JOURNAL OF FUNCTIONAL ANALYSIS 43, 1–31 (1981)

Transient and Recurrent Spectrum*

JOSEPH E. AVRON[†] AND BARRY SIMON[‡]

Department of Mathematics, California Institute of Technology, Pasadena, California 91125

Communicated by Peter D. Lax

Received February 11, 1981

We deal primarily with spectral analysis of an abstract self-adjoint operator, H, on a Hilbert space, $\mathscr{H}$. We propose a further refinement of the absolutely continuous subspace, $\mathscr{H}_{\mathrm{ac}}$, into the transient absolutely continuous subspace, $\mathscr{H}_{\mathrm{tac}}$, which is the closure of those φ with $(\varphi, e^{-itH}\varphi) = O(t^{-N})$ for all N and the recurrent absolutely continuous subspace, $\mathscr{H}_{\mathrm{rac}} = \mathscr{H}_{\mathrm{ac}} \cap \mathscr{H}_{\mathrm{tac}}^{\perp}$. We discuss general features of this breakup. In a subsequent paper, we construct analytic almost periodic functions, V, on $(-\infty, \infty)$ so that $H = -d^2/dx^2 + V(x)$ on $L^2(-\infty, \infty)$ has only recurrent absolutely continuous spectrum in the sense that $\mathscr{H}_{\mathrm{rac}} = \mathscr{H}$.

1. INTRODUCTION

One of the basic questions in analyzing a self-adjoint operator, H, on a Hilbert space, $\mathscr{H}$, is the decomposition of $\mathscr{H}$ obtained by studying the spectral measures for H. In addition to the obvious abstract mathematical interest, it is important in the long time behavior of $e^{-itH}\varphi$ and represents a first step in classification of this behavior.

The standard wisdom (see, e.g., [25]) is that one should decompose the spectrum into three pieces, σ_{pp} (pure point), σ_{ac} (absolutely continuous) and σ_{sc} (singular continuous) corresponding to the decomposition of an abstract measure $d\mu$ into a pure point piece $d\mu_{\mathrm{pp}} = \sum_n a_n \delta(\cdot - x_n)$, an absolutely continuous piece, $d\mu_{\mathrm{ac}} = G(x)\, dx$ and a singular continuous piece, $d\mu_{\mathrm{sc}}$, i.e., a measure with $\mu_{\mathrm{sc}}(\{x\}) = 0$ for all x and so that there is a set A with $\mu_{\mathrm{sc}}(R \backslash A) = 0$ and so that $|A| = 0$ ($|\cdot|$ = Lebesgue measure of $\cdot$).

When H is a Schrödinger operator, the standard wisdom goes a step further: it puts these into two baskets, which, for want of better terms, we

* Research supported by NSF Grant MCS-78-01885.

[†] On leave from Department of Physics, Princeton University.

[‡] On leave from Departments of Mathematics and Physics, Princeton University; Sherman Fairchild Scholar at California Institute of Technology.

0022-1236/81/100001-31$02.00/0

call "ordinary" and "extraordinary." (We must note some dissatisfaction with these terms, having rejected "expected" vs "unexpected" because once understood, the unexpected becomes expected and having rejected "normal" vs "pathological" because we feel the pejorative is unearned. Our unhappiness with the present terms comes from the fact that they are nearly synonymous with "common" and "uncommon" and we feel that extraordinary spectra will become more and more commonly encountered than one might have thought!) In the standard wisdom, pure point and absolutely continuous spectra are "ordinary" and singular continuous is "extraordinary" and is not expected for "reasonable" differential operators.

The standard view has been partly challenged by recent studies. First Pearson [19, 20] has constructed examples of C^∞ functions V on $(-\infty, \infty)$ with all derivatives going to zero at infinity so that $-d^2/dx^2 + V$ has only singular continuous spectrum! Secondly, Goldshtein *et al.* [10], verifying physical insight going back to Anderson [1], have proven that for a large class of "random" V bounded continuous functions on $(-\infty, \infty)$, $-d^2/dx^2 + V$ has only point spectrum dense in a semi-infinite interval (see also [17] for simple looking explicit operators with this property). The latter class of examples shows that in many ways one should refine point spectrum into two types distinguishing between "dense point spectrum" and "ordinary point spectrum." In Section 5, we present a first attempt at such a refinement which we dub "thin" and "thick" point spectrum. The latter is in the extraordinary basket, the former in the ordinary basket. This yields a balance of two kinds of spectra in each basket.

Our main purpose in this paper is to propose a refinement of absolutely continuous spectrum into two pieces: "transient" and "recurrent." The absolutely continuous spectrum that one is used to is "transient" and we place it in the ordinary basket but "recurrent" spectrum is extraordinary. Thus in the end, we have two kinds of ordinary spectrum (transient absolutely continuous and thin point) and three kinds of extraordinary spectrum (singular continuous, recurrent absolutely continuous and thick point).

Let us describe our motivations for refining absolutely continuous spectra. The first comes from considering a class of Cantor-like measures described in some detail in Appendix 1. Consider a set obtained from $[0, 1]$ by removing middle pieces which are not thirds, i.e., first remove the set centered about $\frac{1}{2}$ with size $1/n_1$, then the sets centered about the middle of each of the two remaining intervals of size $1/n_2$ times the size of these intervals, and at the kth step remove the middle $1/n_k$th of the 2^k intervals remaining. Define a Cantor function $C(x)$ by setting it to be $\frac{1}{2}$ on the first interval removed, $\frac{1}{4}$ and $\frac{3}{4}$ on the next pieces removed, etc., and extending by continuity. The associated Cantor measure $d\mu$ on $[0, 1]$ is defined by $C(x) = \int_0^x d\mu(y)$. $d\mu$ is singular with respect to Lebesgue measure if and only if

$$\sum_j n_j^{-1} = \infty, \tag{1.1}$$

otherwise it is absolutely continuous and indeed has the form $|A|^{-1}\mathscr{X}_A(x)\,dx$ for $\mathscr{X}_A$ the characteristic function of the associated Cantor set. Our point is that in many ways these absolutely continuous Cantor measures are closer to singular continuous measures than to ordinary absolutely continuous measures. This becomes clearer if one considers the long times behavior of the Fourier transforms

$$F(t) = \int e^{ixt}\,d\mu(x)$$

which we compute in Appendix 1 in the case where all n_j are odd integers. Of course, for the absolutely continuous case, $F(t) \to 0$ as $t \to \infty$. But also $F(t) \to 0$ so long as

$$n_j \to \infty \qquad \text{as} \quad j \to \infty. \tag{1.2}$$

Obviously (1.2) can hold even when (1.1) holds. Moreover, all $F(t)$ have anomalous bumps at the points

$$2\pi n_1 \cdots n_k,$$

where F is much larger than at most point "nearby" and for all choices of n,

$$\int_0^\infty |F(t)|\,dt = \infty.$$

One can determine whether μ is absolutely continuous by looking at F, since

$$(1.1) \text{ holds if and only if } \int_0^\infty |F(t)|^2\,dt = \infty,$$

but it seems to us that the L^2 norm of F (while "physically natural") is not a particularly critical object for long time behavior and that the measures with $n_j = j$ and $n_j = j(\ln j)^2$ are close relatives even though they are on separate sides of the singular continuous/absolutely continuous barrier.

When we pass to operators one cannot merely look at individual spectral measures. For example, if H is multiplication by x on $L^2(-\infty, \infty)$, and $\varphi(x) = \mathscr{X}_A(x)$, where A is one of these Cantor sets constructed with $\sum n_j^{-1} < \infty$, then $(\varphi, e^{-itH}\varphi)$ will be misbehaved. The point though is that φ can be well approximated by vectors η_n with $(\eta_n, e^{-itH}\eta_n)$ in Schwarz space as a function of t. With this in mind, we define η to be a *transient vector* if $(\eta, e^{itH}\eta) = O(t^{-N})$ for all N and set $\mathscr{H}_{\text{tac}}$ to be the *closure* of the transient vectors. While the transient vectors are not a subspace, we will prove that the closure $\mathscr{H}_{\text{tac}}$ is a subspace. $\mathscr{H}_{\text{tac}}^{\perp}$ is the set of *recurrent vectors* and $\mathscr{H}_{\text{rac}} = \mathscr{H}_{\text{tac}}^{\perp} \cap \mathscr{H}_{\text{ac}}$ is the recurrent absolutely continuous space.

Our second (and actually initial) motivation comes from consideration of Schrödinger operators with almost periodic potentials. Consider, for a moment, $H = -d^2/dx^2 + V$ with V a bounded continuous function but not one going to a constant at infinity. Classically, if the energy is less than $\overline{\lim}_{x \to \infty} V(x)$, then the particle will not leave a bounded set and if it is strictly bigger than sup $V(x)$, it will make it to infinity approaching infinity at a linear rate (not that $t^{-1}x(t)$ approaches a limit but $at \leqslant x(t) \leqslant bt$ for a, b of the same sign at t large). Quantum mechanically, though, this is not true. Not only is tunnelling through barriers possible but more importantly even a very energetic particle which classically sails above the low energy bumps has the possibility of being reflected from each and every bump and therefore the transport to infinity is more complicated. Indeed, barring miracles, we expect the spectrum in such examples to be some isolated eigenvalues of finite multiplicity and mainly some kind of extraordinary spectra! This intuition is borne out by the studies of random [10] and sparse [19] potentials.

Of course, there is a case where a miracle does take place and transport to infinity is normal: If V is periodic, there is nice absolutely continuous spectrum and nice transport (see Section 4). One can understand why this takes place; the coherences in phase that the particle needs to build up to get through the first few bumps are exactly those needed to get through the later bumps. For almost periodic potentials one does not expect this. (There are some *very* special almost periodic potentials constructed by Dubrovin *et al.* [7], whose spectrum has only finitely many gaps where the transport should be normal but these are clearly highly non-generic). A particle in an almost periodic potential will think for a while that it is in a periodic potential. It will sail through a large number of bumps but eventually the bumps will slip out of sync with the coherences and the particle will be reflected back. If the particle attempts to build coherences on a larger scale to get through even more bumps, eventually since the potential is not periodic, reflection takes place. Thus we have a notion of continual return to the origin albeit with possible dispersion (i.e., spreading of the wave packet) and longer and longer runs. This picture is so like the one in Pearson's example (where it is not sufficiently emphasized that even though the spectrum is singular continuous, one expects $(\varphi, e^{itH}\varphi) \to 0$ due to dispersion of the wave packet during the long intervals between bumps), that we initially thought singular continuous spectrum was most likely. We had to reconcile this picture with results of Dinaburg and Sinai [6] (see also Rüssman [24]) that for a larger class of quasiperiodic potentials, one has some absolutely continuous spectrum. Recurrent absolutely continuous spectrum is our synthesis of the thesis of anomalous transport and the antithesis of the results of [6, 24].

In a subsequent paper [2], we will construct some almost periodic potentials on $(-\infty, \infty)$, so that $-d^2/dx^2 + V(x)$ has *only* recurrent absolutely continuous spectrum. It should be mentioned that independently and earlier than we, Moser [18] studied the same class we do in [2]; while he does not prove absolute continuity, his results imply there is no transient absolutely continuous space for his examples.

We close this introduction with a sketch of the remaining contents. As one would expect, since transience vs recurrence is a unitary invariant, it must be possible to describe these spaces in terms of the spectral measure classes. To do this we need some preliminaries on set theory which we put in Section 2. The central section of the paper is Section 3. In Section 4 we discuss the connection with "transport" in the theory of Schrödinger operator and in Section 5 we describe our partition of point spectrum into thick and thin. In Appendix 1, we discuss Cantor measures and in Appendix 2 construct some illustrative sets. In Appendix 3, we discuss the construction of singular measures by Pearson [19] in its relation to a theorem of Kakutani [14]. We emphasize that, while we have not found discussions of the material in Section 2, Appendix 1 and Appendix 2 in the literature, the material is "classical" in spirit; we include it for the reader's convenience.

2. Some Essential Topology

We hasten to begin by noting that "essential" in this section's title is intended in the technical sense of "almost everywhere" rather than in the colloquial sense of "critical."

A *measure class* is an equivalence class of Borel measures on $(-\infty, \infty)$ under the relation of mutually absolutely continuous. An absolutely continuous measure class is a class consisting of measures absolutely continuous with respect to dx. An *event* is an equivalence class, $[A]$, of Borel subsets of $(-\infty, \infty)$ under the relation $A \sim B$ if and only if $|A \mathbin{\Delta} B| = 0$, where Δ is symmetric difference and $|\cdot|$ is Lebesgue measure. By the *support* of an absolutely continuous measure, $d\mu$, we mean the event, $[A]$, determined by writing $d\mu = f(x)\,dx$ and letting

$$A = \{x \mid f(x) > 0\}. \tag{2.1}$$

We have:

LEMMA 2.1. *Support sets up a one-to-one correspondence between events and absolutely continuous measure classes.*

Proof. $d\mu = f(x)\,dx$ is mutually absolutely continuous with respect to $\mathscr{X}(x)\,dx$ if A is given by (2.1). $\mathscr{X}_A\,dx$ is equivalent to $\mathscr{X}_B\,dx$ if and only if $|A \mathbin{\Delta} B| = 0$. ∎

Remark. Every event $[A]$ has a distinguished element B defined to be

$$B = \{x \mid \lim_{t \downarrow 0} (2t)^{-1} |(x-t, x+t) \cap A| = 1\}.$$

That $B \in [A]$ follows from the theory of differentiation of integrals; see [33]. B is called the set of *points of density* of $[A]$. It will play a minimal role in what follows.

Given an event $[A]$, we say that x is an *essential interior point* of $[A]$ if and only if there is a $t > 0$ with $|(x-t, x+t) \cap A| = 2t$. The *essential interior* of $[A]$ is the set (we emphasize that it is not merely an event) of essential interior points of A. x is called an *essential limit point* of $[A]$ if and only if for any $t > 0$, $|(x-t, x+t) \cap A| > 0$ and the set of essential limit points is called the *essential closure.*

PROPOSITION 2.2. *Essential interiors are open sets, essential closures are closed sets, and the complement of the essential interior of* $[A]$ *is the essential closure of* $[R \backslash A]$.

Proof. The last assertion is just chasing negatives. The first assertion implies the second. To prove the first note that if $|(x-t, x+t) \cap A| = 2t$ and if $|y-x| = \varepsilon < t$, then $|(y-s, y+s) \cap A| = 2s$ if $s = t - \varepsilon$. ∎

Remark. Obviously any essential interior point is a point of density. But, in general, the converse is false and the event determined by the essential interior can be much smaller than $[A]$. Indeed, the positive measure Cantor sets of Appendix 1 are sets with empty essential interior and in Appendix 2, we construct an event A whose essential interior is empty but whose essential closure is $(-\infty, \infty)$.

If $[A]$ is an event and if B is its essential interior, we define the event $[A \backslash B]$ to be the *essential frontier* of $[A]$. The decomposition of $[A] = [B] \cup [C]$ with $[B]$ and $[C]$ *disjoint* (i.e., $|B \cap C| = 0$) into the essential interior and essential frontier will be essentially the decomposition $\mathscr{H}_{\mathrm{ac}}$ into $\mathscr{H}_{\mathrm{tac}} \oplus \mathscr{H}_{\mathrm{rac}}$.

We will need one last result of this genre:

PROPOSITION 2.3. *Let* f *be in* $L^1(R)$ *and let the event* $[E] \equiv \{x \mid f(x) \neq 0\}$ *be a subevent of some event* $[A]$. *Suppose* f *has a continuous represetative. Then* $[E]$ *is a subevent of (the event determined by) the essential interior of* $[A]$.

Proof. If f is continuous, $[E]$ contains an open representative $E \in [E]$. Since $E \subset A$, E is in the essential interior of A. ∎

3. Basic Definitions and Properties

DEFINITION. Let H be a self-adjoint operator on a separable Hilbert space, $\mathscr{H}$. $\varphi \in \mathscr{H}$ is called a *transient vector* for H, if and only if, for all $N > 0$,

$$\lim_{|t|\to\infty} |t|^N(\varphi, e^{-itH}\varphi) = 0. \tag{3.1}$$

The *transient subspace*, $\mathscr{H}_{\text{tac}}$, is the closure of the set of transient vectors (which we will eventually prove is a subspace).

PROPOSITION 3.1. *Let $d\mu_\varphi$ be the spectral measure for φ, i.e.,*

$$\int f(x)\, d\mu_\varphi(x) = (\varphi, f(H)\varphi). \tag{3.2}$$

If φ is a transient vector, then $d\mu_\varphi(x) = G(x)\, dx$ with $G \in C^\infty$ and, in particular,

$$\mathscr{H}_{\text{tac}} \subset \mathscr{H}_{\text{ac}}. \tag{3.3}$$

Conversely if $d\mu_\varphi(x) = G(x)\, dx$ and $G \in C_0^\infty$ (note compact support), then φ is a transient vector.

Proof. Let

$$F(t) = (\varphi, e^{-itH}\varphi)$$

so that

$$F(t) = \int e^{-itx}\, d\mu_\varphi(x)$$

and thus μ_φ is up to a factor of $(2\pi)^{1/2}$ the Fourier transform of F. Equation (3.1) and the trivial $|F(t)| \leqslant \|\varphi\|^2$ imply that $(1 + |t|)^N F \in L^1$ for all N so that $d\mu_\varphi(x) = G(x)\, dx$ for $G \in C^\infty$. Conversely, if $G \in C_0^\infty$, its Fourier transform obeys $|t|^N F(t) \to 0$ for all N. ∎

EXAMPLE 3.2. Let $H =$ multiplication by x on $L^2(0, 1)$. Let $h \in C_0^\infty(\frac{1}{3}, \frac{2}{3})$ with $h(\frac{1}{2}) \neq 0$. Let $\mathscr{X}$ be the characteristic function of $[0, \frac{1}{2}]$. Let $\varphi_1 = h$, $\varphi_2 = e^{i\mathscr{X}} h$. Then

$$d\mu_{\varphi_1} = d\mu_{\varphi_2} = |h|^2\, dx$$

and $|h|^2 \in C_0^\infty$, so φ_1, φ_2 are transient vectors. Let $\varphi = \varphi_1 + \varphi_2$ so

$$d\mu_\varphi = |h|^2[2 + 2\cos(\mathscr{X})]\, dx;$$

since $|h|^2[2 + 2\cos \mathscr{X}]$ is discontinuous φ is not a transient vector, so the transient vectors are not a subspace.

The next example, while simple, is so basic, we will call it a proposition.

PROPOSITION 3.3. *Let $\Omega \subset (-\infty, \infty)$ be open and let H be multiplication by x on $\mathscr{H} = L^2(\Omega)$. Then $\mathscr{H} = \mathscr{H}_{\mathrm{tac}}$; i.e., every vector is a limit of transient vectors.*

Proof. As above any $\varphi \in C_0^\infty(\Omega)$ is a transient vector. But $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$. ∎

Given an operator H on a separable space and a trace class oberator B which is strictly positive (i.e., $(\varphi, B\varphi) > 0$ if $\varphi \neq 0$), we can define a measure μ^B by

$$\mu^B(\Delta) = \mathrm{Tr}(BE_\Delta)$$

with E_Δ the spectral projections for H. While μ^B is dependent on B, its measure class is not and we call it the *H-spectral measure class* (it is the union of the spectral multiplicity measure classes defined in the multiplicity theory; see [25]). Every measure class has an absolutely continuous piece and so there is an event, the *H-event* determined by the H-spectral measure class. It is in fact the event $[A]$ determined by A being a "minimal" event with

$$E_A E_{\mathrm{ac}} = E_{\mathrm{ac}} \tag{3.4}$$

when E_{ac} is the projection onto $\mathscr{H}_{\mathrm{ac}}$, the absolutely continuous space for H.

The following identifies $\mathscr{H}_{\mathrm{tac}}$ and $\mathscr{H}_{\mathrm{rac}}$, defined by

DEFINITION. The *recurrent space* is the orthogonal complement of $\mathscr{H}_{\mathrm{tac}}$. $\mathscr{H}_{\mathrm{ac}} \cap \mathscr{H}_{\mathrm{tac}}^\perp \equiv \mathscr{H}_{\mathrm{rac}}$, the recurrent absolutely continuous space.

THEOREM 3.4. *Let H be a self-adjoint spector on $\mathscr{H}$, a separable Hilbert space. Let $[A]$ be the H-event and let $[B]$, $[C]$ be its essential interior and essential frontier. Then*

$$\mathscr{H}_{\mathrm{tac}} = E_B \mathscr{H}_{\mathrm{ac}}; \qquad \mathscr{H}_{\mathrm{rac}} = E_C \mathscr{H}_{\mathrm{ac}}.$$

In particular, $\mathscr{H}_{\mathrm{tac}}$ is a subspace.

Remark. We do not see any easy way of deducing that $\mathscr{H}_{\mathrm{tac}}$ is a subspace without explicitly computing it!

Proof. Let φ be a transient vector so that $\varphi \in \mathscr{H}_{\rm ac}$ and its spectral measure is $f(x)\,dx$ with $f \in C^\infty$. Clearly, the event determined by $\{x \mid f(x) \neq 0\}$ is a subevent to A. Thus by Proposition 2.3, f vanishes on C, i.e., $\varphi \in E_B \mathscr{H}_{\rm ac}$ so $\mathscr{H}_{\rm tac} \subset E_B \mathscr{H}_{\rm ac}$. Because B is open and B is a subset of the A-event, we know that if $\varphi \in E_B \mathscr{H}_{\rm ac}$, then there is an invariant subspace, $\mathscr{H}'$, for H obeying (i) $\varphi \in \mathscr{H}'$, (ii) H restricted to $\mathscr{H}'$ is unitarily equivalent to multiplication on $L^2(B, dx)$. By the last proposition, $\mathscr{H}' \subset \mathscr{H}_{\rm tac}$ so $E_B \mathscr{H}_{\rm ac} = \mathscr{H}_{\rm tac}$. Since $\mathscr{H}_{\rm ac} = E_B \mathscr{H}_{\rm ac} \oplus E_C \mathscr{H}_{\rm ac}$, the theorem is proven. ∎

COROLLARY 3.5. *Let $\varphi \in \mathscr{H}$ with*

$$\int_{-\infty}^{\infty} |(\varphi, e^{-itH}\varphi)|\,dt < \infty.$$

Then $\varphi \in \mathscr{H}_{\rm tac}$.

Proof. Since φ has spectral measure $G(x)\,dx$ with G continuous, G must be supported on B. ∎

Letting $F(t) = (\varphi, e^{-itH}\varphi)$, we see that

$$\mathscr{H}_{\rm tac} = \overline{\{\varphi \mid F(t) \in L^1\}}; \tag{3.5}$$

that is, there is no difference between choosing those vectors with $F(t) \in L^1$ and choosing those with (3.1) holding. This should be compared with a result going back at least to Kato (see, e.g., [26]),

$$\mathscr{H}_{\rm ac} = \overline{\{\varphi \mid F(t) \in L^2\}}, \tag{3.6}$$

so L^2 and L^1 yield, in general, rather distinct spaces. We do not know how to characterize $\overline{\{\varphi \mid F(t) \in L^p\}}$ for any $p \neq 1, 2, \infty$ nor even if this closure is a subspace.

COROLLARY 3.6. *Suppose that H has a nowhere dense spectrum, i.e., $\overline{R \backslash \sigma(H)} = R$. Then*

$$\mathscr{H}_{\rm tac}(H) = 0;$$

i.e., any absolutely continuous space is recurrent.

Proof. Immediate from Theorem 3.4. ∎

If H has nowhere dense spectrum and if $\mathscr{H}_{\rm ac} \neq \{0\}$, then $\sigma_{\rm rac}(H) \equiv \sigma(H \mid \mathscr{H}_{\rm rac})$ is a *Cantor set*, i.e., a perfect nowhere dense set (perfect means closed with no isolated points; $\sigma_{\rm ac}$ always has no isolated points). Recall [16] that perfect sets are everywhere locally uncountable.

Naively, one might expect that $\sigma_{\rm rac}$ is always nowhere dense, but

EXAMPLE 3.7. There exists an operator H with $\mathscr{H} = \mathscr{H}_{\text{rac}}$ and $\sigma_{\text{rac}} = (-\infty, \infty)$. For, in Appendix 2, we construct sets A with empty essential interior and essential closure all of R. Just take $\mathscr{H} = L^2(A, dx)$ and $H =$ multiplication by x.

The decomposition of $\mathscr{H}$ to $\mathscr{H}_{\text{ac}}$, $\mathscr{H}_{\text{sc}}$ and $\mathscr{H}_{\text{pp}}$ behaves nicely under taking direct sums, i.e., $\mathscr{H}_{\text{ac}}(H \oplus H') = \mathscr{H}_{\text{ac}}(H) \oplus \mathscr{H}_{\text{ac}}(H')$, etc. This is not true of $\mathscr{H}_{\text{rac}}$ and $\mathscr{H}_{\text{tac}}$ [although obviously $\mathscr{H}_{\text{tac}}(H \oplus H') \supset \mathscr{H}_{\text{tac}}(H) \oplus \mathscr{H}_{\text{tac}}(H')$].

EXAMPLE 3.8. Let H' be multiplication by x on $L^2(-\infty, \infty)$. Then $\mathscr{H}_{\text{tac}}(H \oplus H') = \mathscr{H}_{\text{ac}}(H \oplus H')$ for any H; i.e., H' is able to eat up *any* recurrent absolutely continuous spectra. This is because the event for H' is $[R]$ and that for $H \oplus H'$ is $[A] \cup [R] = [R]$ for any $A =$ the H-event.

The situation is worse:

EXAMPLE 3.9. There exist H, H' on $\mathscr{H}$ and $\mathscr{H}'$, respectively, so that $\mathscr{H} = \mathscr{H}_{\text{rac}}(H)$, $\mathscr{H}' = \mathscr{H}_{\text{rac}}(H')$ but

$$\mathscr{H} \oplus \mathscr{H}' = \mathscr{H}_{\text{tac}}(H \oplus H'),$$

for by Appendix 2, there exist sets A, B each with empty essential interior so that $A \cup B = R$. Let $\mathscr{H} = L^2(A, dx)$, $\mathscr{H}' = L^2(B, dx)$ and multiply both operators by x.

The above examples say that within a purely *abstract* setting, $\mathscr{H}_{\text{rac}}$ is of limited interest but in concrete examples, where taking direct sums with abandon is not allowed, it is of interest.

The final abstract result we will need is the following:

PROPOSITION 3.10. *Let $\varphi, \varphi_n \in \mathscr{H}$. Suppose that*

(i) $\varphi_n \to \varphi$ *in* $\mathscr{H}$.

(ii) $\int |(\varphi_n, e^{itH}\varphi)|\, dt < \infty$, *each* $n < \infty$.

Then $\varphi \in \mathscr{H}_{\text{tac}}$.

Proof. Let μ_n be the complex measure given by

$$\mu_n(\Delta) = (\varphi_n, E_\Delta \varphi).$$

By (ii) each μ_n is absolutely continuous, so if $|A| = 0$, then $(\varphi_n, E_A \varphi) = 0$. Since $(\varphi, E_A \varphi) = \lim_n (\varphi_n, E_A \varphi)$, we see that φ is absolutely continuous. Let P be the projection on the span of $\{e^{-itH}\varphi\}$. Replacing φ_n by $P\varphi_n$, we see that there exist functions $F(x)$, $F_n(x) \in L^2(R, dx)$, so that

(i) $F_n(x)\bar{F}(x)$ is continuous,

(ii) F_n converges in L^2 to $F(x)$,

(iii) $F^2\,dx$ is the spectral measure for φ.

Let $[A]$ be the event determined by $\{x \mid F(x) \neq 0\}$. Statement (i) says that $F_n(x)\,\bar{F}(x)$ vanishes on the essential frontier of A so (ii) implies that A is essentially open. Statement (iii) and Proposition 3.3 imply that $\varphi \in \mathscr{H}_{\text{tac}}$. ∎

4. Transport and Schrödinger Operators: Some Examples

Abstract spectral theory loses sight of the fact that operators of interest are typically acting on a concrete space, normally $L^2(R^\nu, d^\nu x)$ for some ν. Physics is often connected with the local structure in R^ν and much recent progress (reviewed, e.g., in [9]) uses the geometry in R^ν. Of particular interest is

$$G_R^\varphi(t) = \| F(|x| < R)\, e^{-itH} \varphi \|,$$

where $F(A)$ is multiplication by the characteristic function of A. For example, the celebrated RAGE theorem (see, e.g., [26]) says that $\varphi \in \mathscr{H}_{\text{ac}} \oplus \mathscr{H}_{\text{sc}}$ if and only if, for all R,

$$\frac{1}{2T} \int_{-T}^{T} G_R^\varphi(t)\, dt \to 0$$

as $T \to \infty$ under the sole condition that $F(|x| < R)(H+i)^{-1}$ is compact, something true for virtually all Schrödinger operators.

Let us introduce four subspaces (since $G^{\varphi+\psi} \leqslant G^\varphi + G^\psi$, they are subspaces):

(1) The *transport vectors*, denoted TRANS, being any φ for which there is $a > 0$ so that

$$G_{at}^\varphi(t) = O(t^{-N})$$

for all N.

(2) The *flight vectors*, denoted FLIGHT, being any φ for which

$$G_R^\varphi(t) = O(t^{-N})$$

for each fixed R and N.

(3) The *weak flight vectors*, denoted WEFL, being any φ for which

$$\int_{-\infty}^{\infty} G_R^\varphi(t)\, dt < \infty$$

for each fixed R.

(4) The *Sinha vectors*, denoted SINHA being any φ for which

$$\int_{-\infty}^{\infty} |G_R^{\varphi}(t)|^2\, dt < \infty.$$

For each fixed R we claim inclusions of the form

$$\text{TRANS} \subset \text{FLIGHT} \subset \text{WEFL} \begin{array}{ccc} & \text{SINHA} & \\ \rotatebox{45}{$\subset$} & & \rotatebox{-45}{$\subset$} \\ & & \\ \rotatebox{-45}{$\subset$} & & \rotatebox{45}{$\subset$} \\ & \mathscr{H}_{\text{tac}} & \end{array} \mathscr{H}_{\text{ac}}.$$

The first two inclusions are trivial. $\text{WEFL} \subset \text{SINHA}$ follows from $G_R^{\varphi}(t) \leqslant \|\varphi\|$ and $\text{WEFL} \subset \mathscr{H}_{\text{tac}}$ follows from Proposition 3.10 with $\varphi_n = F(|x| < n)\varphi$. $\text{SINHA} \subset \mathscr{H}_{\text{ac}}$ is a result of Sinha [32], wo proves also that $\overline{\text{SINHA}} = \mathscr{H}_{\text{ac}}$ if $F(|x| < R)(H+i)^{-k}$ is Hilbert–Schmidt for some k.

One might guess that it is always true that $\overline{\text{WEFL}} = \mathscr{H}_{\text{tac}}$ on this basis, but the proof is not obvious. The problem is that Sinha's result depends on the fact that $\mathscr{H}_{\text{ac}}$ has a dense set of ψ for which

$$\int |(\varphi, e^{itH}\psi)|^2\, dt < \infty$$

for *all* φ. But if

$$\int |(\varphi, e^{-itH}\psi)|\, dt < \infty$$

for *all* φ, then $\psi = 0$! Thus one needs to know properties of the eigenfunctions of $(H-i)^{-k} F(|x| < R)(H+i)^{-k}$ to be able to conclude $\overline{\text{WEFL}} = \mathscr{H}_{\text{tac}}$.

Of course, "good transport" corresponds to $\overline{\text{TRANS}} = \mathscr{H}_{\text{ac}}$. Obviously $\mathscr{H}_{\text{rac}} \neq \{0\}$ implies bad transport.

Let us summarize what is known about these Schrödinger operators.

(1) *Free operators.* For $H = -\Delta$, indeed for large class of functions of $-i\nabla$ [11], $\overline{\text{TRANS}} = \mathscr{H}$. This observation is critical to the Enss theory [8].

(2) *Two-body operators.* It follows from work of Jensen and Kato [12, 13] that if $\nu \geqslant 3$, if V falls faster than any power, then $\overline{\text{TRANS}} = \mathscr{H}_{\text{ac}}$ for $-\Delta + V = H$. From work of Perry [22], if V is globally smooth and dilation analytic, the same is true. For no other V's is there even information on FLIGHT although Jensen and Kato prove $\overline{\text{WEFL}} = \mathscr{H}_{\text{ac}}$ if V has fixed power falloff at a rate depending only on ν. For a large class of V's we know that $H \upharpoonright \mathscr{H}_{\text{ac}} \cong -\Delta$ [4, 8, 21, 30], so $\mathscr{H}_{\text{tac}} = \mathscr{H}_{\text{ac}}$.

(3) *N-body operators.* Nothing is known about even WEFL but we claim that the spectrum is ordinary under very general circumstances because $\sigma_{ac} \subset [\Sigma, \infty)$ whenever the HVZ theorem holds. If wave operators exist, then $\mathscr{H}_{ac}$ has a component unitary equivalent to $-\Delta + \Sigma$ for some Laplacian. It follows that the H-event is that determined by (Σ, ∞) and this has empty essential frontier. Thus existence of wave operators implies that $\mathscr{H}_{rac} = \{0\}$. Perry *et al.* [23] have proven under wide circumstances that there is no thick point spectrum and no singular continuous spectrum.

(4) *One-dimensional periodic potentials.* Davies and Simon [5] prove that $\overline{\text{TRANS}} = \mathscr{H}$. It also follows from their ideas that if V is asymptotically periodic as $x \to +\infty$ or $x \to -\infty$ (distinct periodic potentials allowed), with $O(|x|^{-1-\epsilon})$ approach, then $\mathscr{H}_{rac} = \{0\}$.

(5) *v-dimensional periodic potentials.* Using the Davies–Simon method [5] and results of Wilcox [34], one can show $\overline{\text{TRANS}} = \mathscr{H}$; the difference between $\nu = 1$ and $\nu > 1$ is that for $\nu = 1$ one knows that for many f's, $f(H)\mathscr{S} \subset \text{TRANS}$ but this is not known if $\nu \geqslant 2$ (see [30]).

(6) *Limit periodic potentials.* A limit periodic potential is one that is a uniform limit of periodic potentials, e.g., $\sum a_n \cos(x/n)$ with $\sum |a_n| < \infty$. In [2], we will prove that generic limit periodic potentials have a spectrum which is a Cantor set. Such operators *must* have extraordinary spectrum and bad transport. By Corollary 3.6, $\mathscr{H}_{tac} = \{0\}$, so WEFL $= \{0\}$; thus bad transport. Moreover, since $\sigma(H)$ is locally uncountable, most of $\sigma(H)$ cannot be the thin point spectrum. All that is left is extraordinary spectrum of one type or another. In [2], we will construct examples with $\mathscr{H} = \mathscr{H}_{rac}$; in general, we suspect that the low energy spectrum might have some thick point and the high enegy spectrum recurrent absolutely continuous.

5. Thin and Thick Point Spectrum

In [25], σ_{pp} is *defined* to be the set of eigenvalues so $\bar{\sigma}_{pp}$ is the spectrum of $H \upharpoonright \mathscr{H}_{pp}$. We make the following refinement.

Definition. $\lambda \in \bar{\sigma}_{pp}$ is said to lie in the *thin point spectrum* if and only if $(\lambda - t, \lambda + t) \cap \bar{\sigma}_{pp}$ is countable for some $t > 0$. $\lambda \in \bar{\sigma}_{pp}$ is said to lie in the *thick point spectrum* if $(\lambda - t, \lambda + t) \cap \bar{\sigma}_{pp}$ is uncountable for all $t > 0$.

In this section, we will prove some elementary but illuminating results:

Theorem 5.1. *The thin point spectrum is countable.*

Corollary 5.2. *The thick point spectrum is empty if and only if $\bar{\sigma}_{pp}$ is countable.*

THEOREM 5.3. *The thick point spectrum is a perfect set, i.e., closed with no isolated points.*

Before proving these, we note an illustrative example,

EXAMPLE 5.4. Let B be the operator with a complete set of eigenfunctions with distinct eigenvalues λ_n which are the middle points of the intervals removed in the construction of the conventional Cantor set, A; i.e., λ_n are those points in $[0, 1]$ whose base three expansion consists of an arbitrary *finite* sequence of 0's and 2's followed only by 1's. Thus $\bar{\sigma}_{\mathrm{pp}} = \sigma_{\mathrm{pp}} \cup A$. The thin point spectrum is σ_{pp}, the thick is A. This shows that the thin spectrum may not have countable closure and that σ_{pp} and the thick point spectrum can be disjoint.

Proof of Theorem 5.1. (Actually a standard result [14, 15] in disguise.) For each point λ in the point spectrum, pick an interval I_λ, *with rational end points* so that $\lambda \in I_\lambda$ and $I_\lambda \cap \bar{\sigma}_{\mathrm{pp}}$ is countable. Clearly $\bigcup (I_\lambda \cap \bar{\sigma}_{\mathrm{pp}})$ contains the thin point spectrum. But since there are only countably many intervals with rational end points, there can be only countably many distinct I_λ. Thus $\bigcup (I_\lambda \cap \bar{\sigma}_{\mathrm{pp}})$ is countable. ∎

Proof of Corollary 5.2. If $\bar{\sigma}_{\mathrm{pp}}$ is countable, trivially there is no thick spectrum. If there is only thin spectrum, $\bar{\sigma}_{\mathrm{pp}}$ is countable by Theorem 5.1. ∎

Proof of Theorem 5.3. It is trivial that a limit of thick point spectrum is in the thick point spectrum. Thus we need only show it has no isolated points. Every neighborhood contains uncountably many points of $\bar{\sigma}_{\mathrm{pp}}$ but, by Theorem 5.1, only countably many points of the thin point spectrum. ∎

APPENDIX 1: SOME CANTOR MEASURES

In this appendix, we describe some properties of certain Cantor measures illustrating aspects of recurrent absolutely continuous spectrum. The sets we are mainly interested in are fatter than the original Cantor set. There is an enormous literature on much thinner Cantor sets (see Carleson [3] and its extensive bibliography) but other than Salem's beautiful notes [27], and references therein, we have located no literature on the question of most interest to us, the falloff of the Fourier transform. Nevertheless, given the classical nature of these questions, we have little doubt additional literature exists. We provide our discussion here primarily for the reader's convenience.

Let $n_1, n_2, \ldots$ be a sequence of real numbers $1 < n_j < \infty$ (eventually we will take the n_j to be odd integers).

We define a *Cantor set* $S(n_j)$ as follows:

From $[0, 1]$ remove the open interval of size n_1^{-1} about the point $\frac{1}{2}$. Then remove the two open intervals of size $\frac{1}{2}(1 - n_1^{-1})n_2^{-1}$ about the middle of each of the two remaining intervals. After j removals, there are 2^j intervals left and at the $(j+1)$ step we remove the intervals of size $2^{-j}[\prod_1^j(1 - n_j^{-1})]n_{j+1}^{-1}$ about the center of these intervals. $S(n_j)$ is the complement in $[0, 1]$ of the union of these open intervals. Occasionally, we will use $S^{(j)}(n_j)$ to denote the union of the 2^j closed intervals left after j removals.

We define a *Cantor function* $C(n_j, t)$ by setting it equal to $\frac{1}{2}$ on the first interval removed, $\frac{1}{4}$ and $\frac{3}{4}$ on the two pieces removed at stage two, $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$ on the next removal, etc. (the values chosen in the obvious way so that C is monotone increasing). C is extended to $[0, 1]$ by continuity. The *Cantor measure*, $d\nu$, is defined by

$$\nu([0, t]) = C(t). \tag{A.1.1}$$

We have:

PROPOSITION A.1.1. (a) *The Lebesgue measure of the Cantor set* $S(n_j)$ *is*

$$|S(n_j)| = \prod_{j=1}^{\infty} (1 - n_j^{-1}), \tag{A.1.2}$$

where the product always has a limit, the limit being zero if and only if

$$\sum n_j^{-1} = \infty. \tag{A.1.3}$$

(b) *S is always nowhere dense, i.e.,* $[0, 1]\backslash S$ *has closure* $[0, 1]$; *indeed given any* $x \in [0, 1]$, *we can obtain it as a limit of both larger and smaller numbers in* $[0, 1]\backslash S$.

(c) *S is a perfect set.*

(d) *If* (A.1.3) *holds, then ν is mutually singular with respect to Lebesgue measure.*

(e) *If* (A.1.3) *fails, then ν is absolutely continuous with respect to Lebesgue measure; indeed*

$$d\nu = |S|^{-1} \mathscr{X}_S \, dx$$

with $\mathscr{X}_S$ the characteristic function of S.

Proof. (a) The measure of $S^{(j)}(n_j)$ is clearly $\prod_1^j(1 - n_k^{-1})$. Since the $S^{(j)}$ are decreasing and countable, and their intersection is S, the result follows.

(b) $S^{(j)}$ consists of 2^j intervals of equal Lebesgue measure of total size at most 1 so given any point in S, there is a point of $[0, 1]\backslash S^{(j)} \subset [0, 1]\backslash S$ within 2^{-j-1} of it.

(c) Let I be any interval of $S^{(j)}$. Then $I \cap S^{(k)} \neq \varnothing$ for each k, so by compactness, $I \cap S \neq \varnothing$. Given any point, x, in S, and m, first, find $y_0 \notin S$ with $|x - y_0| \leqslant m^{-1}$ (use (b) for this). y_0 is contained in some open interval, O_0 removed at step S_0. We claim that without loss, we can suppose that $x \notin \bar{O}_0$, for x is a limit of points both above and below it and it cannot be a boundary point of open intervals both above and below it (for these intervals will be both removed at some finite step at which point all intervals of $S^{(j)}$ have non-zero size). Since $x \notin \bar{O}_0$, we can find y_1 between x and O_0 so that $y_1 \notin S$ (use (b)). Let O_1 be the interval in $[0, 1]\backslash S$ containing y_1. Both O_0 and O_1 are removed at some finite step j. $S^{(j)}$ will have a piece I between O_0 and O_1, so there is an $x' \in S \cap I$. $x' \neq x$ since x is not between O_0 and O_1 and clearly $|x - x'| \leqslant m^{-1}$:

$$\begin{array}{ccccc} & O_1 & & O_0 & \\ \cdot & (\cdot) & \cdot & (\cdot) & \cdot \\ x. & y_1 & x' & y_0 & \end{array}$$

FIGURE 1.

(d) Since $\nu([0, 1]\backslash S^j) = 0$, we have $\nu([0, 1]\backslash S) = 0$. If $|S| = 0$, ν and dx are mutually singular.

(e) Let $C^{(j)}(t)$ be defined by $C^{(j)}(0) = 0$, $dC^{(j)}/dt = 0$ if $t \notin S^{(j)}$; $|S^{(j)}|^{-1}$ if $t \in S^{(j)}$. Then $C^{(j)}(t) = C(t)$ on $[0, 1]\backslash S^{(j)}$ from which we conclude that $C(t) = \lim_{j\to\infty} C^{(j)}(t)$ for all t. It follows that ν is the weak limit of $|S^{(j)}|^{-1} \mathscr{X}_{S^{(j)}}\, dx$. If $|S|^{-1} < \infty$, then by the dominated convergence theorem, $|S^{(j)}|^{-1} \mathscr{X}_{S^{(j)}} \to |S|^{-1} \mathscr{X}_S$ in L^1-norm so $|S|^{-1} \mathscr{X}_S\, dx$ is also the weak limit of $|S^{(j)}|^{-1} \mathscr{X}_{S^{(j)}}\, dx$. ∎

Remarks. 1. The dichotomy that $d\nu$ is either absolutely continuous or singular with respect to dx is illuminated (and proven if the n_j are odd integers) by Kukurani's theorem discussed in the next appendix.

2. If $n_j \to x$ as $j \to \infty$, it is not hard to see that the Hausdorff dimension of S is exactly $\log 2/[\log 2 - \log(1 - x^{-1})]$ since $S^{(j)}$ is made of 2^j intervals of size $2^{-j} \prod_1^j (1 - n_i^{-1})$. In particular, for the sets discussed by Carleson [3], where $x = 1$, $\dim(S) = 0$ and for those that will most interest us, where $x = \infty$, $\dim(S) = 1$.

Of particular concern for us is the large t behavior of

$$F(t) \equiv \int e^{ixt}\, d\nu(x).$$

We have:

COROLLARY A.1.2. (a) *$F(t) \in L^2$ if and only if* (A.1.3) *fails; explicitly*:

$$\int_{-\infty}^{\infty} |F(t)|^2 \, dt = (2\pi) \left[\prod_{j=1}^{\infty} (1 - n_j^{-1}) \right]^{-1}. \tag{A.1.4}$$

(b) *$F(t)$ is never L^1.*

Proof. Equation (A.1.4) is just the Plancherel theorem. If $F \in L^1$, $F \in L^2$ (since $|F(t)| \leqslant 1$ is trivial), so $d\nu = |S|^{-1} \mathscr{X}_s \, dx$. Thus $F \in L^1$ would imply $\mathscr{X}_s$ is continuous, which is obviously false. ∎

Henceforth, let us suppose that each n_j is an odd integer

$$n_j = 2l_j + 1; \qquad l_j \in \{1, 2, \ldots\}. \tag{A.1.5}$$

It will be convenient to shift the Cantor set, measure, etc., by $\frac{1}{2}$ unit so it is now in $[-\frac{1}{2}, \frac{1}{2}]$. Thus $d\nu$ is even and F is real. *We henceforth do this without changing notation.*

Let $\Omega^{(j)}$ be the set of the n_j points $-l_j, -l_j + 1, \ldots, l_j$ and map $\Omega = \text{X}_{j=1}^{\infty} \Omega^{(j)}$ into $[-\frac{1}{2}, \frac{1}{2}]$ by

$$f(a) \equiv \sum_{j=1}^{\infty} a_j / n_1 \cdots n_j$$

(i.e., use a variable base expansion). If we remove countable sets from Ω and $[-\frac{1}{2}, \frac{1}{2}]$, f is 1–1 and onto (it is two-to-one on these removed countable sets). Let $d\mu^{(j)}$ be the measure on $\Omega^{(j)}$ giving weight 0 to $a_j = 0$ and weight $1/(n_j - 1)$ to each of the other points. Let $d\mu$ be the infinite product measure on Ω. We claim that

$$\nu(A) = \mu(f^{-1}[A]) \tag{A.1.6}$$

for any set A. To see this, let μ_j be the infinite product of $\mu^{(1)}, \ldots, \mu^{(j)}$ and normalized counting measure on $\Omega^{(j+1)}, \ldots$, let ν_j be the measure $|S^j|^{-1} \mathscr{X}_{S^j} \, dx$ and note that

$$\nu_j(A) = \mu_j(f^{-1}[A]).$$

Equation (A.1.6) comes from taking $j \to \infty$. As a result of this formula,

$$F(t) \equiv \int \exp\left(it \sum a_j/n_1 \cdots n_j\right) d\mu(a)$$

is an infinite product; i.e., we have proven

THEOREM A.1.3. *If* (A.1.5) *holds then*

$$F(t) = \prod_{j=1}^{\infty} G_j(t), \tag{A.1.7}$$

where

$$G_j(t) = (2l_j)^{-1} \left\{ \left[\sum_{a=-l_j}^{l_j} \exp(ita/n_1 \cdots n_j) \right] - 1 \right\}.$$

Remark. By abstract nonsense the infinite product (A.1.7) must converge; this can also be seen concretely if we note that $(1 - \cos S) \leqslant \frac{1}{2}S^2$, and

$$G_j(t) = l_j^{-1} \sum_{a=1}^{l_j} \cos(ta/n_1 \cdots n_j) \tag{A.1.8}$$

so

$$|G_j(t) - 1| \leqslant \tfrac{1}{8}t^2/[n_1 \cdots n_{j-1}]^2. \tag{A.1.9}$$

Thus (A.1.7) converges absolutely since $n_j \geqslant 3$.
We immediately prove

COROLLARY A.1.4. *Let* $t_j = 2\pi n_1 \cdots n_{j-1}$. *Then*

$$|F(t_j)| \geqslant \tfrac{1}{2} d l_j^{-1}, \tag{A.1.10}$$

where

$$d = \prod_{k=1}^{\infty} (1 - \pi^2/3^{2k}) > 0.$$

If $n_j \to \infty$, *then*

$$\lim_{j \to \infty} (2l_j)\, F(t_j) = -1. \tag{A.1.11}$$

Proof. By (A.1.9)

$$1 \geqslant G_{j+k}(t_j) \geqslant 1 - \pi^2/n_{j+k-1}^2$$

so

$$\prod_{k=1}^{\infty} G_{j+k}(t_j) \geqslant d$$

and if $n_j \to \infty$, then

$$\lim_{j\to\infty} \prod_{k=1}^{\infty} G_{j+k}(t_j) = 1.$$

The corollary follows from noting

$$G_k(t_j) = 1; \qquad k = 1,\dots, j-1,$$

by (A.1.8), and by (A.1.7)

$$G_j(t_j) = -(2l_j)^{-1}. \quad \blacksquare$$

The above result shows that if n_j does not go to infinity, then $F(t)$ does not go to zero at infinity. The next result will show the converse:

THEOREM A.1.5. *Let $t_j = 2\pi n_1 \cdots n_{j-1}$ as above. Then for*

$$\tfrac{1}{2}t_j \leqslant t \leqslant \tfrac{1}{2}t_{j+1} \tag{A.1.12}$$

we have

$$|f(t)| \leqslant (9/4)[\min(l_{j-1}, l_j]^{-1}. \tag{A.1.13}$$

Proof. We claim that if

$$\tfrac{1}{2}t_j \leqslant t \leqslant t_{j+1} - \tfrac{1}{2}t_j \tag{A.1.14}$$

then

$$|G_{j-1}(t)\, G_j(t)| \leqslant (9/4)[\min(l_{j-1}, l_j]^{-1}. \tag{A.1.15}$$

Since $t_j < t_{j+1}$, (A.1.12) implies (A.1.14). Moreover (A.1.15) implies (A.1.13) since $|G_k(t)| \leqslant 1$. Thus, we need only prove the above claim.

By summing the geometric series, write

$$G_j(t) = (2l_j)^{-1}[b_j(t) - 1],$$

where

$$b_j(t) = \sin(\tfrac{1}{2}\alpha_{j-1}t)/\sin(\tfrac{1}{2}\alpha_j t)$$

with

$$\alpha_j = (n_1 \cdots n_j)^{-1}.$$

Notice first that (as a sum of n_j exponentials)

$$|b_j(t)| \leqslant n_j \leqslant 3l_j.$$

Thus

$$\begin{aligned}|G_{j-1}(t)\,G_j(t)| &\leqslant (4l_jl_{j-1})^{-1}[|b_{j-1}(t)\,b_j(t)| + 1 + 3l_j + 3l_{j-1}] \\ &\leqslant (4l_jl_{j-1})^{-1}[|b_{j-1}(t)\,b_j(t)|] + 7[4\min(l_j, l_{j-1})]^{-1}\end{aligned}$$

so we need only prove that (A.1.14) implies

$$|b_{j-1}(t)\,b_j(t)| \leqslant 2l_j. \tag{A.1.16}$$

Note that

$$|b_{j-1}(t)\,b_j(t)| \leqslant [\sin(\tfrac{1}{2}\alpha_j t)]^{-1}$$

and that (A.1.14) implies that

$$\tfrac{1}{2}\pi n_j^{-1} \leqslant \tfrac{1}{2}\alpha_j t \leqslant \pi - \tfrac{1}{2}\pi n_j^{-1}.$$

Notice also that for $0 < \alpha < \pi/2$,

$$\max_{\alpha \leqslant \theta \leqslant \pi - \alpha} (\sin\theta)^{-1} \leqslant (\sin\alpha)^{-1}$$

and that since $\sin x/x$ is decreasing on $[0, \pi/2]$ and $\frac{1}{2}\pi n_j^{-1} \leqslant \pi/6$, we have

$$\sin\left(\frac{1}{2}\pi n_j^{-1}\right) \geqslant \frac{1}{2}\pi n_j^{-1}\frac{6}{\pi}\sin\left(\frac{\pi}{6}\right) = \frac{3}{2}n_j^{-1}.$$

Thus

$$|b_{j-1}(t)\,b_j(t)| \leqslant 2n_j/3 \leqslant 2l_j$$

as required. ∎

COROLLARY A.1.6. $\lim_{t\to\infty} F(t) = 0$ *if and only if* $\lim_{j\to\infty} n_j = \infty$.

There are two morals we want to draw from above:

(a) $F(t)$ has strange large t behavior of occasional bumps where $F(t)$ is anomalously larger than at nearby points. The shift from ac to sc is not easy to see in qualitative behavior (except for the L^2 norm).

(b) If for any $1 > \alpha > 0$

$$\lim_{j\to\infty} n_j/(n_1 \cdots n_j)^\alpha = 0$$

[e.g., $n_j \sim Cj^k$ for some k], then $F(t)$ does not even have power falloff. However, if n_j grows fast enough, we can have power falloff; e.g., if $n_j = 3^{2^j}$, then $t_j \sim n_j$ and $n_{j-1} = (n_{j+1})^{1/4}$ so our result says that $F(t)$ falls at least as fast as $t^{-1/4}$.

We close this appendix by noting an intriguing connection, suggested by the above, which is misleading. Namely, for the above if S has Hausdorff dimension smaller than 1, then $F(t) \nrightarrow 0$. This is an artifact of our choice if n_j integral, for Salem [27] has proven that is $n_j = \theta$, a constant, and θ is not an algebraic integer than $F(t) \to 0$ (but for such a choice, $\dim(S) < 1$).

APPENDIX 2: CONSTRUCTION OF A SET

We will construct two sets A, B which are disjoint, $A \cup B = R$, so that each is essentially dense in R (equivalently, so that A is essentially dense in R but has empty essential interior). Let $l_j = 2^{j-1}$; $n_j = 2l_j + 1$, $j = 1, 2, \ldots$ so that $\sum n_j^{-1} < \infty$ and each n_j is odd. Any $x \in R$ has a unique expansion

$$x = m + \sum_{j=1}^{\infty} a_j/n_1 \cdots n_j,$$

where a_j is one of $-l_j$, $-l_j$, $-l_j + 1, \ldots, l_j$ and by convention we cannot have $a_j = -l_j$ for all large j (to get uniqueness) and m is an integer. We call these coordinates $m(x)$, $a_j(x)$. These are clearly measurable functions of x. We will let

$$A = \{x \mid \text{An odd number of } a_j(x) \text{ are } 0\},$$

$$B = \{x \mid \text{An even number or an infinite number of } a_j(x) \text{ are } 0\}.$$

(Actually, by a Borel Cantelli lemma, the set with infinitely many $a_j = 0$ has Lebesgue measure zero.)

Let S be the set of points, x, with $a_j(x) = 0$ for all large j. S is clearly dense in R. If we show that for every $x \in S$ and every j sufficiently large

$$|(x - l_j/n_1 \cdots n_j, x + l_j/n_1 \cdots n_j) \cap A| > 0,$$

$$|(x - l_j/n_1 \cdots n_j, x + l_j/n_1 \cdots n_j) \cap B| > 0,$$

then both A and B are essentially dense in S and so in R. Given $x \in S$, pick j_0 so that $a_j(x) = 0$ for $j \geqslant j_0$. Let $I_j = (x - l_j/n_1 \cdots n_j,\ x + l_j/n_1 \cdots n_j)$. We shall prove that $|I_j \cap A| > 0$, $|I_j \cap B| > 0$ for $j > j_0$. We suppose that the number of $a_k(x)$, $k = 1, \ldots, j-1$, which are zero is odd (a similar argument works if it is even). Consider all y with $m(x) = m(y)$, $a_k(y) = a_k(x)$,

$k = 1,..., j - 1$; and $a_k(y) \neq 0$, $k \geqslant j$. The set of such y's lies in $I_j \cap A$. Its Lebesgue measure is

$$\prod_1^{j-1} n_k^{-1} \prod_j^{\infty} (1 - n_k^{-1}) > 0.$$

Similarly, the set of y with $m(y) = m(x)$, $a_k(y) = a_k(x)$, $k = 1,..., j$, and $a_k(y) \neq 0$ for $k \geqslant j + 1$ lies in $I_j \cap B$ and has measure

$$\prod_1^{j} n_k^{-1} \prod_{j+1}^{\infty} (1 - n_k^{-1}) > 0.$$

APPENDIX 3: Kakutani's Theorem and Pearson's Conctruction

In his construction of Schrödinger operator with singular continuous spectrum, Pearson [19] proved and applied the following result:

THEOREM A.3.1 [19]. *Let* $f_n(k, y)$; $0 \leqslant k \leqslant 1$, $-\infty < y < \infty$, $n = 1, 2,...,$ *be a sequence of function periodic in* y *of period* 1 *so that*

(i) $\inf_{n,y,k} f_n(k, y) \equiv d > 0$.

(ii) $\int_0^1 f_n(k, y)\, dy = 1$ *for all* n, k.

(iii) *For all sufficiently large* $N, f_n(k, Nk)$ *is analytic in* $[0, 1]$; $f_n(k, y)$ *is* C^1 *in* k *and* y *jointly.*

Given a sequence N_i *of positive numbers, let* μ_n *be the measure on* $[0, 1]$ *given by*

$$d\mu_n(k) = \prod_1^n f_i(k, N_i k)\, dk.$$

Let

$$\alpha_n(k) \equiv -\int_0^1 \log f_n(k, y)\, dy. \qquad \text{(A.3.1)}$$

Then the N_i *may be chosen so that* $d\mu_n$ *has a weak limit* $d\mu_\infty$ *and so that* $d\mu_\infty$ *is absolutely continuous with respect to Lebesgue measure if*

$$\sum_n \alpha_n(k) < \infty, \qquad 0 \leqslant k \leqslant 1,$$

and mutually singular if

$$\sum_n \alpha_n(k) = \infty, \qquad 0 \leqslant k \leqslant 1.$$

Our point here is that this result is intimately related to a beautiful theorem of Kakutani [14] which is not as widely known among mathematical physicists as it should be (although its special case for Gaussian measures, which is a large chunk of the Feldman–Hajek–Shale–Bogolubov theorem is well known; see, e.g., [29, 31]). This not only illuminates Pearson's theorem but also allows two improvements: minimal "smoothness" applies and more importantly (i) can be dropped if α_n is replaced by β_n (below), which is the "correct" term.

We begin by stating and proving Kakutani's theorem. The mutually singular part of the proof is from Kakutani [14]; the absolutely continuous part uses ideas of Segal [28].

THEOREM A.3.2 [14]. *Let $\Omega_1, \Omega_2, \ldots$ be a sequence of measure spaces, and let μ_n and ν_n be probability measures on Ω_n so that $d\nu_n = f_n\, d\mu_n$. Let μ, ν be the infinite product measure on $\Omega = \times_n \Omega_n$ and let*

$$\gamma_n = \int \sqrt{f_n}\, d\mu_n. \tag{A.3.2}$$

Then, if $\prod_n \gamma_n = 0$, μ and ν are mutually singular and if $\prod_n \gamma_n \neq 0$, then ν is absolutely continuous with respect to μ.

Remarks. 1. Since $\gamma_n \leqslant 1$ by the Schwartz inequality, $\lim_{N\to\infty} \prod_1^N \gamma_n$ always exists.

2. We emphasize the remarkable fact that ν is always either entirely μ-absolutely continuous or entirely μ-singular (this is a kind of 0–1 law).

3. If μ_n is absolutely continuous with respect to ν_n, i.e., if f_n is ac nonzero, then $d\mu_n = g_n d\nu_n (g_n = f_n^{-1})$ and

$$\tilde{\gamma}_n \equiv \int \sqrt{g_n}\, d\nu_n = \int f_n^{-1/2} f_n\, d\mu_n = \gamma_n$$

so that if $\prod_n \gamma_n \neq 0$, the measures are mutually absolutely continuous.

Proof. Suppose first that $\prod_n \gamma_n = 0$. If we can find $A_k \subset \Omega$, so that $\nu(A_k) \leqslant 2^{-k}$, $\mu(\Omega \backslash A_k) \leqslant 2^{-k}$, we take $A = \bigcap_m \bigcup_{k=m}^\infty A_k$ and find $\nu(A) = 0$, $\mu(\Omega \backslash A) = 0$ so we have mutual singularity. Let

$$A_k = \left\{ \omega \mid \prod_1^{N(k)} f_n(\omega_n) < 1 \right\},$$

where $N(k)$ will be chosen below. Then

$$\nu(A_k) = \int_{A_k} \prod_1^{N(k)} f_n \, d\mu_n \leqslant \int_{A_k} \sqrt{\prod_1^N f_n} \, d\mu_n$$
$$\leqslant \int_\Omega \sqrt{\prod_1^N f_n} \, d\mu = \sqrt{\prod_1^N \gamma_n},$$

while

$$\mu(\Omega \backslash A_k) = \int_{\Omega \backslash A_k} \prod_1^N d\mu_n \leqslant \int_{\Omega \backslash A_k} \sqrt{\prod_1^N f_n} \, d\mu_n$$
$$\leqslant \int_\Omega \sqrt{\prod_1^N f_n} d\mu_n = \sqrt{\prod_1^N \gamma_n}.$$

Picking $N(k)$ so $\prod_1^N \gamma_n \leqslant 2^{-2k}$, we have mutual singularity.

On the other hand, suppose that $\prod_n \gamma_n > 0$. We will prove that $G_N = \prod_1^N f_n$ convergence in $L^1(d\mu)$-norm as $N \to \infty$ from which absolute continuity follows easily. Let $N \leqslant M$ and write

$$\| G_N - G_M \|_1^2 = \| (\sqrt{G_N} - \sqrt{G_M})(\sqrt{G_N} + \sqrt{G_M} \|_1^2$$
$$\leqslant \| (\sqrt{G_N} - \sqrt{G_M}) \| \, [\| \sqrt{G_N} \|_2 + \| \sqrt{G_M} \|_2]$$
$$= 2 \int [G_N + G_M - 2\sqrt{G_N G_M}] \, d\mu$$
$$= 4 - 4 \prod_{N+1}^M \gamma_n \leqslant 4 \left(1 - \prod_{N+1}^\infty \gamma_n\right).$$

Since $\prod_N \gamma_n$ is absolutely convergent, it is guaranteed that $\lim_{N \to \infty} \prod_{N+1}^\infty \gamma_n = 1$. ∎

There is some similarity between the first half of this proof and Pearson's proof. The relation between (A.3.1) and (A.3.2) is seen in:

PROPOSITION A.3.3. (i) *Let* $0 < \theta < 1$. *Let* $\gamma_n^{(\theta)} \equiv \int f_n^\theta \, d\mu_n$. *Then* $\prod_1^\infty \gamma_n^{(\theta)} = 0$ *if and only if* $\prod_1^\infty \gamma_n = 0$.

(ii) *Let* $\alpha_n \equiv -\int \log f_n \, d\mu_n$. *Then* $\prod_1^\infty \gamma_n = 0$ *implies that* $\sum_n \alpha_n = \infty$.

(iii) *Suppose* $\inf_n f_n(\omega_n) = d > 0$. *Then* $\sum_n \alpha_n = \infty$ *implies that* $\prod_1^\infty \gamma_n = 0$.

(iv) *There exist* Ω_n, μ_n, ν_n, *so that* $\sum_n \alpha_n = \infty$ *but* $\prod_1^\infty \gamma_n > 0$.

Proof. (i) Holder's inequality says that $\ln \gamma_n^{(\theta)}$ is convex in θ and of course, $\gamma_n^{(\theta=0)} = \gamma_n^{(\theta=1)} = 1$. Thus, if $\theta < \frac{1}{2}$, we use $\gamma_n^{(\theta)} \leqslant [\gamma_n]^{2\theta}$, $\gamma_n \leqslant [\gamma_n^{(\theta)}]^\alpha$

with $\alpha = \frac{1}{2}(1-\theta)$. If $\theta > \frac{1}{2}$, we use $\gamma_n^{(\theta)} \leqslant [\gamma_n]^{2(1-\theta)}$ and $\gamma_n \leqslant [\gamma_n^{(\theta)}]^{\alpha}$ with $\alpha = \frac{1}{2}\theta$.

(ii) By Jensen's inequality and the convexity of $-\log x$, we have

$$-\log \gamma_n \leqslant \tfrac{1}{2}\alpha_n .$$

(iii) We have that

$$\sup_{\substack{y \geqslant \sqrt{d} \\ y \neq 1}} \{-\log y + \tfrac{1}{2}(y^2 - 1)\}/(y-1)^2 = c < \infty$$

since the function is obviously continuous away from 0, 1, ∞, and has nice limits at 1, ∞.

Thus, if $f_n(\omega_n) \geqslant d$

$$-\log \sqrt{f_n} + \tfrac{1}{2}(f_n - 1) \leqslant 2c(1 - \sqrt{f_n} + \tfrac{1}{2}(f_n - 1)),$$

integrating, we get

$$\tfrac{1}{2}\alpha_n \leqslant 2c(1-\gamma_n).$$

Since $\prod_1^\infty \gamma_n = 0$ if and only if $\sum_1^\infty (1-\gamma_n) = \infty$, the result is proven.

(iv) Let Ω_n be the two point space $\{0, 1\}$. Let $\mu_n(0) = n^{-2}$; $\mu_n(1) = 1 - n^{-2}$; $\nu_n(0) = 2^{-n}n^{-2}$; $\nu_n(1) = 1 - \nu_n(0)$ so $f_n(0) = 2^{-n}$; $f_n(1) = 1 + n^{-2} + O(n^{-4})$. Then

$$\gamma_n = 1 + O(1/n^2)$$

so $\sum(1-\gamma_n) < \infty$ and $\prod \gamma_n > 0$. But

$$\alpha_n = (\log 2)n^{-1} + O(n^{-2})$$

so $\sum \alpha_n = \infty$. ∎

The only reason that Theorem A.3.1 requires hypothesis (i) is that Pearson uses condition (A.3.1) in place of the "right" condition (A.3.2). As Pearson notes, the key aspect of the $N_i \to \infty$ limit is that $f_i(k, N_i k)$ and $f_{i+1}(k, N_{i+1} k)$ are "almost" independent. This is shown by the following result:

LEMMA A.3.4. *Let g, h be two measurable functions on $[0, 1)$ with $h \in L^\infty$ and $g \in L^1$. Extend g to be periodic on $(-\infty, \infty)$ with period 1. Then*

$$\lim_{N\to\infty} \int_0^1 h(x)\, g(Nx)\, dx = \left[\int_0^1 h(x)\, dx\right]\left[\int_0^1 g(x)\, dx\right].$$

Proof. By adding a constant to g we may suppose $\int_0^1 g(x)\,dx = 0$. Since

$$\int_0^1 |f(Nx) - g(Nx)|\,dx \leqslant [N+1]/N \int_0^1 |f(x) - g(x)|\,dx$$

we see that without loss we can take g to be C^∞. Since g is now L^∞ we can approximate h by a $\mathbb{C}^\infty$ also without loss. But, clearly, breaking up $[0, 1]$ into $[N]$ intervals of size $1/N$ and one extra interval

$$\left| \int_0^1 h(x)\, g(Nx)\,dx \right| \leqslant \frac{[N]}{N} \| g\|_\infty \sup_{(x-y) \leqslant N^{-1}} [|h(x) - h(y)|] + \frac{1}{N} \| g\|_\infty \|h\|_\infty$$

goes to zero as $N \to \infty$. ∎

As a warmup to an improved version of Pearson's theorem, we note:

THEOREM A.3.5. *Let $f_n(y)$ be a sequence of non-negative functions periodic in y of period 1 so that*

(i) $\int_0^1 f_n(y)\,dy = 1$ *for all* n,

(ii) *each* $f_n \in L^\infty$.

Given a sequence N_i of positive numbers, let μ_n be the measure on $[0, 1]$ given by

$$d\mu_n = \prod_1^n f_i(N_i y)\,dy.$$

Let

$$\beta_n = -\log \int_0^1 \sqrt{f_n(y)}\,dy \tag{A.3.3}$$

Then the N_i may be chosen so that $d\mu_n$ has a weak limit $d\mu_\infty$ with no pure points and so that $d\mu_\infty$ is absolutely continuous with respect to Lebesgue measure if

$$\sum_n \beta_n < \infty$$

and is mutually singular if

$$\sum_n \beta_n = \infty.$$

Proof. We choose $1 = N_1 \leqslant N_2 \leqslant \cdots$ inductively as follows. Having chosen $N_1, \ldots, N_j$ and $m_1, \ldots, m_{j-1}$ first choose m_j so that

$$\int_{\alpha/m_j}^{(\alpha+1)/m_j} d\mu_j \leqslant 2^{-j}, \qquad \alpha = 0, 1, \ldots, m_j - 1.$$

This can be done since $d\mu_j$ is dy absolutely continuous. Then, using Lemma A.3.4, pick N_{j+1} so that the following four inequalities hold:

$$\text{(a1)} \quad \int_0^1 \left[\prod_{n=1}^{m} f_n(N_n y)\right]\left[\prod_{n=m+1}^{l} \sqrt{f_n(N_n y)}\right]\left[\prod_{l+1}^{j} f_n(N_n y)\right] \times [f_{j+1}(N_{j+1} y) - 1]\, dy \leqslant 2^{-j-2} \prod_{m+1}^{l} \gamma_n$$

for $m = 1, \ldots, j$, $l = m, \ldots, j$;

$$\text{(a2)} \quad \int_0^1 dy \prod_{n=1}^{m} f_n(N_n y) \prod_{n=m+1}^{j} \sqrt{f_n(N_n y)}\, [\sqrt{f_n(N_{j+1} y)} - \gamma_{j+1}]\, dy \leqslant 2^{-j-2} \prod_{m+1}^{j+1} \gamma_n$$

for $m = 1, \ldots, j$.

$$\text{(b)} \quad \int_0^1 y^l \left[\prod_{n=1}^{j} f_n(N_n y)\right] [f_{j+1}(N_{j+1} y) - 1]\, dy \leqslant 2^{-j-2}$$

for $l = 1, \ldots, j$.

$$\text{(c)} \quad \int_{\alpha/m_l}^{(\alpha+1)/m_l} \left[\prod_{n=1}^{j} f_n(N_n y)\right] [f_{j+1}(N_{j+1} y) - 1]\, dy \leqslant 2^{-j}$$

for $l = 1, 2, \ldots, j$; $\alpha = 0, 1, \ldots, m_l - 1$.

By (b), the measures $d\mu_n$ have a weak limit $d\mu_\infty$. By (c) and the choice of m_l

$$\int_{\alpha/m_l}^{(\alpha+1)/m_l} d\mu_\infty \leqslant 2^{-l} + 2^{-l-1} + \cdots \leqslant 2^{-l+1}$$

so that $d\mu_\infty$ has no pure points. Finally, we will show that (a) and (b) imply that

$$\frac{1}{2} \prod_{m+1}^{l} \gamma_n \leqslant \int_0^1 \prod_{m+1}^{l} \sqrt{f_n(N_n y)} \prod_{\substack{1 \leqslant n \leqslant m \\ l+1 \leqslant n \leqslant j}} f_n(N_n y)\, dy \leqslant \frac{3}{2} \prod_{m+1}^{l} \gamma_n, \text{(A.3.4)}$$

so, by the proof of Kakutani's theorem, we can conclude the mutual singularity or absolute contrinuity.

To prove (A.3.4), let $A(m, l, j)$ denote the integral in that inequality. Inequality (a1) implies that

$$|A(m, l, j) - A(m, l, j+1)| \leqslant 2^{-j-2} \prod_{m+1}^{l} \gamma_n$$

so we see that

$$\begin{aligned} |A(m, l, l) - A(m, l, j)| &\leqslant \prod_{m+1}^{l} \gamma_n \sum_{j=l}^{\infty} 2^{-j-2} \\ &= \frac{1}{2} 2^{-l} \prod_{m+1}^{l} \gamma_n. \end{aligned} \tag{A.3.5}$$

Next note that (a2) implies that

$$|A(m, j, j)\,\gamma_{j+1} - A(m, j+1, j+1)| \leqslant 2^{-j-2} \prod_{m+1}^{j+1} \gamma_n$$

so

$$|A(m, l, l) - \left[\prod_{m+1}^{l} \gamma_n\right] A(m, m, m)| \leqslant \frac{1}{2}(2^{-m} - 2^{-l}) \prod_{m+1}^{l} \gamma_n. \tag{A.3.6}$$

Finally (b) implies that

$$|A(m, m, m) - 1| \leqslant \tfrac{1}{2}(1 - 2^{-m}); \tag{A.3.7}$$

(A.3.5)–(A.3.7) imply (A.3.4). ∎

To extend Theorem A.3.1, we need

LEMMA A.3.6. *Let $f(k, y)$ be a continuous function on* $[0, 1]$, $x(-\infty, \infty)$ *periodic in y with period* 1. *Let* $g \in L^\infty[0, 1]$. Then

$$\lim_{N \to \infty} \int_0^1 g(k) f(k, Nk)\, dk = \int_0^1 \int_0^1 g(k) f(k, y)\, dk\, dy.$$

Proof. If f is a finite sum of products $h(k)\, l(y)$, this is just Lemma A.2.6. But every continuous function is a uniform limit of such sums. ∎

From this result, by just mimicking the proof of Theorem A.3.5, we obtain:

THEOREM A.3.7. *Let $f_n(k, y)$ be a sequence of nonnegative continuous functions on $[0, 1] \times (-\infty, \infty)$ periodic in y with period 1 so that*

$$\int_0^1 f_n(k, y)\, dy = 1 \qquad \text{all } n, \text{all } k \text{ in } [0, 1].$$

Given a sequence N_i of positive numbers, let μ_n be the measure on $[0, 1]$ given by

$$d\mu_n = \prod_1^n f_i(y, N_i y)\, dy.$$

Let

$$\bar{\beta}_n = \sup_k \left[-\log \int_0^1 \sqrt{f_n(k, y)}\, dy \right],$$

$$\underline{\beta}_n = \inf_k \left[-\log \int_0^1 \sqrt{f_n(k, y)}\, dy \right].$$

Then the N_i may be chosen so that $d\mu_n$ has a limit $d\mu_\infty$ with no pure points so that $d\mu_\infty$ is dk absolutely continuous if $\sum \bar{\beta}_n < \infty$ and so that it is singular if $\sum \underline{\beta}_n = \infty$.

Remark. This is not, strictly speaking, stronger than Pearson's result since we take inf and sup over k but improvements are possible.

ACKNOWLEDGMENTS

It is a pleasure to thank the Caltech Mathematics Department for its hospitality and Barry Simon would like to thank the Fairchild Scholar Visiting Program for its support. We thank W. Littman and P. Sarnak for valuable conversations.

REFERENCES

1. P. W. ANDERSON, *Phys. Rev.* **109** (1958), 1492–1505.
2. J. AVRON AND B. SIMON, Cantor sets and Schrödinger operators. II. Limit periodic potentilas, *Comm. Math. Phys.*, in press.
3. L. CARLESON, "Exceptional Sets," Van Nostrand, Princeton, N. J., 1967.
4. E. B. DAVIES, On Enss' approach to scattering theory, *Duke. Math. J.* **47** (1980), 171–186.
5. E. B. DAVIES AND B. SIMON, Scattering theory for systems with different spatial asymptotics on the left and right, *Comm. Math. Phys.* **63** (1978), 277–301.

6. E. I. DINABURG AND JA. G. SINAI, On the one dimensional Schrödinger equation with quasiperiodic potential, *Funkcial. Anal. i Priložen* **9**(4) (1975), 8–21.
7. D. A. DUBROVIN, V. B. MATVEEV, AND S. P. NOVIKOV, Nonlinear equations of *K dV* type, finite zone linear operators and Abelian varieties, *Russian Math. Surveys* **31** (1976), 59–146.
8. V. ENSS, Asymptotic completeness for quantum mechanical scattering. I. Short range potentials, *Comm. Math. Phys.* **61** (1978), 285.
9. V. ENSS, *in* "Rigorous Atomic and Molecular Physics" (G. Velo and A. S. Wightman Eds.), Plenum, New York, 1981.
10. YA. GOLDSHTEIN, S. A. MOLCHANOV, AND L. A. PASTUR, *Funkcial Anal. i Priložen.* **11** (1977), 1–10.
11. L. HORMANDER, The existence of wave operators in scattering theory, *Math. Z.* **146** (1976), 69–91.
12. A. JENSEN, Spectral properties of Schrödinger operators and time decay of wave functions in $L^2(\mathbb{R}^n)$, $n \geqslant 5$, *Duke Math. J.* **47** (1980), 57–80.
13. A. JENSEN AND T. KATO, Spectral properties of Schrödinger operators and time decay of the wave functions, *Duke Math. J.* **46** (1979), 583–611.
14. S. KAKUTANI, On equivalence of infinite product measures, *Ann. of Math.* **49** (1948), 214–224.
15. J. L. KELLY, "General Topology," Van Nostrand, New York, 1955.
16. K. KARATOWSKI, "Topologie I: Espaces métrisable espaces complet," Warsaw, 1933.
17. K. MILLER AND B. SIMON, Quantum magnetic Hamiltonians with remarkable spectral properties, *Phys. Rev. Lett.* **44** (1980), 1706.
18. J. MOSER, An example of a Schrödinger equation with almost periodic potential and nowhere dense spectrum, ETH preprint.
19. D. PEARSON, Singular continuous measures in scattering theory, *Comm. Math. Phys.* **60** (1978), 13–36.
20. D. PEARSON, Pathological spectral properties, *in* "Mathematical Problems in Theoretical Physics," pp. 49–51, Lecture Notes in Physics No. 80, Springer-Verlag, New York/Berlin, 1979.
21. P. PERRY, Mellin transforms and scattering theory. I. Short range potentials, *Duke Math. J.* **47** (1980), 187–193.
22. P. PERRY, Propagation of stats in dilation analytic potentials and asymptotic cmpleteness, *Comm. Math. Phys.*, in press.
23. P. PERRY, I. SIGAL, AND B. SIMON, Spectral analysis of multiparticle Schrödinger operators, *Ann. of Math.*, in press.
24. H. RÜSSMAN, On the one dimensional Schrödinger equation with a quasiperiodic potential, *Ann. N.Y. Acad. Sci*, 1980.
25. M. REED AND B. SIMON, "Methods of Modern Mathematical Physics," Vol. I, "Functional Analysis," Academic Press, New York, 1972; 2nd ed., 1980.
26. M. REED AND B. SIMON, "Methods of Modern Mathematical Physics," Vol. III, "Scattering Theory," Academic Press, New York, 1979.
27. R. SALEM, "Algebraic Numbers and Fourier Analysis," Heath, Boston, 1963.
28. I. SEGAL, Tensor algebraic over Hilbert spaces, I, *Trans. Amer. Math. Soc.* **81** (1956), 106–134.
29. B. SIMON, "The $P(\phi)_2$ Euclidean (Quantum) Field Theory," Princeton Univ. Press, Princeton, N. J., 1974.
30. B. SIMON, Phase space analysis of simple scattering systems. Extensions of some work of Enss, *Duke Math. J.* **46** (1979), 119–168.
31. B. SIMON, "Functional Integration and Quantum Physics," Academic Press, New York, 1979.

32. K. B. SINHA, On the absolutely and singularly continuous subspaces in scattering theory. *Ann. Inst. H. Poincaré* **26** (1977), 263–277.
33. E. M. STEIN, "Singular Integrals and Differentiability Properties of Functions," Princeton Univ. Press, Princeton, N. J., 1970.
34. C. WILCOX, The theory of Bloch waves, *J. Analyse Math.* **33** (1978), 146–167.