# *The Theory of Semi-Analytic Vectors: A New Proof of a Theorem of Masson and McClary*

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**1. Introduction.** Suppose $A$ is a Hermitian operator on a Hilbert space, $\mathcal{H}$. If $\psi \in D(A^n)$ for all $n$, we say $\psi \in C^\infty(A)$. Let us define:

***Definition.*** A vector $\psi \in C^\infty(A)$ is called a *semi-analytic vector* for $A$ if and only if

$$\sum_{n=0}^{\infty} \frac{||A^n\psi||}{(2n)!} t^n < \infty$$

for some $t > 0$.

The name comes from the similarity to the notion of analytic vector [7] for which the *stronger* condition that $\sum_{n=0}^{\infty} (||A^n\psi||/n!)t^n < \infty$ for some $t > 0$ is required. Our major goal in this note is the proof of the theorem:

***Theorem 1.*** *If $A$ is a Hermitian operator which is bounded below so that $D(A)$ contains a dense set of semi-analytic vectors, then $A$ is essentially self-adjoint.*

This theorem is in the genre of the by-now classic theorem of Nelson [7] that a Hermitian operator with a dense set of analytic vectors is essentially self-adjoint (although, we shall see it lacks a certain deep aspect of Nelson's theorem). Theorem 1 is a special case of a theorem of Masson and McClary (Theorem 2, below) which was proven in a recent interesting paper [6]. To state their theorem, let us first recall two definitions:

***Definition.*** A vector $\psi \in C^\infty(A)$ is called *quasi-analytic* if and only if

$$\sum_{n=1}^{\infty} ||A^n\psi||^{-1/n} = \infty,$$

$\psi \in C^\infty(A)$ is called *Stieltjes* if and only if

$$\sum_{n=1}^{\infty} ||A^n\psi||^{-1/2n} = \infty.$$

The relations of the four kinds of vectors is shown by the diagram:



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$$\begin{array}{ccc} \psi \text{ Analytic} & \Rightarrow & \psi \text{ Quasi-analytic} \\ \Downarrow & & \Downarrow \\ \psi \text{ Semi-analytic} & \Rightarrow & \psi \text{ Stieltjes} \end{array}$$

The top row is stronger than the bottom row in that the allowed growth rate of $||A^n\psi||$ is the square root of the growth rate allowed on the bottom. The left is only slightly stronger than the right in that $\psi$ can only be quasi-analytic (Stieltjes) without being analytic (semi-analytic) if $||A^n\psi||$ does not have a fairly regular growth. The Masson–McClary theorem says:

***Theorem 2.*** *(Masson–McClary) If $A$ is a Hermitian operator which is bounded below so that $D(A)$ contains a dense set of Stieltjes vectors, then $A$ is essentially self-adjoint.*

This theorem is to Theorem 1 exactly as Nussbaum's generalization of Nelson's theorem [10] (which says a Hermitian operator with a dense set of quasi-analytic vectors is essentially self-adjoint) is to Nelson's theorem.

In §3, we provide a new proof of Theorem 2 which we feel is more transparent than the proof of Masson–McClary [6] which is somewhat computational and which has a somewhat magical quality. Despite the fact that Theorem 1 follows from Theorem 2, we provide a separate proof of Theorem 1 in §2. This proof is elementary making no use of moment problem lore and we feel explains exactly why theorems of the type we consider here are true. §2 is thus the heart of this note.

**2. The semi-analytic vector theorem.** The proof of Theorem 1 consists of four simple remarks:

(1) Since $A$ is semi-bounded, it possesses self-adjoint extensions by a theorem of Von Neumann [9], in particular, the Friedrichs extension [4, 5] exists. Thus Theorem 1 is not an existence but only a uniqueness theorem; as a result, the deep part of Nelson's Theorem (existence of self-adjoint extensions) is bypassed.

(2) As we show in Appendix 1, a semi-bounded Hermitian operator with a unique semi-bounded self-adjoint extension has a unique self-adjoint extension. Thus, we need only show that $A$ in the theorem has a unique *semi-bounded* self-adjoint extension.

(3) If $A$ has a dense set of semi-analytic vectors, then $A$ has at most one self-adjoint $\tilde{A} > 0$. For let $\psi \in D(A)$ be a semi-analytic vector. Pass to a spectral representation [8] for $\tilde{A}$ so $\tilde{A}$ is multiplication by $x$ on $\bigoplus_{n=1}^{N} L^2(R, d\mu_n)$ ($N = 1, 2, \cdots$ or $\infty$). We know that

$$\sum_{n=0}^{N} \int_0^\infty x^{2m} \, |\psi(x, n)|^2 \, d\mu_n \equiv ||A^m\psi||^2 < (ab^m(2m!))^2 < cd^m(4m)!$$

Thus

$$\sum_{m=0}^{\infty} \sum_{n=1}^{N} \int_0^\infty x^{m/2} \frac{t^m}{m!} \, |\psi(x; n)|^2 \, d\mu_n < \infty \quad \text{if} \quad |t| < d^{-1/4}$$

so by the Dominated Convergence Theorem $\sum_{n=1}^{N} \int_0^\infty e^{x^{1/2}t} |\psi(x, n)|^2 \, d\mu_n < \infty$ if $|t| < d^{-1/4}$. Let $f(y) = \cos(y^{1/2}) = \sum_{n=0}^\infty (-y)^n/(2n)!$. Since $|\cos y^{1/2}| < \exp(|\text{Im } y^{1/2}|)$, we see $\sum_{n=1}^{N} \int_0^\infty |f(t^2x)| \, |\psi(x, n)|^2 \, d\mu_n < \infty$ if $|\text{Im } t| < d^{-1/4}$. Thus $g_\psi(t) = \sum_{n=1}^{N} \int_0^\infty f(t^2x) \, |\psi(x, n)|^2 \, d\mu_n$ is analytic in the strip $|\text{Im } t| < d^{-1/4}$ and is given by the power series $\sum_{m=0}^\infty t^{2n}/2n! \, \langle \psi, (-A)^n\psi \rangle$ if $|t| < d^{-1/4}$. As a result $\langle \psi, \cos(t\tilde{A}^{1/2})\psi \rangle$ is uniquely determined by $\langle \psi, A^n\psi \rangle$ for $t$ real if $\psi$ is semi-analytic and $\tilde{A}$ is a positive self adjoint extension. If $A$ has a dense set of semi-analytic vectors, $\cos(t\tilde{A}^{1/2})$ is uniquely determined independent of $\tilde{A}$. Since the Spectral Theorem implies

$$(\tilde{A} + 1)^{-1} = \int_0^\infty e^{-t} \cos(t\tilde{A}^{1/2}) \, dt.$$

We see that $\tilde{A}$ is uniquely determined.

(4) If $\psi$ is a semi-analytic vector for $A$, it is a semi-analytic vector for $A + M$ for any real number $M > 0$. Thus (3) implies an operator $A$ with a dense set of semi-analytic vectors has at most one extension $\tilde{A}$ with $\tilde{A} > -M$. To see the claim made above, note that if $||A^m\psi|| < ab^m(2m)!$, then

$$||(A + M)^m\psi|| \leqq \sum_{n=0}^{m} \binom{m}{n} M^{m-n} \, ||A^n\psi||$$

$$\leqq a(2m)! \sum_{n=0}^{m} \binom{m}{n} M^{m-n}b^n \leqq a(b + M)^m(2m)!$$

**3. The Masson–McClary Theorem.** Basic to the proof we give of the Masson–McClary Theorem is the notion of vector of uniqueness and a simple lemma of Nussbaum.

***Definition.*** [10]. Let $\psi \in C^\infty(A)$. Let $D_\psi$ be the algebraic linear span of $\psi, A\psi, \cdots, A^n\psi, \cdots$ and let $\mathcal{H}_\psi = \bar{D}_\psi$. Let $A_\psi$ be the operator in $\mathcal{H}_\psi$ with domain $D_\psi$ given by $A_\psi = A|D_\psi$. $\psi$ is called a *vector of uniqueness* if and only if $A_\psi$ is essentially self-adjoint in $\mathcal{H}_\psi$.

***Lemma 1.*** (Nussbaum [10]). *If $A$ has a total set (i.e. a set $S$ whose finite linear combinations are dense) of vectors of uniqueness, then $A$ is essentially self-adjoint.*

*Proof.* (Masson–McClary [6]). We need only show Ran $(A \pm i)$ are dense [8]. Given $\psi \in \mathcal{H}$ and $\epsilon$, find $c_1, \cdots, c_n \in \mathbf{C}$ and $\phi_1, \cdots, \phi_n$ vectors of uniqueness so that $||\psi - \sum_{i=1}^n c_i\phi_i|| < \epsilon/2$. Since $\phi_i$ is a vector of uniqueness we can find $\eta_i \in D_{\psi_i}$ so that $||\phi_i - (A + i)\eta_i|| < \epsilon/2(\sum |c_i|)$. Then $\sum_{i=1}^n c_i\eta_i \in D(A)$ and $||\psi - (A + i) \sum_{i=1}^n c_i\eta_i|| < \epsilon$. As a result, we see Ran $(A + i)$ is dense. Similarly Ran $(A - i)$ is dense.

Nussbaum's theorem on quasi-analytic vectors follows from this lemma by noting: (a) $\psi \in C^\infty(A)$ is a vector of uniqueness if and only if $\langle \psi, A^n\psi \rangle = a_n$ is a sequence for which the Hamburger moment problem has a unique solution.

(b) Employing the well-known uniqueness criterion of Carleman for the

Hamburger Moment Problem. We can provide a parallel proof for the Masson–McClary Theorem:

***Lemma 2.*** *Let $\psi \in C^\infty(A)$ where $A$ is Hermitian and semi-bounded. If the moment problem for $a_n = \langle \psi, A^n \psi \rangle$ has a unique solution in every interval $(-M, \infty)$, then $\psi$ is a vector of uniqueness for $A$.*

*Proof.* Suppose $A_1$ and $A_2$ are two self-adjoint extensions of $A_\psi$ (since $A_\psi$ is bounded from below, it has self-adjoint extensions). If $dE_{1,2}$ are the spectral resolutions of $A_{1,2}$, then $\langle \psi, dE_1 \psi \rangle$ and $\langle \psi, dE_2 \psi \rangle$ both solve the moment problem for $a_n$ in an interval $(-M, \infty)$ ($M = \inf(\sigma(A), \sigma(A_2))$) so they are equal. Thus, also $\langle A_1^m \psi, dE_1(\lambda) A_1^n \psi \rangle = \lambda^{n+m} \langle \psi, dE_1(\lambda)\psi \rangle$ and $\langle A_2^m \psi, dE_2(\lambda) A_2^n \psi \rangle = \lambda^{n+m} \langle \psi, dE_2(\lambda)\psi \rangle$ have identical matrix elements on a dense set in $\mathcal{H}_\psi$, *i.e.* $A_1 = A_2$.

***Lemma 3.*** *If $\sum_{n=0}^\infty |a_n|^{-1/2n} = \infty$, then there is at most one measure $\mu$ on $(-M, \infty)$ with*

$$a_n = \int_{-M}^\infty x^n \, d\mu_n .$$

*Proof.* See Appendix 2

*Proof of Theorem 2.* Let $\psi$ be a Stieltjes vector. Then by Lemma 3, the conditions of Lemma 2 hold so $\psi$ is a vector of uniqueness. Then, by Lemma 1, $A$ is essentially self-adjoint.

The proof of Theorem 1 is related to this proof as follows: steps (3) and (4) of the proof of Theorem 1 (using $(x+1)^{-1} = \int_0^\infty e^{-t} \cos(tx^{\frac{1}{2}})dt$) are essentially a proof of Lemma 3 in case $|a_n| \leqq C d^n (2n)!$

**Appendix 1.** *Self-adjoint extensions of semi-bounded operators.* We wish to prove here that if $A > 0$ and if $A$ has at most one semi-bounded extension, then $A$ is essentially self-adjoint. The proof is elementary but does not appear to be in the basic literature. We also note the result is not true if $A$ has only one positive self-adjoint extension; *e.g.* $A = -(d^2/dx^2) - 1$ on $L^2[0, \pi]$ with $D(A) = C_0^\infty(0, \pi)$ has only one positive self-adjoint extension but has deficiency indices (2, 2).

***Lemma.*** *If $A > 0$ and $A$ has deficiency indices $[m, m]$ $(m < \infty)$ then every self-adjoint extension of $A$ is semibounded.*

*Proof.* See [1], pp. 115–116. The idea is that if $\tilde{A}$ is any self-adjoint extension of $A$, $D(\tilde{A})/D(A)$ has dimension $m$ so that $\tilde{A}$ has a spectral projection on $(-\infty, 0)$ of dimension at most $m$.

***Theorem.*** *If $A > 0$ and has a unique semibounded self-adjoint extension, then $A$ is essentially self-adjoint.*

*Proof.* Suppose not and let $A$ have deficiency indices $[m, m]$ (all the arguing is in case $m = \infty$). Let $A_F$ be the Friedrichs extension. Then, if $m \neq 0$, we can find a Hermitian operator $\tilde{A}$ with deficiency indices [1, 1] so $A \subset \tilde{A} \subset A_F$. Since $A_F > 0$, $\tilde{A} > 0$ and thus all the other self-adjoint extensions of $\tilde{A}$ are semibounded (by the Lemma). Thus, $A$ has more than one semibounded extension if $m \neq 0$. We conclude $m = 0$, *i.e.* $A$ is essentially self-adjoint.

**Appendix 2.** *On the Moment Problem.* We wish to prove that if $\sum |a_n|^{-1/2n} = \infty$, then the Moment Problem on $(-M, \infty)$ has at most one solution for any $M$. Again the proof is easy but doesn't seem to be in the standard texts [2, 11].

We need Carleman's standard criterion: *Carleman's Theorem* ([3]). If $g(r)$ is analytic in the region $\{z \mid |\arg z| \leqq \pi/2;\ |z| \geqq r_0\}$ and in that region $|g(z)| \leqq a_n |z|^{-n}$ with $\sum_{n=0}^{\infty} a_n^{-1/n} = \infty$, then $g$ is identically 0.

By rotation of the sector translation upwards plus a simple "interpolation" argument, we see:

***Corollary.*** *Let $g(z)$ be analytic in the region $\{z \mid \operatorname{Im} z > R\}$ and in that region suppose*

$$|g(z)| \leqq a_n |z|^{-2n} = 0, 1, 2, \cdots$$

*with*

$$\sum_{n=0}^{\infty} |a_n|^{-1/2n} = \infty. \textit{ Then } g \equiv 0.$$

***Theorem.*** *Let $a_n$ be a set of numbers with $\sum_0^\infty |a_n|^{-1/2n} = \infty$. Suppose $\rho_1$, $\rho_2$ are two measures on $(-M, \infty)$ $(M > 0)$ with $\int_{-M}^{\infty} \lambda^n d\rho_i(\lambda) = a_n (i = 1, 2; n = 0, 1, 2, \cdots)$. Then $\rho_1 = \rho_2$.*

*Proof.* Let $f_i(z) = z^2 \int_{-M}^{\infty} d\rho_i(\lambda)/(z^2 - \lambda)$ which are analytic in the region cut by $z^2 \in (-M, \infty)$. Since $\rho_i$ may be recovered as a boundary value of $f_i$ we need only show $f_1 = f_2$ to prove the theorem. We will first prove that if $\operatorname{Im} z > (3M)^{\frac{1}{2}}$ then

$$\left| f_i(z) - \sum_{n=0}^{N-1} a_n z^{-2n} \right| \leqq |z|^{-2N+2}(a_N + 2M^N a_0)M^{-1}.$$

It then follows that

$$|z^{-2}|\ |f_1(z) - f_2(z)| \leqq |z|^{-2n} b_n$$

in the region $\operatorname{Im} z > (3M)^{\frac{1}{2}}$ with $b_n = 2M^{-1}(a_n + 2M^n a_0)$. We will finally show that $\sum_{n=0}^{\infty} b_n^{-1/2n} = \infty$. This implies $z^{-2} f_1 = z^{-2} f_2$.

By the geometric series with remainder formula

$$g_i^{(N)}(z) \equiv f_i(z) - \sum_{n=0}^{N-1} a_n z^{-2n} = z^{-2N+2} \int_{-M}^{\infty} \frac{\lambda^N\, d\rho_i(\lambda)}{z^2 - \lambda}.$$

Writing $z = x + iy$ we see that $|\lambda - z^2| > M$ if $\lambda > -M$ and if $y^2 > 3M$

(for if $y^2 > 3M$, $x^2 \leqq M$, then $|\lambda - z^2| > |\mathrm{Re}\,\lambda - z^2)| = y^2 + \lambda - x^2 > M$ and if $y^2 > 3M$, $x^2 \geqq M$, then $|\lambda - z^2| > |\mathrm{Im}\,(\lambda - z^2)| > 2(3M)^{\frac{1}{2}}$. Thus, if $\mathrm{Im}\, z > (3M)^{\frac{1}{2}}$,

$$|g^{(N)}(z)| < |z|^{-2N+2} \int_{-M}^{\infty} \frac{|\lambda|^N}{|z^2 - \lambda|}\, d\rho_i(\lambda)$$

$$< M^{-1}\, |z|^{-2N+2}\left(\int_{-M}^{\infty} \lambda^N\, d\rho_i + 2\int_{-M}^{0} |\lambda|^N\, d\rho_i\right) < \tfrac{1}{2}\, |z|^{-2N+2} b_n\, .$$

Finally, we note either $|a_n| < 2M^n a_0$ for infinitely many $n$, in which case $b_n^{-1/2n} > (M/4a_0)^{1/2n} M^{-\frac{1}{2}}$ for infinitely many $n$ so $\sum |b_n|^{-1/2n} = \infty$ or else $|a_n| > 2M^n a_0$ for $n > N$ so that $|b_n|^{-1/2n} > (\frac{1}{2}M)^{1/2n}\, |a_n|^{-1/2n}$ for $n > N$ so $\sum_{n=0}^{\infty} |b_n|^{-1/n} = \infty$.

We have seen how the Moment Problem can be used to study Hilbert space problems in §3. The train of ideas can be turned around to tell us something about the Moment Problem. Explicitally:

(1) By looking at the operator $-d^2/(dx^2) - 1$ on $L^2(0, 2\pi)$ we see there exist moment sequences for which the Stieltjes Moment Problem has a unique solution but for which there are many solutions in $(-M, \infty)$. This illustrates (given our last theorem) the existence of Stieltjes moment sequences for which $\sum_{n=0}^{\infty} a_n^{-1/2n} < \infty$ but where the Moment Problem has a unique solution.

(2) This last theorem combined with the basic theorem of appendix one proves the following theorem about the Hamburger Moment Problem:

Suppose $a_n$ is a sequence of numbers obeying $\sum_{n=0}^{\infty} a_n^{-1/2n} = \infty$ so that $a_n = \int_{-M}^{\infty} \lambda^n d\rho(\lambda)$ for some measure $\rho$ on $(-M, \infty)$. Then the Hamburger Moment Problem has a unique solution *also!*

*Notes added in proof*:

1. The theorem we have called the Masson McClary Theorem appeared previously in A. E. Nussbaum, *Studia Math.* **33** (1969) 305–309. Because of our remarks at the end of appendix 2, his objection to the use of Lemma 1 is not valid.

4. The operator mentioned at the beginning of appendix 1 has two positive self-adjoint extensions. By taking $f(0) = f(\pi) = f'(0) = 0$ boundary conditions, we get an operator with deficiency indices (1, 1) and a unique positive self-adjoint extension.

## References


[1] N. I. Akhiezer & I. M. Glazman, *Theory of Operations in Hilbert Space*, Vol. II Ungar 1963.

[2] N. I. Akhiezer & M. Krein, *Some Questions in the Theory of Moments*, A.M.S. Translations, 1962.

[3] T. Carleman, *Les Fonctions Quasianalytiques*, Gauthies-Villars, Paris, 1926.

[4] H. Freudenthal, Uber die Friedrichssche Fortstzung halbbeschankter Hermitescher Operatoren., *Proc. Acad. Amsterdam*, **39** (1936) 832–833.

[5] K. O. Friedrichs, Spectraltheorie halbeschrankter Operatoren I-II *Math. Ann.*, **109** (1934) 465–487, 685–713, **110** (1935) 777–779.

[6] D. MASSON & W. K. MCCLARY, *Classes of $C^\infty$ Vectors and Essential Self-Adjointness*, Toronto Preprint, October, 1970.
[7] E. NELSON, Analytic Vectors, *Ann. of Math.*, **70** (1959) 572.
[8] E. NELSON, *Topic in Dynamics, I* Princeton University Press (1969).
[9] J. VON NEUMANN, Allgemeine Eigenwert theorie Hermitescher Funtinaloperatoren, *Math. Ann.*, **102** (1929) 49–131.
[10] A. E. NUSSBAUM, Quasianalytic Vectors, *Arkiv fur Mat.*, **6** (1965) 179.
[11] J. A. SHOHAT & J. D. TAMARKIN, *The Problem of Moments*, A.M.S. 1943.

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