The Codimension of Degenerate Pencils

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ABSTRACT

Let d_n $[d_n(r)]$ denote the codimension of the set of pairs of $n \times n$ Hermitian [really symmetric] matrices (A, B) for which $\det(\lambda I - A - xB) = p(\lambda, x)$ is a reducible polynomial. We prove that $d_n(r) \leqslant n-1$, $d_n \leqslant n-1$ (n odd), $d_n \leqslant n$ (n even). We conjecture that the equality holds in all three inequalities. We prove this conjecture for $n = 2, 3$.

1. INTRODUCTION

The calculation of the codimension of various varieties of matrices has been a useful device in understanding various qualitative aspects of eigenvalue perturbation theory. The most famous and the first of these results is the

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theorem of Wigner and von Neumann [4] which states that the codimension of the variety of $n \times n$ Hermitian matrices with a degenerate eigenvalue in the space of all $n \times n$ Hermitian matrices is independent of n and is equal to three. This implies that "in general," a one-parameter family of Hermitian matrices will not contain a matrix with a degenerate eigenvalue. This result is called in quantum physics "the no-crossing rule".

Consider a pair of complex square matrices (A, B). We identify this pair with the pencil $A(x) = A + xB$, where x belongs to the complex field C. A pencil $A + xB$ is called *nondegenerate* if the polynomial

$$p(\lambda, x) = \det(\lambda I - A - xB)$$

is irreducible over $C[\lambda, x]$. If $A(x)$ is a nondegenerate pencil, all eigenvalues $\lambda_1(x), \ldots, \lambda_n(x)$ of $A(x)$ can be obtained from a single eigenvalue [for example $\lambda_1(x)$] by all possible analytic continuations in x. $A(x)$ is a *degenerate* pencil if $p(\lambda, x)$ is a reducible polynomial. "In general" all the eigenvalues of a reducible pencil cannot be obtained from one eigenvalue. (More precisely, all the eigenvalues of a reducible pencil can be generated from a single eigenvalue if and only if $p(\lambda, x) = q(\lambda, x)^m$, where $q(\lambda, x)$ is irreducible and $m \geqslant 2$. It can be shown that such pencils form a proper subvariety in reducible pencils. See for example [2].)

Let M_n $[M_n(r)]$ denote the set of pairs (A, B) of Hermitian [real symmetric] matrices, and let D_n $[D_n(r)]$ be the set of pairs for which $A + xB$ is a degenerate pencil. Since reducibility of $p(\lambda, x) = \Sigma_{k+j \leqslant n} a_{kj} \lambda^k x^j$, $a_{n0} = 1$, is equivalent to a set of polynomial conditions on a_{kj}, clearly D_n and $D_n(r)$ are varieties in M_n and $M_n(r)$. Here we view M_n and $M_n(r)$ as real spaces of dimension $2n^2$ and $n(n+1)$ respectively. In [1] Avron and Simon gave an explicit example of a real symmetric nondegenerate pair (A, B). Thus D_n and $D_n(r)$ are clearly proper subvarieties, so

$$d_n = \operatorname{codim} D_n = \dim M_n - \dim D_n > 0,$$

$$d_n(r) = \operatorname{codim} D_n(r) > 0.$$

In order to understand some results in the analytic theory of bands in state quantum Hamiltonians, Avron and Simon asked for the exact values of d_n. By identifying a component of D_n they proved $d_n \leqslant 2n - 2$ and conjectured equality, although they emphasized that the evidence for the equality sign was weak. In this paper we will prove that

$$d_n \leqslant n-1 \qquad (n \text{ odd}), \tag{1.1a}$$

$$d_n \leqslant n \qquad (n \text{ even}), \tag{1.1b}$$

$$d_n(r) \leqslant n-1 \qquad (\text{all } n). \tag{1.1c}$$

Thus, the Avron-Simon conjecture is false if $n \geqslant 3$. We believe that the equality holds for (1.1), in part for reasons explained in [2]. In Section 2 we show

$$d_2 = 2, \qquad d_2(r) = 1,$$

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In Section 3 we discuss (1.1) for odd n, and in Section 4 for even n.

We should mention the relevance of (1.1) to the result of Avron and Simon we are trying to understand. They were interested in a theorem of Kohn [3], who considered a class of pencils $A + xB$, where A and B are specific differential operators, B is fixed, and A depends on a function V periodic on $(-\infty, \infty)$ with period 1. For this particular class, Kohn showed that if V is not constant, then all eigenvalues of $A(x)$ can be obtained from any fixed eigenvalue of $A(x)$ by analytic continuation. In a natural n-point difference-equation approximation, A and B are $n \times n$ matrices and V is replaced by an $n \times n$ diagonal matrix. Thus, the intersection of this n-dimensional family with D_n is one-dimensional "when $n = \infty$," as can be understood if $d_n \geqslant n-1$ (the constant function plays a special role in Kohn's analysis, so even if d_n were strictly larger than $n-1$, the one dimensional intersection would not be disturbed). If our conjecture is true, one can understand Kohn's result as a specific case of a generic phenomenon.

2. THE CASES $n = 2, 3$

In the case that $n = 2, 3$, $p(\lambda, x) = \det(\lambda I - A - xB)$ is reducible if and only if $p(\lambda, x)$ is divisible by a linear factor $\lambda - a - xb$. Let $\tilde{A} = A - aI$, $\tilde{B} = B - bI$. Then $p(\lambda, x)$ is divisible by $\lambda - a - xb$ if and only if

$$\det(\tilde{A} + x\tilde{B}) = 0. \tag{2.1}$$

LEMMA 2.1. *The pair (A, B) belongs to D_n $[D_n(r)]$ if and only if A and B commute.*

Proof. Assume first that A and B commute. Then there exists a unitary matrix U such that $A_1 = U^{-1}AU$ and $B_1 = U^{-1}BU$ are diagonal. So $\det(\lambda I - A - xB) = \det(\lambda I - A_1 - xB_1) = (\lambda - a_1 - xb_1)(\lambda - a_2 - xb_2)$. Vice versa, suppose that $\det(\lambda I - A - xB)$ splits to a product of two linear factors. Let $\tilde{A}$ and $\tilde{B}$ be defined as above. It is enough to show that $\tilde{A}$ and $\tilde{B}$ commute. By changing basis we can suppose that

$$\tilde{A} = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} b_1 & c \\ \bar{c} & b_2 \end{pmatrix},$$

since $\det \tilde{A} = 0$. Then (2.1) becomes $b_1\alpha = 0$, $b_1 b_2 - |c|^2 = 0$. If $\alpha = 0$, then $\tilde{A} = 0$, so $[A, B] = AB - BA = 0$ trivially. If $\alpha \neq 0$, then $b_1 = 0$ and the second equality implies $c = 0$. That is, $\tilde{B}$ is diagonal and $\tilde{A}$ and $\tilde{B}$ commute. ■

THEOREM 2.2. *Let D_2 $[D_2(r)]$ be pairs of degenerate 2×2 Hermitian (real symmetric) matrices. Then*

$$\begin{aligned} \dim D_2 &= 6, & d_2 &= 8 - 6 = 2 \\ \dim D_2(r) &= 5, & d_2(r) &= 6 - 5 = 1. \end{aligned} \tag{2.2}$$

Proof. According to Lemma 2.1, $A, B \in D_2$ [or $D_2(r)$] and if and only if $[A, B] = 0$. Either $A = aI$ and B is arbitrary, leading to a component of dimension 5 [or 4], or A is arbitrary and $B = b_1 I + b_2 A$, leading to a component of dimension 6 [or 5]. ■

REMARKS.

(1) The codimension-$(2n-2)$ component found by Avron and Simon consists of pairs (A, B) with a common invariant subspace. For $n = 2$ all degenerate pencils have a common invariant subspace, which explains why they got the correct answer in that case.

(2) Let $M_n(c)$ denote the complex space of all (A, B) where A and B are $n \times n$ complex symmetric matrices. Denote by $d_n(c)$ the complex codimension of the degenerate pencils. Then

$$d_2(c) = 1.$$

The extra condition on D_2 comes from the fact that the single condition $|c|^2=0$ (which is replaced by $c^2=0$ in the complex symmetric case) implies $\operatorname{Re} c=0$ and $\operatorname{Im} c=0$. This example reveals the extra difficulty in computing dimensions of polynomial varieties in R^n as opposed to C^n.

THEOREM 2.3. *Let D_3 $[D_3(r)]$ be pairs of degenerate 3×3 Hermitian (real symmetric) matrices. Then*

$$\begin{aligned} \dim D_3 &= 16, \qquad & d_3 &= 18-16=2, \\ \dim D_3(r) &= 10, \qquad & d_3(r) &= 12-10=2. \end{aligned} \tag{2.3}$$

Proof. Let $(A,B)\in D_3$ $[D_3(r)]$. As in the case $n=2$, $\det(\lambda I - A - xB)$ has a linear factor, so (2.1) holds. After a change of basis,

$$A=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix}, \qquad \tilde{B}=\begin{pmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \tilde{b}_{13} \\ \tilde{b}_{21} & \tilde{b}_{22} & \tilde{b}_{23} \\ \tilde{b}_{31} & \tilde{b}_{32} & \tilde{b}_{33} \end{pmatrix}, \quad \bar{\tilde{b}}_{ij}=\tilde{b}_{ji}.$$

Let us assume the generic case, i.e., $\alpha_1\neq\alpha_2$, $\alpha_1\alpha_2\neq 0$. Then (2.1) becomes

$$\det(\tilde{B})=0, \qquad \alpha_1\alpha_2\tilde{b}_{11}=0,$$

$$\alpha_1\left(\tilde{b}_{11}\tilde{b}_{33}-|\tilde{b}_{13}|^2\right)+\alpha_2\left(\tilde{b}_{11}\tilde{b}_{22}-|\tilde{b}_{12}|^2\right)=0.$$

Since $\alpha_1\alpha_2\neq 0$, the equalities reduce to

$$\det\tilde{B}=0, \qquad \tilde{b}_{11}=0, \qquad \alpha_1|\tilde{b}_{13}|^2+\alpha_2|\tilde{b}_{12}|^2=0. \tag{2.4}$$

The equations (2.4) give rise to two distinct components. For $\alpha_1\alpha_2>0$ the last equality in (2.4) implies $\tilde{b}_{13}=\tilde{b}_{12}=0$. In that case (2.4) reduces to $\tilde{b}_{11}=\tilde{b}_{12}=\tilde{b}_{13}=0$. Taking into account that $\alpha_3=0$ (A has zero eigenvalue), we see that we have lost 6 real parameters (in the real case we lost 4 real parameters). By letting $A=\tilde{A}+aI$, $B=\tilde{B}+bI$ we recover two real parameters. If we denote this component of D_3 $[D_3(r)]$ by A_3 $[A_3(r)]$, then we get

$$\begin{aligned} \operatorname{codim} A_3 &= 4, \qquad & \dim A_3 &= 18-4=14, \\ \operatorname{codim} A_3 &= 2, \qquad & \dim A_3 &= 12-2=10. \end{aligned} \tag{2.5}$$

However, if $\alpha_1\alpha_2<0$, then the last equation in (2.4) eliminates only one real parameter. In that case the conditions (2.4) reduce 3 real parameters in B. If we denote the second component of D_3 $[D_3(r)]$ by B_3 $[B_3(r)]$, then the above arguments show

$$\begin{aligned} \operatorname{codim} B_3 &= 2, & \dim B_3 &= 18-2=16, \\ \operatorname{codim} B_3(r) &= 2, & \dim B_3(r) &= 12-2=10, \end{aligned} \tag{2.6}$$

It is left to consider the case where A has a multiple eigenvalue. Then by the Wigner–von Neumann theorem $\operatorname{codim} W_n = 3$, and one can easily show that $\operatorname{codim} W_n(r) = 2$. Clearly

$$\operatorname{codim}(W_3 \cap D_3) > 3, \qquad \operatorname{codim}[W_3(r) \cap D_3(r)] > 2. \tag{2.7}$$

This establishes the equalities (2.3). ■

REMARK. The A_3 component is precisely the one found by Avron and Simon. It has codimension $4 = 2n - 2$, as they computed.

3. ODD n

To get lower bounds on $\dim D_n$ we need only to find a component of D_n with the required dimension. While not every component of D_n will have a linear factor in $p(\lambda, x) = \det(\lambda I - A - xB)$ when $n \geqslant 4$, according to [2] the component of D_n with the highest dimension is the component for which $p(\lambda, x)$ has a linear factor. Motivated by (2.1) and the proof of Theorem 2.3, we try A with an index $[n/2]$. By considering the matrices $Q\tilde{A}Q^t$, $Q\tilde{B}Q^t$, we may assume that

$$A_0 = \operatorname{diag}(0, 1, -1, 1, -1, \ldots, 1, -1), \tag{3.1}$$

where $n = 2m + 1$.

PROPOSITION 3.1. *Let $A = A_0$ as in (3.1). Then the dimension of the set B of Hermitian matrices B with* $\det(A_0 + xB) = 0$ *is of dimension $n^2 - n$ at least.*

Accepting this result for the moment, let us prove

THEOREM 3.2. *Let D_n be the set of $n\times n$ Hermitian degenerate pairs. Then*

$$\dim D_n \geqslant 2n^2-(n-1), \qquad d_n \leqslant n-1$$

if n is odd.

Proof. Let A be a generic matrix with $2m+1$ distinct eigenvalues. Let $\lambda_1(A)>\lambda_2(A)> \cdots >\lambda_{2m+1}(A)$ be the eigenvalues of A. Then there exists a unitary matrix $U(A)$ which can be chosen to depend smoothly on A in some neighborhood of a A_1, with distinct eigenvalues, such that

$$U(A)^*AU(A)=\operatorname{diag}(\lambda_{m+1}(A),\lambda_1(A),\lambda_{2m+1}(A),\dots,\lambda_m(A),\lambda_{m+2}(A)).$$

Define

$$D(A)=\operatorname{diag}(d_1(A),\dots,d_{2m+1}(A)),$$

$$d_1(A)=1,$$

$$d_{2i}(A)=[\lambda_i(A)-\lambda_{m+1}(A)]^{1/2},$$

$$d_{2i+1}(A)=[\lambda_{m+1}(A)-\lambda_{2m-i+2}(A)]^{1/2}, \qquad i=1,2,\dots,m.$$

Let B be any matrix satisfying $\det(A_0+xB)=0$, where A_0 is given by (3.1). Put

$$C=U(A)D(A)BD(A)U(A)^*+cI. \tag{3.2}$$

Then

$$\det\{A+xC-[\lambda_{m+1}(A)-cx]I\}=\det[U(A)D(A)(A_0+xB)D(A)U^*(A)]$$
$$=0$$

That is, $\det(\lambda I-A-xC)$ has a linear factor $\lambda-\lambda_{m+1}(A)-cx$. A direct count of the parameters shows that this component of degenerate pencils has at least the dimension $2n^2-(n-1)=n^2+n^2-n+1$. ■

Let

$$\det(A_0 + xB) = \sum_{j=1}^{n} q_j(B)x^j. \tag{3.3}$$

Thus, the condition $\det(A + xB) \equiv 0$ is equivalent to n polynomial equations

$$q_j(B) = 0, \qquad j = 1,\ldots,n. \tag{3.4}$$

Therefore over the complex numbers this algebraic variety has codimension n at most. However, since B is taken to be Hermitian, we have to show explicitly that the codimension of (3.4) is at most n. It is easy to see that $q_1(B) = b_{11}$. So (3.4) yields that $b_{11} = 0$. The matrix B is parametrized by $n^2 - 1$ real numbers $\xi_{ij} = \operatorname{Re} b_{ij}$, $\eta_{ij} = \operatorname{Im} b_{ij}$ for $i < j$ and $\xi_{ii} = b_{ii}$ for $1 < i$. For simplicity of notation we denote these parameters by $y_1,\ldots,y_{n^2-1}$, and we view the numbers $q_2(B)$, $q_n(B)$ as the elements of R^{n-1}. Thus the equality (3.3) ($b_{11} = 0$) defines a polynomial map F: $R^{n^2-1} \to R^{n-1}$, $F = (F_1,\ldots,F_{n-1})$. If we can find $y^{(0)}$ with $F(y^{(0)}) = 0$ such that

$$\operatorname{rank} \frac{\partial F_\alpha}{\partial y_\beta}(y^0) = n - 1,$$

then by the implicit-function theorem $\{y \mid F(y) = 0\} \cap$ (a neighborhood y_0) is a smooth manifold of dimension $n^2 - n$. Obviously, it suffices to find $n - 1$ independent parameters $z_1,\ldots,z_{n-1}$ such that the square matrix $\frac{\partial F_\alpha}{\partial z_\beta}(y^0)$ is nonsingular, i.e., the kernel of this matrix is trivial.

Now let $P_i(x)$ be the polynomial $P_i(x) = (\partial/\partial z_i)[\det(A_0 + xB)]$. Then the corresponding kernel is trivial if and only if $P_1(x),\ldots,P_{n-1}(x)$ are linearly independent. Thus we seek n Hermitian matrices $B_0, B_1,\ldots,B_{n-1}$ (the last $n - 1$ matrices linearly independent) such that $\det(A_0 + xB_0) \equiv 0$, and

$$P_i(x, z) = \frac{\partial}{\partial z_i} \det(A_0 + xB_0 + xz_iB_i), \qquad i = 1,\ldots,n-1, \tag{3.5}$$

are linearly independent for $z = 0$. To this end we need the following observation.

LEMMA 3.3. *Let $C=(c_{ij})$ be an $n\times n$ matrix with $c_{ij}=0$ if $i\geqslant 2$, $j\geqslant 2$, and $i\neq j$. Suppose that $c_{jj}\neq 0$ for $j\geqslant 2$. Then*

$$\det C = -\prod_{j=2}^{n} c_{jj}\left(\sum_{j=\alpha}^{n} c_{jj}^{-1}c_{1j}c_{j1}-c_{11}\right). \tag{3.6}$$

Proof. For $j=2,\dots,n$, from the first row, subtract the jth row multiplied by $c_{1j}c_{jj}^{-1}$. The result is a lower triangular matrix with diagonal elements $c_{11}-\sum_{j=2}^{n}c_{jj}^{-1}c_{1j}c_{j1}, c_{22},\dots,c_{nn}$. ■

Proof of Proposition 3.1. We will let B_0 be the Hermitian matrix of the form given by Lemma 3.3 having the diagonal elements $0,1,-1,2,-2,\dots,m,-m$ and the first row $0,1,\dots,1$. By the above lemma

$$\det(A_0+xB_0)=-x^2\prod_{j=1}^{m}(1+jx)(-1-jx)\left[\sum_{j=1}^{m}\frac{1}{1+jx}+\frac{1}{-1-jx}\right]\equiv 0.$$

Let $\sum_{i=1}^{2m}z_iB_i$ be a real symmetric matrix satisfying the conditions of Lemma 3.3 with the diagonal elements $0,z_1,0,z_2,\dots,z_m,0$ and the first row $0,z_{m+1},0,z_{m+2},\dots,z_{2m},0$.

By (3.6)

$$\det\left(A+xB_0+\sum_{i=1}^{n-1}xz_iB_i\right)=Q(x,z)\left[\sum_{j=1}^{m}\frac{(1+z_{m+j})^2}{1+x(j+z_j)}-\frac{1}{1+xj}\right],$$

where

$$Q(x,z)=-x^2\prod_{j=1}^{m}\big(1+x(j+z_j)\big)(-1-xj).$$

So

$$P_j(x,0)=-\frac{Q(x,0)x}{(1-xj)^2},\qquad P_{j+m}=\frac{2Q(x,0)}{1+xj}.$$

The equality

$$\sum_{i=1}^{2m} \alpha_i P_i(x,0) \equiv 0$$

implies

$$\sum_{j=1}^{m} -\alpha_j x(1+xj)^{-2} + 2\alpha_{j+m}(1+xj)^{-1} \equiv 0.$$

Multiplying this identity by $(1+xj)^2$ and letting $x=-1/j$, we deduce that $\alpha_j=0$ for $j=1,\dots,m$. A similar argument implies that $\alpha_{j+m}=0$, $j=1,\dots,m$. This establishes the linear independence of $P_1(x,0),\dots,P_{2m}(x,0)$ and completes the proof of the theorem. ■

Notice that in the above proof all the matrices involved were real symmetric. That is, the set of all real symmetric matrices B satisfying $\det(A_0+xB)\equiv 0$ is at most of codimension n. Then for any generic real symmetric matrix A we construct the matrix C given (3.2), where $U(A)$ is a real orthogonal matrix. As before, we conclude that the codimension of all pencils $A+xC$ such that $\det(\lambda I - A - xC)$ has a linear factor has at most codimension $n-1$.

THEOREM 3.4. *Let $D_n(r)$ be the set of $n\times n$ real symmetric degenerate pairs. Then $d_n(r)\leqslant n-1$, $\dim D_n(r)\geqslant n^2+1$ if n is odd.*

4. EVEN n

The results of Section 2 show that for an even n there is a distinction between the codimension of real symmetric and Hermitian degenerate pencils. A technical reason for that is that if a singular Hermitian matrix A has the equal number of positive and negative eigenvalues, then A has at least a double zero eigenvalue. According to Wigner and von Neumann, the codimension of all such Hermitian matrices is 4. However if we consider all real symmetric matrices with a double zero eigenvalue, the codimension of this set is 3.

In order to prove the inequalities (1.1b) and (1.1c) for an even n we must give the correct analog to the key result of Section 3—Proposition 3.1. The explanation we gave above suggests the "right" form of A_0 for $n=2m+2$:

$$A_0=\text{diag}(0,0,1,-1,1,-1,\dots,1,-1). \tag{4.1}$$

PROPOSITION 4.1. *Let A_0 be as defined above. Then the dimension of the set B $[B(r)]$ of Hermitian [real symmetric] matrices satisfying* $\det(A_0+xB)\equiv 0$ *is of codimension $n-1$ at most.*

Proof. Consider the equality (3.3). Clearly

$$q_1(B)=0, \qquad q_2(B)=(-1)^m(b_{11}b_{22}-|b_{12}|^2). \tag{4.2}$$

Thus if we restrict ourselves to all $B=(b_{ij})$ such that

$$b_{11}b_{22}=|b_{12}|^2, \tag{4.3}$$

then

$$\det(A_0+xB)=\sum_{j=3}^{n} q_j(B)x^j. \tag{4.4}$$

Choose B_0 to be a Hermitian matrix satisfying the assumptions of Lemma 3.3, with the diagonal elements $b_{11}^{(0)}, b_{22}^{(0)}, 1, -1,\dots,m,-m$ and the first row $b_{11}^{(0)}, b_{12}^{(0)}, 1, 1,\dots,1$. Here we assume that $b_{11}^{(0)}b_{22}^{(0)}=|b_{12}^{(0)}|^2>0$.

Again using Lemma 3.3, we easily deduce $\det(A_0+xB_0)=0$.

Now let $\sum_{i=1}^{2m} z_iB_i$ be a real symmetric matrix satisfying the conditions of Lemma 3.3 with the diagonal elements $0,0,z_1,0,z_2,\dots,z_m,0$ and the first row $0,0,z_{m+1},\dots,z_{2m},0$. The calculations carried out in the previous section show that the polynomials $P_1(x,0),\dots,P_{2m}(x,0)$ are linearly independent. That is, the set of all Hermitian matrices $B=(b_{ij})$ satisfying $b_{ij}=b_{ij}^{(0)}$ for $1\leqslant i,j\leqslant 2$ and the equality $\det(A_0+xB)\equiv 0$ is of codimension $4+2m$ at most. However, since we allowed to choose $b_{ij}^{(0)}$, $1\leqslant i,j\leqslant 2$, free within the restriction (4.3), the codimension of B is at most $2m+1$. In the real symmetric case we choose $b_{12}^{(0)}$ to be real, and we deduce as before that the codimension of $B(r)$ is at most $n-1$. ■

THEOREM 4.2. *Let D_n $[D_n(r)]$ be the set of $n\times n$ Hermitian [real symmetric] degenerate pairs. Then*

$$d_n \leqslant n, \qquad \dim D_n \geqslant 2n^2 - n,$$

$$d_n(r) \leqslant n-1, \qquad \dim D_n(r) \geqslant n^2+1$$

if n is even.

Proof. Let A be a Hermitian matrix with a double middle eigenvalue

$$\lambda_1(A) > \cdots > \lambda_m(A) > a = \lambda_{m+1}(A) = \lambda_{m+2}(A) > \cdots > \lambda_{2m+2}(A) \tag{4.5}$$

The Wigner–von Neumann result implies that the codimension of such sets of matrices is 3. Let

$$U(A)^*AU(A) = \operatorname{diag}(a, a, \lambda_1(A), \lambda_{2m+2}(A), \ldots, \lambda_m(A), \lambda_{m+3}(A)),$$

$$D(A) = \operatorname{diag}(d_1(A), \ldots, d_{2m+2}(A)),$$

$$d_1(A) = d_2(A) = 1,$$

$$d_{2i+1}(A) = [\lambda_i(A) - a]^{1/2},$$

$$d_{2i+2}(A) = [a - \lambda_{2m+3-i}(A)]^{1/2}, \qquad i = 1, \ldots, m.$$

Let B be any matrix satisfying $\det(A_0 + xB) \equiv 0$. Define C by (3.3). As in the proof of Theorem 3.2, $\det(\lambda I - A - xC)$ has a linear factor. So the codimension of all pairs (A, C) is at most $3+(n-1)-1 = n+1$. Finally we consider all pencils of the form $(A+\alpha C, C)$, where α is a real parameter and (A, C) is the pencil described above. Clearly $(A+\alpha C, C)$ is also a degenerate pencil. It is left to show that the set of all degenerate pencils of the form $(A+\alpha C, C)$ is not contained in the original set (A, C). To this end it is enough to show that $A+\alpha C$ has n distinct eigenvalues for some A and α. By the definition of C, $A+\alpha C-(a+c)I$ is equivalent to the matrix $A_0+\alpha B$, where $\det(A_0+\alpha B) = 0$. Choose $B = B_0$ as in the proof of Proposition 4.1.

Since $b_{11}^{(0)} b_{22}^{(0)} = |b_{12}^{(0)}|^2 > 0$ for a small $\alpha \neq 0$, $A_0 + \alpha B$ will have only one eigenvalue which is equal to zero. Therefore $A+\alpha C$ has pairwise distinct eigenvalues. Thus the algebraic set of all degenerate pairs of the form $(A+\alpha C, C)$ has at least one codimension less than the set (A, C). That is, the codimension of $(A+\alpha C, C)$ is at most n. In the real case the codimension of all real symmetric degenerate pairs of the form $(A+\alpha C, C)$ is $n-1$, since the codimension of all real symmetric matrices with a multiple eigenvalue is 2.

The proof of the theorem is completed. ■

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