsilon)V$ is not positive. We determine the exact rate of divergence of the norm of e^{-tH} as a map from L^∞ to L^∞. A number of related problems are discussed.

1. INTRODUCTION

This paper is a sequel to our earlier paper [8]. For a large class of potentials, V, Carmona [1] and Simon [7] independently showed that the Schrödinger Semigroup e^{-tH}, $H = -\frac{1}{2}\Delta + V$ initially defined on $L^2(R^\nu)$ is an exponentially bounded semigroup on each $L^p(R^\nu)$. If $\|A\|_{p,q}$ is the norm of A as a map from L^p to L^q, then it was proven in [8] that the rate of exponential divergence of $\|e^{-tH}\|_{p,p}$ is independent of p. This p-independence is only for the lending order as $t \to \infty$. In particular, in [8], we gave examples with $\|e^{-tH}\|_{2,2} = 1$ but $\lim_{t\to\infty} \|e^{-tH}\|_{\infty,\infty} = \infty$. One of our main goals in this paper is to determine the precise rate of divergence of this $\|\cdot\|_{\infty,\infty}$-norm for these examples of "critical potentials."

To describe things more precisely, let us begin by recalling some of the results of [1, 7, 8].

DEFINITION. A function f on R^μ is called *uniformly locally* L^p (written L^p_u) if and only if

$$\sup_x \left[\int_{|x-y|\leqslant 1} |f(y)|^p \, dy\right]^{1/p} < \infty.$$

* Research partially supported by USNSF under Grant MCS-78-01885.

0022-1236/81/010066–18$02.00/0

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DEFINITION. $\mathscr{V}_1$ is the smallest class of real-valued functions on R^ν closed under sums and containing (i) any positive L^1_{loc} function: (ii) any function of the form $V(Tx)$, where T is a linear map from R^ν to R^μ and $V \in L^p_u(R^\mu)$ for some $p > (\mu/2)$ ($\mu \geqslant 2$; $p \geqslant 1$ if $\mu = 1$).

DEFINITION. Let $\nu \geqslant 3$. $\mathscr{V}_2$ is the set of real-valued functions on R^ν with $V \in L^p \cap L^q$ for some p, q with $p < \nu/2 < q$.

Later, we will need a third class for some special purposes.

DEFINITION. We say that $V \in \mathscr{V}_3$ if V is $\mathscr{V}_1$ and for any R, there exists a $W_R \in \mathscr{V}_1$ of the form $\sum_{\alpha=1}^{N(R)} f_\alpha(T_\alpha r)$ with T_α a linear map from R^ν to R^{μ_α} and $f_\alpha \in L^p(R^{\mu_\alpha})$ some $p > \frac{1}{2}\mu_\alpha$, so that $V(x) = W_R(x)$ for $|x| < R$.

Thus $V \in \mathscr{V}_3$ has anegative part consisting of potentials of type (ii) in the definition of $\mathscr{V}_1$ and its positive part has *locally* the same structure.

THEOREM 1.1 ([7]). *Let $V \in \mathscr{V}_1$. Then for any p and $t \geqslant 0$, e^{-tH} (defined for $p < \infty$ by density from $L^2 \cap L^p$ and for $p = \infty$ by duality on L^1) is bounded from L^p to L^p with*

$$\|e^{-tH}\|_{p,p} \leqslant Ce^{+tA}. \tag{1.1}$$

For any $t > 0$, $p \leqslant q$, e^{-tH} is bounded from L^p to L^q.

THEOREM 1.2 ([8]). *Let $V \in \mathscr{V}_1$. Then*

$$\alpha_p \equiv \lim_{t\to\infty} t^{-1} \ln \|e^{-tH}\|_{p,p} \tag{1.2}$$

is independent of p.

Remark. In Appendix 1, we extend Theorem 1.2 to include the norm for any $p \leqslant q$. There we also prove that for $V \in \mathscr{V}_3$ and for any positive function $f \in L^1 + L^\infty$, and *any* x

$$\alpha_f(x) \equiv \lim_{t\to\infty} t^{-1} \ln [(e^{-tH}f)(x)] \tag{1.3}$$

exists and equals α_∞ independently of f and x.

DEFINITION. Let $V \in \mathscr{V}_2$. We say V is *supercritical* if $\alpha_\infty(V) > 0$, V is subcritical if $\alpha_\infty((1+\varepsilon)V) = 0$ for some $\varepsilon > 0$ and V is critical if $\alpha_\infty(V) = 0$ but $\alpha_\infty((1+\varepsilon)V) > 0$ for all $\varepsilon > 0$.

DEFINITION. If $\alpha_\infty(V) = 0$, we define $\beta_\infty(V) = \sup_t \|e^{-tH}\|_{\infty,\infty}$.

THEOREM 1.3 ([8]). *Let $V \in \mathscr{K}_2$. If V is subcritical, then $\beta_\infty(V) < \infty$. If V is critical, $\beta_\infty(V) = \infty$.*

In this paper we want to concentrate mainly on the question of how fast $\|e^{-tH}\|_{\infty,\infty}$ diverges as $t \to \infty$ when V is critical. Reference [8] has some information on this:

THEOREM 1.4 ([8]). *Let $V \in \mathscr{K}_1$. Then*

$$\|e^{-tH}\|_{\infty,\infty} \leqslant C(1+t)^{\nu/2}\, e^{\alpha_2(V)t}. \tag{1.4}$$

Remark. Since $\|e^{-tH}\|_{2,2} \leqslant e^{\alpha_\infty(V)t}$ by self-adjointness, duality and interpolation, (1.4) is the main step in the proof of Theorem 1.2.

Let us begin by presenting a convincing argument, using Brownian motion, that in the critical case, $\|e^{-tH}\|_{\infty,\infty}$ diverges at least as fast as $t^{(\nu-2)/2}$ and which strongly suggests this is the precise behavior. Suppose that $V \leqslant 0$ and $V \in C_0^\infty(R^\nu)$. If V is critical, then [8] there exists a function $\eta > 0$ in L^∞ with $H\eta = 0$ which solves the homogeneous integral equation

$$\eta = (-\Delta)^{-1}(-V\eta). \tag{1.5}$$

Since V has compact support and $-V\eta \geqslant 0$, (1.5) implies that

$$\eta \sim C\,|x|^{-(\nu-2)}, \qquad |x| \to \infty, \tag{1.6}$$

for some $C \neq 0$. The Feynman–Kac formula implies that

$$\eta(0) = E\left(\exp\left(-\int_0^t V(b(s))\,ds\right)\eta(b(t))\right) \tag{1.7}$$

with $b(s)$ Brownian motion [7] and E expectation with respect to Brownian motion.

Let $\chi_R(\,\cdot\,)$ be the characteristic function of $\{x \mid |x| < R\}$ and $\chi_R' = 1 - \chi_R$. Brownian motion goes a distance of order $t^{1/2}$ in time t in the sense that

$$E(\chi_{\delta t^{1/2}}(b(t)))$$

can be made small by taking δ small. Thus, it is reasonable to suppose that paths with $|b(t)| \leqslant \delta t^{1/2}$ make a small contribution to (1.7) so that for δ small and fixed and t large:

$$\tfrac{1}{2}\eta(0) \leqslant E\left(\exp\left(-\int_0^t V(b(s))\,dx\right)\eta(b(t))\,\chi'_{\delta t^{1/2}}(b(t))\right). \tag{1.8}$$

But

$$\begin{aligned}\text{RHS of (1.8)} &= (e^{-tH}\eta\chi'_{\delta t^{1/2}})(0)\\ &\leqslant \|e^{-tH}\|_{\infty,\infty}\,\|\eta\chi'_{\delta t^{1/2}}\|_\infty\\ &= \|e^{-tH}\|_{\infty,\infty}\,O(t^{-(\nu-2)/2})\end{aligned}$$

on account of (1.6). Since $\eta(0)\neq 0$, we get the required $t^{(\nu-2)/2}$ lower bound. Some thought about this argument suggests also that $t^{(\nu-2)/2}$ will be an upper bound.

There is one major defect in the above argument; its conclusion is FALSE (except for $\nu=3$, where $t^{1/2}$ is the right answer but preseumably not on account of the above)! (1.8) is false: While not many paths have $|b(t)|\leqslant \delta t^{1/2}$, the contribution of the exponential is sufficiently large to overcome this: the usual attempt to use Hölder's inequality to estimate

$$E\left(\exp\left(-\int_0^t V(b(s))\right)\chi_{\delta t^{1/2}}(b(t))\right)$$

by

$$E\left(\exp\left(-p\int_0^t V(b(s))\right)ds\right)^{1/p} E(\chi_{\delta t^{1/2}}(b(t)))^{1/q}$$

fails because $E(\exp(-p\int_0^t —))$ diverges exponentially in t for any $p>1$. Indeed, for $\nu=5$, our results in this paper show that the small fraction of paths with $|b(t)|\leqslant t^\alpha$ contribute all of the expectation in (1.7) as $t\to\infty$ so long as $\alpha>(\nu-2)^{-1}$.

To end the suspense, let us say we will show that for $\nu\geqslant 5$

$$\lim_{t\to\infty} t^{-1}\|e^{-tH}\|_{\infty,\infty} \tag{1.9}$$

exists an is a non-zero finite number. For $\nu=4$, t^{-1} must be replaced by $(t/\ln t)^{-1}$ and for $\nu=3$, t^{-1} must be replaced by $t^{-1/2}$.

What distinguishes $\nu\geqslant 5$, from $\nu=3$, 4 is that for critical V's, the η we discuss above is in L^2 if $\nu\geqslant 5$ and is not in L^2 if $\nu=3,4$. As we will describe in Section 2, the growth of $\|e^{-tH}\|_{\infty,\infty}$ is intimately related to the growth of

$$\left\|\int_0^t (e^{-sH}V)\,ds\right\|_2. \tag{1.10}$$

If $V\in L^2$ and if $H>0$, as it is in $\nu=3,4$, then $\|e^{-sH}V\|_2\to 0$ so (1.10) is $o(t)$ but if $\eta\in L^2$, then $\|(e^{-sH}V)-(V,\eta)\eta\|_2\to 0$ and if $(V,\eta)\neq 0$, then (1.10) is $O(t)$. We make these arguments precise in Section 2.

To treat the cases $\nu = 3, 4$ and $V \in L^2$ we will have to be a little less direct. By Tauberian theorems the large t divergence of $\int_0^t e^{-sH} V\, ds$ is related to the small α divergence of $\int_0^\infty e^{-s\alpha} e^{-sH} V\, ds = (H+\alpha)^{-1} V$. This divergence is essentially the same as the divergence of $\|(1 + (H_0+\alpha)^{-1/2} V (H_0+\alpha)^{-1/2})^{-1}\|_{2,2}$, a divergence which has been analyzed by Klaus and Simon [2]. These arguments are described in Section 3.

In Sections 2 and 3, we completely analyze the divergence of $\|e^{-tH}\|_{\infty,\infty}$ for cases where $V \in \mathscr{Y}_2$ and $\alpha_\infty(V) = 0$, $\beta_\infty(V) = \infty$. This leaves open the situation when $\alpha_\infty(V) = 0$, $\beta_\infty(V) = \infty$ for more general $V \in \mathscr{Y}_1$. This is something we discuss further in Sections 4 and 5.

2. C_0^∞-Negative Potentials, $\nu \geqslant 5$

In this section, we will prove the following pair of theorems:

Theorem 2.1. *Let $V \in C_0^\infty(R^\nu)$ with $H = -\frac{1}{2}\Delta + V > 0$ (i.e., $(\phi, H\phi) > 0$ for all $\phi \neq 0$, $\phi \in L^2$). Then*

$$\lim_{t\to\infty} t^{-1} \|e^{-tH}\|_{\infty,\infty} = 0. \tag{2.1}$$

Theorem 2.2. *Let $V \in C_0^\infty(R^\nu)$ $\nu \geqslant 5$. Suppose that $V \leqslant 0$, that $H = -\frac{1}{2}\Delta + V \geqslant 0$ and that there is $\eta \in L^2$ with $H\eta = 0$, $\|\eta\|_2 = 1$. Then*

$$\lim_{t\to\infty} t^{-1} \|e^{-tH}\|_{\infty,\infty} = \|\eta\|_\infty \int \eta(x)\, V(x)\, d^\nu x \neq 0. \tag{2.2}$$

Remarks 1. The hypotheses $V \in C_0^\infty$ (and $V \leqslant 0$ in Theorem 2.2) are very restrictive and one can easily use the methods to handle much more general V's. Since we will prove stronger results for general $V \in \mathscr{Y}_2$ in the next section (except that we will not check we get the constant $-\|\eta\|_\infty (\eta, V)$ of (2.2)), we do not bother to make this explicit.

2. We do not use $\nu \geqslant 5$ in our proof of Theorem 2.2 but the existence of such an η *implies* that $\nu \geqslant 5$.

3. Theorem 2.1 is only interesting in the critical case with $\nu = 3, 4$ since, in all other cases where $H > 0$, we know by Theorem 1.3 that $\sup_t \|e^{-tH}\|_{\infty,\infty} < \infty$.

We begin by noting that e^{-tH} is positivity preserving and thus $|e^{-tH} f| \leqslant e^{-tH}|f|$. Moreover, since $|f| \leqslant \|f\|_\infty 1$, we have that $e^{-tH}|f| \leqslant \|f\|_\infty (e^{-tH} 1)$. Thus

$$\|e^{-tH}\|_{\infty,\infty} = \|e^{-tH} 1\|_\infty. \tag{2.3}$$

Next, we note that

$$e^{-tH} = e^{-tH_0} + \int_0^t e^{-sH}(-V)\, e^{(t-s)H_0}\, ds.$$

Applying this to the function 1, we find that

$$e^{-tH}1 = 1 + \int_0^t e^{-sH}(-V)\, ds. \tag{2.4}$$

Proof of Theorem 2.1. Since $H > 0$, $s - (L^2) - \lim e^{-sH} = 0$. Thus $\lim_{s\to\infty} \|e^{-sH}(-V)\|_2 = 0$. But e^{-H} is bounded from L^2 to L^∞ (Theorem 1.1) so $\lim_{s\to\infty} \|e^{-(s+1)H}(-V)\|_\infty = 0$. Since $V \in L^\infty$, $\|e^{-sH}(-V)\|_\infty$ is bounded on $[0, 1]$. It follows that

$$\frac{1}{t}\int_0^t \|e^{-sH}(-V)\|_\infty\, ds \to 0 \tag{2.5}$$

as $t \to \infty$. (2.3), (2.4) and (2.5) imply (2.1). ∎

Proof of Theorem 2.2. Write $-V = a\eta + W$ with $a = (\eta, -V)$ and $W = -V - a\eta$ so that W is orthogonal to η. By general principles [4] η is a simple eigenvector so $H > 0$ on $\{\phi \mid (\phi, \eta) = 0\}$. As in the above proof, we conclude that $\|e^{-tH}W\|_\infty \to 0$. Since (2.4) and $e^{-tH}\eta = \eta$ implies that

$$e^{-tH}1 = 1 + ta\eta + \int_0^t (e^{-sH}W)\, ds$$

we have that

$$\frac{1}{t}\|e^{-tH}1 - a\eta\|_\infty \to 0. \tag{2.6}$$

From (2.6), $a \neq 0$, and (2.3), we conclude (2.2) holds. ∎

We cannot resist pointing out a lucky irony in our choice of the terms "subcritical," etc., in [8]. We had in mind an analogy with the scattering theory for the linearized Boltzmann equation [5], which is a characature of neutron scattering off a chunk of uranium. In that problem there is an L^1-semigroup, $W(t)$, depending on the size of the chunk of uranium. As that size passes from below a critical one to above the semigroup is expected to change from being bounded to becoming exponentially unbounded. This analogy motivated our choice of names. It did not occur to me that there was an even closer analogy in that the behavior was the same at the critical size. In the Boltzman case, there is a simple picture suggesting linear growth

Thus

$$\lim_{\alpha \downarrow 0} \| B_\alpha(x) - (1 + e_0(\alpha))^{-1} P(\alpha)[(H_0 + \alpha)^{-1}(-V)] \|_\infty < \infty, \tag{3.7}$$

i.e., the small α divergence of $B_\alpha(x)$ is uniformly in x due to the small α divergence of $(1 + e_0(\alpha))^{-1}$.

In order to apply Tauberian theorems, one needs a positive integrand. Therefore, write $-V = V_- - V_+$, where $V_\pm = \max(\pm V, 0)$. Equation (3.3) with $-V$ replaced by V_+ we denote as $B_\alpha^\pm$ so that

$$B_\alpha = B_\alpha^- - B_\alpha^+$$

Equation (3.7) remains true with $-V$ replaced by $V_\pm$ and B_α by $B_\alpha^\pm$ by the exact same arguments.

The small α behavior of $e_0(\alpha)$ was analyzed by Klaus and Simon [2] (actually, [2] considers the eigenvalue of the L^2 operator $K_\alpha = V^{1/2}(H_0 + \alpha)^{-1}|V|^{1/2}$, but in [8] it is proven the eigenvalues of K_α on L^2 and L_α on L^∞ are the same).

$$e_0(\alpha) = -1 + c\alpha + \cdots \ (\nu \geqslant 5),$$
$$e_0(\alpha) = -1 + c\alpha \ln \alpha + \cdots \ (\nu = 4),$$
$$e_0(\alpha) = -1 + c\alpha^{1/2} + \cdots \ (\nu = 3).$$

Since $P(\alpha = 0)\psi = \eta l(\psi)$, where l is a suitable linear functional and η obeys $H\eta = 0$ as usual, we find by (3.7) that

$$B_\alpha^\pm(x) = c_\pm \eta(x) f(\alpha) + C_\alpha^\pm(x), \tag{3.8}$$

where

(i) $\| C_\alpha^\pm(\cdot) \|_\infty$ is bounded as $\alpha \to 0$.

(ii) $f(\alpha) \sim \alpha^{-1}$ $(\nu \geqslant 5)$, $(\alpha \ln \alpha)^{-1}$ $(\nu = 4)$, $\alpha^{-1/2}$ $(\nu = 3)$

(iii) $c = c_- - c_+ \neq 0$. This follows because, a calculation shows that $c = (\text{const})\langle \eta, V \rangle$ with the constant non-zero. But

$$\langle \eta, V \rangle = \int \eta(-\Delta\eta/\eta)$$

$$= \int (-\Delta\eta) \neq 0 \text{ since } \eta \sim d\,|x|^{-\nu-2}\, d \neq 0 \text{ as } x \to \infty \ [2]$$

We can now apply the Karamata Tauberian theorem suitably modified to include logarithms (see [7] for a discussion and proof of this theorem) to conclude:

THEOREM 3.1. *Let $V \in \mathscr{T}_2^{\cdot}$ be a critical potential. Then*

$$\lim_{t\to\infty} g(t)^{-1} \|e^{-tH}\|_{\infty,\infty} = d \neq 0, \infty,$$

$$\begin{aligned} g(t) &= t && (\nu \geqslant 5) \\ &= t/(\ln t) && (\nu = 4) \\ &= t^{1/2} && (\nu = 3). \end{aligned}$$

4. SOME MORE RESULTS

In this section, we want to report some additional results about the