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Absence of Continuous Symmetry Breaking in a One-Dimensional n^{-2} Model

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For a one-dimensional array of S^{N-1} spins ($N \geqslant 2$) with isotropic pair interactions (and more general systems) with $J(j-i)$ obeying $\sup_n[n^{-1}\sum_1^n j^2|J(j)|] < \infty$, we prove that every equilibrium state is invariant under the natural action of $SO(N)$. In particular, there is no long-range order of the conventional type. Included is the case $J(n) = n^{-2}$.

KEY WORDS: Continuous symmetry; one-dimensional model; n^{-2} model.

There has been considerable interest in long-range one-dimensional lattice gases, in part because of formal connections with the Kondo problem, and in part because of an analogy with higher-dimensional models: continuous variation of rate of falloff is somewhat akin to continuous variation of dimension.

For pair-interacting ferromagnetic models with coupling $J(j) = j^{-\alpha}$, it has been known for some time that $\alpha = 2$ is the borderline. Ruelle[7] showed if $\alpha > 2$, neither the Ising or multicomponent models have multiple phases; if $\alpha < 2$, then Dyson[1] proved that the Ising model has multiple phases and Frohlich *et al.*[3] proved the same thing for the multicomponent models.

Naturally, interest has focused on the borderline case $\alpha = 2$. Recently, Frohlich and Spencer[4] proved the *existence* of discrete symmetry breaking for the Ising model with this value of α. Our main goal here is to prove the *absence* of continuous symmetry breaking in models of the same type.

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For falloff near n^{-2}, Ruelle's method shows no symmetry breaking for

$$J(j) \lesssim j^{-2}(\log j)^{-\alpha}(\log_2 j)^{-\beta} \tag{1}$$

if $\alpha > 1$ or if $\alpha = 1, \beta > 1$; Dyson[2] allows $\alpha = 1, \beta > 0$, and recent results of Rogers and Thompson[6] allow $\alpha > 0$ or $\alpha = 0, \beta > 1$.

The condition we will need is

$$\sup_n \left[n^{-1} \sum_1^n j^2 |J(j)| \right] < \infty \tag{2}$$

For J's obeying (1), our condition is strictly weaker than those in Refs. 2 or 6, but as we show in an appendix, there are J's which fail to obey (2) but which obey the condition of Ref. 6:

$$\sum_1^n j|J(j)| = o(\log n / \log_2 n)$$

We do emphasize that since Refs. 2 and 6 use correlation inequalities, there are restrictions on the model which we don't need.

The proof is embarrassingly simple; indeed it should be viewed as a postscript on two recent proofs of the absence of continuous symmetry in two dimensions which allow long-range interactions in that dimension.[5,8] We will use Pfister's method here because it is technically somewhat simpler but we emphasize that the Simon–Sokal method would prove our theorems also; indeed their method proves that if $\sum_1^n j|J(j)| = O(n(\log n)^\alpha)$ with $\alpha < 1$, then finite susceptibility would imply no continuous symmetry breaking. This *suggests* that the borderline for continuous symmetry breaking is $n^{-2}(\log n)$ and that at that point there might be a Thouless effect (discontinuous magnetization); we recall that it is known[3] that there *is* continuous symmetry breaking for $n^{-2}(\log n)^\beta$ if $\beta > 1$.

Lemma 1. Let J obey (2) and let $\theta(j)$ be the function which is 1 for $j = 1, \ldots, n$; 0 for $j \geqslant 2n$ or $j \leqslant -n+1$ and which obeys $\theta(j) = 2 - (j/n)$ if $n \leqslant j \leqslant 2n$; $\theta(j) = 1 + [(-1+j)/n]$ if $-n+1 \leqslant j \leqslant 0$ (i.e., middle region of width n and two linear falloff regions of size n). Then

$$\sum_{i \neq j} |J(i-j)| [\theta(i) - \theta(j)]^2 \tag{3}$$

is bounded independently of n.

Proof. Call the region where $\theta = 1$ region I, the region where $\theta = 0$ region II, and the intermediate region, region III. As a preliminary, we note that in the Appendix we show that (2) implies (indeed is equivalent to)

$$\sup_n \left[n \sum_n^\infty |J(j)| \right] < \infty \tag{4}$$

The contribution to (3) from $i \in \mathrm{I}, j \in \mathrm{II}$ is bounded by a multiple of

$$n \sum_{k=n}^{\infty} |J(k)|$$

the n coming from the number of i values and the $k > n$ from the distance between regions I and II. The interaction between regions I and III is bounded by a multiple of

$$\sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \sum_{k=i}^{\infty} |J(k)| \leqslant n \sum_{n}^{\infty} |J(k)| + n^{-1} \sum_{1}^{n} k^2 |J(k)|$$

and a similar bound on the II–III interaction. Thus (2) and (4) show (3) is bounded. ∎

Theorem 1. Consider a model with spins σ_i in S^1 and pair interactions $J(i-j)$ obeying (2). Then every equilibrium state is invariant under the action of $SO(2)$.

Proof. Given any angle ϕ_0, any configuration σ and any n, we can form two configurations σ' and σ'' by rotating spin i by angle $\theta(i)\phi_0$ and $\theta(i)(2\pi - \phi_0)$, respectively (θ as in the lemma). The lemma controls the second-order energy shift so since the first-order shifts have opposite signs either

$$-H(\sigma') \leqslant H(\sigma) + c$$

or

$$-H(\sigma'') \leqslant H(\sigma) + c$$

with c independent of n and σ. From this one concludes the result as in Pfister's paper.[5] ∎

By the same argument, one proves the following result.

Theorem 2. Consider a one-dimensional lattice gas with spins $s_i \in \Omega$ some compact space. Let G be a compact *connected* Lie group which acts on Ω by $(g,s) \to \tau_g s$. Suppose that for each finite volume Λ and each assignment, t, of spins external to Λ, we have

(a) $H_\Lambda(\tau_g s \mid \tau_g t) = H_\Lambda(s \mid t)$ (same g at all sites)

(b) The map $\{g_i\}_{i \in \Lambda} \mapsto H_\Lambda(t_{\tau g_i} s \mid t)$ is C^2 for each s and t

(c) $J(i) \equiv \sup_{\Lambda, t, s} \left| \partial^2 H_\Lambda(\tau_{g_i} s \mid t) / \partial g_i \, \partial g_0 \right|$

obeys (2). Then every equilibrium state is G invariant.

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APPENDIX. CONDITIONS ON $J(j)$

Theorem A.1. Let $J(j), j = 1, 2, \ldots$ be given. Then the two conditions

(a) $$\sup_n \left[n \sum_n^\infty |J(j)| \right] = a < \infty$$

(b) $$\sup_n \left[n^{-1} \sum_1^n j^2 |J(j)| \right] = b < \infty$$

are equivalent.

Proof. Let

$$c(n) = 2^n \sum_{2^n}^{2^{n+1}-1} |J(j)|$$

We will prove (a) and (b) are each equivalent to

(c) $$\sup_n c(n) = c < \infty$$

Clearly $c(n) \leqslant a$ and $c(n) \leqslant 2b$ so (a) or (b) implies (c). Conversely, if $2^n \leqslant k \leqslant 2^{n+1}$, then

$$k \sum_k^\infty |J(j)| \leqslant 2 \left[2^n \sum_{2^n}^\infty |J(j)| \right] \leqslant 2 \left[c(n) + \tfrac{1}{2} c(n+1) + \cdots \right] \leqslant 4c$$

and

$$\begin{aligned} k^{-1} \sum_1^k j^2 |J(j)| &\leqslant 2^{-n} \sum_1^{2^{n+1}-1} j^2 |J(j)| \\ &\leqslant 2^{-n} \sum_{l=1}^n \left[\sum_{2^l}^{2^{l+1}-1} j^2 |J(j)|^2 \right] \\ &\leqslant 4 \sum_{l=1}^n 2^{-n+l} c(l) \leqslant 8c \end{aligned}$$

so (c) implies (a) or (b). ∎

Remark. In Ref. 6, Rogers and Thompson consider the condition

$$\sum_1^n j |J(j)| = o([\log n / \log_2 n])$$

This is as above seen to be equivalent to

$$\sum_1^n c(n) = o(n / \log n) \tag{A.1}$$

If c is not too misbehaved, this is stronger than c bounded but there are $J(j)$'s, e.g., with $c(n) = k$ if $n = 2^k$ and zero otherwise with (A.1) holding but c unbounded. Thus our condition is not strictly weaker than in Ref. 6.

REFERENCES

1. F. J. Dyson, *Commun. Math. Phys.* **12**:212 (1969).
2. F. J. Dyson, *Commun. Math. Phys.* **21**:269 (1971).
3. J. Frohlich, R. Israel, E. Lieb, and B. Simon, *Commun. Math. Phys.* **62**:1 (1978).
4. J. Frohlich and T. Spencer, Bures preprint.
5. C. Pfister, Ecole Polytechnique Fédérale, preprint.
6. J. Rogers and C. Thompson, *J. Stat. Phys.*, **25**:669.
7. D. Ruelle, *Commun. Math. Phys.* **9**:267 (1968).
8. B. Simon and A. Sokal, *J. Stat. Phys.*, **25**:679.