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Almost Periodic Schrödinger Operators

I. Limit Periodic Potentials*

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Abstract. We study $H = -d^2/dx^2 + V(x)$ with $V(x)$ limit periodic, e.g. $V(x) = \sum a_n \cos(x/2^n)$ with $\sum |a_n| < \infty$. We prove that for a generic V (and for generic a_n in the explicit example), $\sigma(H)$ is a Cantor ($\equiv$nowhere dense, perfect) set. For a dense set, the spectrum is both Cantor and purely absolutely continuous and therefore purely recurrent absolutely continuous.

1. Introduction

This is the first of several papers on the spectral properties of operators $-d^2/dx^2 + V(x)$ (and its higher dimensional analogs) with $V(x)$ an almost periodic function. Two themes will recur throughout:

(1) There is a tendency for the spectrum to be a Cantor set ($\equiv$nowhere dense, closed set with no isolated points), albeit one with positive Lebesgue measure.

(2) If V is multiplied by a suitably large constant, there are "mobility edges", in the sense that the spectrum in certain intervals is pure point and otherwise is absolutely continuous (however, if (1) holds the absolutely continuous spectrum must be recurrent in the sense of [2] so that the states are not exactly "mobile"; since Cantor sets are locally uncountable, the point spectrum will be "thick" in the sense of [2]).

We emphasize that while we believe both these phenomena occur for certain almost periodic potentials, we have not yet proven this. In the present paper, we prove (1) for generic limit periodic potentials. We recall

Definition. A function $V(x)$ on $(-\infty, \infty)$ is called *limit periodic* if there exist continuous periodic functions $V_n(x)$ of period L_n so that $\sup_x |V_n(x) - V(x)| \to 0$ as $n \to \infty$.

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We review some of the basic facts about almost periodic functions in Appendix 1; in particular, the V_n can always be chosen so that L_{n+1} is an integral multiple of L_n.

In fact, any limit periodic V can always be written

$$V(x)=\sum_{j=1}^{\infty} W_j(x/n_1 \dots n_j), \tag{1.1}$$

where $\sum_1^{\infty} \|W_j\|_\infty < \infty$, each W_j has the same period L_1 and $n_j = L_{j+1}/L_j$ is an integer greater than or equal to 2. A typical example is

$$V(x)=\sum_{j=1}^{\infty} a_j \cos(x/2^j) \tag{1.2}$$

with

$$\sum_j |a_j| < \infty . \tag{1.3}$$

We will let $\mathscr{L}$ be the set of all limit periodic functions with the metric induced by $\|\cdot\|_\infty$. We let $\mathscr{M}$ be the set of all $V\in\mathscr{L}$ of the form (1.2) with the norm induced by (1.3). $\mathscr{M}$ is a Banach space. $\mathscr{L}$ is not a vector space but it is a complete metric space. As such, it is a Baire space [5] and any countable intersection of dense open sets is itself dense. Such dense G_δ's are called generic sets. Our two main theorems are:

Theorem 1. *For a generic element, V, of $\mathscr{L}$ (respectively $\mathscr{M}$), the spectrum of $H \equiv -d^2/dx^2 + V$ is a Cantor set.*

Theorem 2. *There is a dense set of V in $\mathscr{L}$ (respectively $\mathscr{M}$), for which* (i) *the spectrum of H is a Cantor set.* (ii) *The spectrum of H is purely absolutely continuous. For the $\mathscr{M}$ case, the spectrum is moreover of uniform multiplicity* 2.

We emphasize that the set in Theorem 2 is *not* claimed to be generic. Indeed, our expectation (2) above says that we think the behavior will not be generic. Theorem 2 says that the possibility of "recurrent absolutely continuous spectrum" which we introduced in [2] occur for many simple looking differential operators.

In a recent preprint, Moser proved that for a dense set in $\mathscr{L}$, the spectrum is a nowhere dense set [17]. While our work is independent of Moser's, his work predates ours by several months. In comparing Theorem 1 with the result of Moser, note first that while Moser states and proves nowhere dense, the lack of isolated points is a general property that is not difficult to prove (see Appendix 2). Second, Moser only handled a dense set of V, but he remarks that genericity would follow if the rotation number he discusses always exists. The existence has been proven by Russell Johnson [13] and subsequently from a rather different point of view ("integrated density of states") by us [3]. Moreover as we shall see shortly (Lemma 1.1), it is very easy to prove directly that $\{V| -d^2/dx^2 + V(x)$ has a nowhere dense spectrum$\}$ is a G_δ.

We emphasize that not only is our result in Theorem 1 close to Moser's, our proof is related. The overall strategy is the same but the tactics are rather different and we feel somewhat simpler. There is an interesting distinction in the approach.

While the use of Floquet theory and discriminants is not unknown in the physics literature, it is unusual – Bloch wave analysis over quasi-momentum space is the usual tool because it extends to higher dimension. (This is particularly unfortunate in studying the Kronig-Penny model where the Floquet theory approach is much superior.) On the other hand, the mathematical theory tends to use almost exclusively Floquet theory and discriminants. It's as if everyone insisted on studying these problems with one hand tied behind their back but half chose their right hand and half their left. While we primarily exploit the Bloch wave analysis, certain estimates critical to Theorem 2 are proven using the discriminant. In physicist's language we prove an upper bound on the widths of energy bands. While Moser exploits manipulation of the differential equation or discriminant, we exploit eigenvalue perturbation theory.

Moser uses mainly Floquet theory. However, by introducing the "rotation number", α, as a function, of E, he is introducing a quasimomentum analysis in disguised form. A main difference in his analysis and ours is that he tends to study α as a function of E and we study E as a function k (which is related to α). In the fully a.p. case it is very useful to think of α as a function of E and this we do in a future publication [3].

We should explain why the limit periodic potentials play a special role. Because V can be approximated in uniform norm by periodic functions, H is a limit in the *norm* resolvent sense of periodic Schrödinger operators while general a.p. potentials only lead to H's which are limits in the *strong* resolvent sense of periodic Schrödinger operators. Since spectrum can be lost in taking strong limits, the difference is significant. Because of that fact, to prove Theorems 1 and 2, one needs mainly detailed information about periodic operators. We do this in Sect. 2; the most interesting new result concerns control on the L^p norms of spectral densities and the density of states ($p<2$). With this background, it is fairly easy to prove Theorems 1 and 2 in Sect. 3. In Appendix 1, we present some of the basic features of a.p. functions and in Appendix 2 some simple general features of the a.p. case.

We want to note that Theorem 1 has various extensions. First there is nothing sacred about $\mathcal{L}$ and $\mathcal{M}$; Theorem 1 will hold for any complete metric space, $\mathcal{V}$, of limit periodic functions with metric ϱ obeying:

(i) ϱ dominates $\|\cdot\|_\infty$, i.e. $\|f-g\|_\infty \leqq \varrho(f,g)$.

(ii) Let $\mathcal{P}_\ell$ denote the periodic functions of period ℓ. We suppose there is a distinguished set, $\mathcal{S}$, of reals so that for $\ell\in\mathcal{S}$, $\mathcal{P}_\ell\cap\mathcal{V}\neq\emptyset$ and $\bigcup_{\ell\in\mathcal{S}}(\mathcal{P}_\ell\cap\mathcal{V})$ is dense in $\mathcal{V}$.

(iii) For a dense set in $\mathcal{P}_\ell\cap\mathcal{V}$, all gaps associated to period ℓ (see Sect. 2) are open.

(iv) For any $\ell\in\mathcal{S}$, there is $m\in\mathcal{S}$ with $m>\ell$ and m/ℓ integer.

For example, for any k, we can take $\mathcal{V}$ to be the limits in C^k norm of periodic functions with ϱ the C^k norm. Or we could consider the functions analytic in the strip $|\mathrm{Im}\, z|\leqq a$ with sup norm over the strip as norm.

In another direction, one can replace $\|\cdot\|_\infty$ by the norm

$$\|V\| = \sup_{-\infty<n<\infty} \int_n^{n+1} |V(x)|dx \tag{1.4}$$

and prove Theorem 1 for $\mathscr{L}^1$, the space of limits of continuous periodic functions in this norm, with metric induced by this norm. To see this one need only note that the operator norm of $V(-d^2/dx^2+1)^{-1}$ is dominated by a multiple of $\|\cdot\|$ so the various norm resolvent arguments are still applicable.

As a final remark in this introduction, we want to note that the set of V's leading to nowhere dense spectrum is always a G_δ. Since $\sigma(-d^2/dx^2+V)$ is always perfect (Corollary A.2.3), this reduces Theorem 1 to a density theorem.

Lemma 1.1. *Let $\mathscr{V}$ be a metric space of continuous (not necessarily almost periodic) functions on $\mathbb{R}^\nu$ with metric ϱ obeying $\varrho(V,W \geqq \|V-W\|_\infty$. Then $N \equiv \{V\in\mathscr{V} \mid -\Delta+V$ has nowhere dense spectrum$\}$ is a G_δ (countable intersection of open sets).*

Proof. For a,b fixed with $a<b$, let

$$S_{(a,b)}=\{V\in\mathscr{V} \mid (a,b)\cap\text{resolvent set }(-\Delta+V)\neq\emptyset\}.$$

Then

$$N=\bigcap_{a,b\text{ rational},\,a<b} S_{(a,b)}$$

so it suffices to prove that S is open. Let $V\in S_{(a,b)}$. Then since resolvent-set $\varrho(H)$ is open, there exists c,δ so that

$$(c-\delta,c+\delta)\subset(a,b)\cap\text{resolvent set }(-\Delta+V).$$

By general principles if $\|V-W\|_\infty<\delta$, then c is contained in the resolvent set of $-\Delta+W$. [For $\|(-\Delta+V-c)^{-1}\|<\delta^{-1}$ by the spectral theorem, so $1+(W-V)(-\Delta+V-c)^{-1}$ is invertible if $\|V-W\|_\infty<\delta$.] □

2. Some Features of Periodic Potentials

The analysis of limit periodic potentials depends on having enough information about periodic potentials. We develop this information in the present section taking care to compare our results and methods with those of Moser [17], where applicable. For additional background on periodic potentials see [26], Sect. XIII.16 of [21] or the pretty book of Eastham [9].

We begin by briefly reviewing quasimomentum analysis to settle notation. In analyzing potentials of period L, we introduce $K\equiv\pi/L$ and the Brilluoin zone $\mathscr{B}=(-K,K]$. For each k ($\equiv$quasimomentum) in $\mathscr{B}$, we introduce a Hilbert space, $\mathscr{H}_k$, with orthonormal basis $\{\Psi_n^{(0)}(k)\}_{n=-\infty}^{\infty}$. We regard $\Psi_n^{(0)}(k)$ as the plane wave $L^{-1/2}\exp(i[2nK+k]x)$, either as a function all of $(-\infty,\infty)$ or only on $[0,L]$. Viewing it in the former way, the Fourier transform gives us a realization of $L^2(-\infty,\infty)$ as the direct integral $\int_{\mathscr{B}}^{\oplus}\mathscr{H}_k\left[\frac{dk}{2K}\right]$. Explicitly, given $\varphi\in C_0^\infty(-\infty,\infty)$, we define

$$\begin{aligned}\tilde{\varphi}^{(0)}(n,k)&=\int_{-\infty}^{\infty}dx\,\overline{\Psi_n^{(0)}(k;x)}\varphi(x)\\&=(2\pi/L)^{1/2}\hat{\varphi}(2nK+k),\end{aligned}\tag{2.1}$$

where $\hat{\varphi}$ is the conventional Fourier transform. The Plancherel theorem then reads

$$\int_{-K}^{K} \sum_{n=-\infty}^{\infty} |\tilde{\varphi}^{(0)}(n,k)|^2 \frac{dk}{2K} = \int |\varphi(x)|^2 dx. \tag{2.2}$$

If we define a vector $\varphi^{\#}(k) = \sum_n \tilde{\varphi}^{(0)}(n,k)\Psi_n^{(0)}(k)$ in $\mathscr{H}_k$, (2.2) says that the map $\#$ is a unitary form $L^2(-\infty,\infty)$ to the direct integral.

If we view the $\Psi_n^{(0)}$ as functions on $[0,L]$, they are the natural Fourier basis for functions obeying the boundary condition:

$$\varphi(L) = e^{i\theta}\varphi(0); \qquad \varphi'(L) = e^{i\theta}\varphi'(0), \tag{2.3}$$

where

$$\theta(k) = kL = \pi(k/K), \tag{2.3'}$$

so that θ runs from $-\pi$ to π as k runs through the Brilluoin zone.

The operator $-d^2/dx^2$ on $L^2(0,L)$ with boundary condition (2.3) is self-adjoint with eigenfunctions $\Psi_n^{(0)}(k;x)$ with eigenvalues $(2nK+k)^2$. Thus, we define operators $H_0(k)$ on $\mathscr{H}_1$ by

$$H_0(k)\Psi_n^{(0)}(k) = \varepsilon_n^{(0)}(k)\Psi_n^{(0)}(k), \tag{2.4}$$

$$\varepsilon_n^{(0)}(k) = (2nK+k)^2, \tag{2.4'}$$

and $-d^2/dx^2$ on $L^2(-\infty,\infty)$ is just the direct integral of the $H_0(k)$.

If V is a continuous function of period L, we introduce an operator which we call V on $\mathscr{H}_k$ with matrix elements

$$(\Psi_n^{(0)}(k), V\Psi_m^{(0)}(k)) = \int_0^L V(k)\exp[2iK(m-n)x]dx/L. \tag{2.5}$$

Under the association of $L^2(0,L)$ with $\mathscr{H}_k$ given by the $\Psi_n^0(k)$ Fourier basis, multiplication by V goes over to V. The operator

$$H(k) = H_0(k) + V \tag{2.6}$$

is related to $H = -d^2/dx^2 + V$ on $L^2(-\infty,\infty)$ by

$$H = \int_{\mathscr{B}}^{\oplus} H(k)\frac{dk}{2K}. \tag{2.7}$$

The basis $\Psi_n^0(k)$ defines a natural association of $\mathscr{H}_k$ with ℓ_2, and the image of $H(k)$ we will denote be $H^b(k)$. It is somewhat pedantic to do this but useful because $H(k)$ viewed as an operator on $L^2(0,L)$ with boundary condition (2.3) has a k dependent domain but $H^b(k)$ has a k-independent domain, i.e. those $\{a_n\}\in\ell_2$ with $\sum n^4 a_n^2 < \infty$. The k dependence of $H^b(k)$ [but not of the $H(k)$ viewed as operators on $L^2(0,L)$] is:

$$H^b(k) = H^b(0) + 4kKN + k^2, \tag{2.8}$$

where $(Na)_n = na_n$.

$H(k)$ is a bounded-operator perturbation of $H_0(k)$ which has compact resolvent so $H(k)$ has a compact resolvent and thus there is an orthonormal complete set

$\Psi_n(k)$ of eigenfunctions of $H(k)$ with eigenvalues $\varepsilon_n(k)$. We can think of $\Psi_n(k)$ as functions on $(-\infty, \infty)$. They will obey the boundary condition (2.3) and the differential equation

$$\left(\frac{-d^2}{dx^2} + V\right)\Psi_n(k,x) = \varepsilon_n(k)\Psi_n(k,x). \tag{2.9}$$

since $\bar{\Psi}_n$ obeys the same equation and a different boundary condition if $k \neq 0, K$ and since (2.9) has at most two independent solutions we see that $\varepsilon_n(k)$ is a simple eigenvalue of $H(k)$ for $k \neq 0, K$ and that

$$\varepsilon_n(-k) = \varepsilon_n(k). \tag{2.10}$$

By the simplicity, $\varepsilon_n(k)$ is analytic on $(0, K)$ and by first order perturbation ("Feynman-Hellman" theorem) and (2.8):

$$\begin{aligned}\frac{d\varepsilon_n}{dk} &= 2(\Psi_n^b, (2KN+k)\Psi_n^b)\\ &= 2\left(\Psi_n(k), \left[\frac{1}{i}\frac{d}{dk}\right]\Psi_n(k)\right],\end{aligned} \tag{2.11}$$

since under the map from $L^2(0, L)$ to $\mathscr{H}_k$ and then to ℓ_2, $\frac{1}{i}\frac{d}{dx}$ goes to $2KN+k$. This is known as the velocity theorem in solid state [27] because it relates the expectation of $\dot{x} = 2p$ with the group velocity $\frac{d\varepsilon_n}{dk}$.

Notational Warning: $\Psi_n^{(0)}(k)$ were indexed by n running from $-\infty$ to ∞ but we will index $\Psi_n(k)$, $\varepsilon_n(k)$ from $n=0$ to $n=\infty$ so that $\varepsilon_0(k) \leqq \varepsilon_1(k) \leqq \dots$ (strict inequality if $k \neq 0, K$). For $k \geqq 0$, the counting is that the labeling $\Psi_n^{(0)}$ ordered by $(0, -1, 1, -2, 2, \dots)$ go into $(0, 1, \dots)$ under the Ψ_n labeling.

Floquet theory depends on the discriminant, $F(E)$, defined to be the trace of the 2×2 matrix $M(E)$ given by $M(E)(a, b)$ is the pair (c, d) with $c = u(L)$, $d = u'(L)$ where u obeys

$$-u'' + Vu = Eu \tag{2.12}$$

with initial condition $u(0) = a$, $u'(0) = b$. If $E = \varepsilon_n(k)$, (2.9) plus the boundary condition (2.3) says that $M(E)$ has eigenvalue $e^{i\theta}$. By complex symmetry it has a second eigenvalue $e^{-i\theta}$, so

$$F(\varepsilon_n(k)) = 2\cos(\theta(k)). \tag{2.13}$$

This argument can be turned around and implies that any solution of $F(E) = 2\cos(\theta(k))$ is a value of $\varepsilon_n(k)$. Further analysis of F (see e.g. [21]) shows that on $(0, K)$, $d\varepsilon_n/dk > 0$ for $n = 0, 2, \dots$ and $d\varepsilon_n/dk < 0$ for $n = 1, \dots$. Thus the spectrum of H which is the union of the spectrum of the $H(k)$ is

$$[\varepsilon_0(0), \varepsilon_0(K)] \cup [\varepsilon_1(K), \varepsilon_1(0)] \cup \dots,$$

and the complement is a semi-infinite interval $(-\infty, \varepsilon_0(0))$ and a set of "gaps" (perhaps empty). The nth gap ($n=1,2,\ldots$) is given by

$$(\varepsilon_{n-1}(K), \varepsilon_n(K)) \quad n \text{ odd}, \tag{2.14}$$

$$(\varepsilon_{n-1}(0), \varepsilon_n(0)) \quad n \text{ even}. \tag{2.14'}$$

Implicit in the above is a complete spectral decomposition of H as an operator on $L^2(-\infty, \infty)$ (going back to Gel'fand [10]):

Define for $k\in\mathscr{B}$, $n=0,1,\ldots$

$$\tilde{\varphi}(n;k)=\int_{-\infty}^{\infty} dx\overline{\Psi_n(k;x)}\varphi(x). \tag{2.15}$$

Then by the direct integral decomposition:

$$\int\left[\sum_{n=0}^{\infty}|\tilde{\varphi}(n,k)|^2\right]\frac{dk}{2K}=\|\varphi\|^2, \tag{2.16}$$

$$\int g(\varepsilon_n(k))\left[\sum_{n=0}^{\infty}|\tilde{\varphi}(n,k)|^2\right]\frac{dk}{2K}=(\varphi, g(H)\varphi). \tag{2.17}$$

From this and the strict monotonicity of ε_n on $(0,K)$, we read off the fact that H has purely absolutely continuous spectrum and that φ has a spectral measure

$$d\mu_\varphi(E)\equiv G_\varphi(E)dE, \tag{2.18a}$$

$$G_\varphi(E)=[|\tilde{\varphi}(n,k)|^2+|\tilde{\varphi}(n,-k)|^2]\varrho(E), \tag{2.18b}$$

$$\varrho(E)=[|d\varepsilon_n/dk|]^{-1}, \tag{2.18c}$$

where n and k are determined by

$$E=\varepsilon_n(k); \qquad 0\leqq k\leqq K \tag{2.18d}$$

[$G=0$ if (2.18d) has no solution]. $\varrho(E)$ is called the *density of states* for reasons described in [21] or [3].

We conclude our review of this material by noting that Moser's rotation number [17], $\alpha(E)$, is expressible in terms of (2.18d) by

$$\alpha(E)=k+nK \qquad n=0,2,4,\ldots \tag{2.19a}$$

$$=-k+nK \quad n=1,3,\ldots, \tag{2.19b}$$

so that

$$\frac{d\alpha}{dE}=\varrho(E). \tag{2.20}$$

We do not claim that (2.19) is obvious but since we only require it for comparison purposes we defer its proof to a later paper, where we systematically discuss α as "an integrated density of states".

Having completed this review, we turn to the new results:

(a) Upper bound on $d\varepsilon_n/dk$ and bounds on band size.

(b) Upper bound on $\varrho(E)$ and L^p properties of the spectral density G_φ.

(c) Bounds on total gap size.

Finally, we will recall the results on genericity of open gaps found in [22] and which were rediscovered by Moser [17] in a slightly different form.

Theorem 2.1. *Let* $a \equiv -\inf_x (V(x))$. *Then*

$$\left|\frac{d\varepsilon_n}{dk}\right| \leqq 2\sqrt{\varepsilon_n(k)+a}\,. \tag{2.21}$$

Proof. By (2.11) and the Schwarz inequality

$$\pm\left[\frac{i}{2}\frac{d\varepsilon_n}{dk}\right]^2 \leqq \left(\Psi_n, \frac{-d^2}{dx^2}\Psi_n\right) \leqq (\Psi_n, (H+a)\Psi_n) = \varepsilon_n(k)+a\,. \quad \square$$

Remarks. 1. One can compare this with Moser's result [17] that $d\alpha^2/dE \geqq 1$. By (2.19), in Moser's language, (2.21) reads

$$\frac{dE}{d\alpha} \leqq 2\sqrt{E+a}\,. \tag{2.22}$$

Since $H(k) \leqq H_0(k)+b$ we have $\varepsilon_n(k) \leqq b+\varepsilon_n(k;V=0)$ $\left[\text{with } b=\sup_x V(x)\right]$ and since (2.19) reads $\alpha(E)=\sqrt{\varepsilon_n(k;V=0)}$, we have

$$E \leqq b+\alpha^2\,, \tag{2.23}$$

so that

$$\frac{dE}{d\alpha} \leqq 2\sqrt{\alpha^2+(a+b)} \tag{2.24}$$

to be compared with Moser's $dE/d\alpha \leqq 2\alpha$. Pushed through to this form, Moser's result is slightly stronger but on the other hand it is not clear how to get (2.22) from his result. For the applications any of his results, (2.22) or (2.24) will suffice.

2. It is worth seeing that the perturbation theory result (2.11) is equivalent to Dubrovin's result [8]

$$\frac{d\alpha}{dE} = \int_0^L D(x)dx\,,$$

where $D(x)=(2L)^{-1}|w(x)|^2$ and w is the multiple of Ψ normalized by $\bar{w}w' - w\bar{w}' = 2i$. Let $\Psi = aw$. Then integrating by parts

$$\left(\Psi, \frac{1}{i}\frac{d}{dx}\Psi\right) = \frac{a^2}{2}\int_0^L i^{-1}[\bar{w}w' - \bar{w}'w]dx = a^2 L\,.$$

On the other hand

$$1 = (\Psi, \Psi) = a^2\int_0^L w^2 dx = 2La^2\int_0^L D(x)dx\,,$$

proving the equivalence.

Corollary 2.2. *An individual band* $[E, E']$ $\{$*with* $E=\varepsilon_n(0)$, $E'=\varepsilon_n(K)$ $(n=0,2,4,\ldots)$ *or* $E=\varepsilon_n(K)$, $E'=\varepsilon_n(0)$ $(n=1,3,\ldots)\}$ *obeys*

$$|E'-E| \leqq \pi\sqrt{E'+a}/L\,. \tag{2.25}$$

Proof. Integrate $d\varepsilon_n/dk$ from 0 to K or K to 0. $\square$

The point of (2.25), is that it says that if all gaps are open, the connected components of $\sigma(H)$ in some fixed energy range go to zero $L\to\infty$.

As a preliminary to studying L^p properties of spectral measures, we need to get L^p properties of $\varrho(E)$ and therefore to control $d\varepsilon_n/dk$. We do this using the discriminant:

Lemma 2.3. (i) $|f(E)|\leqq 2\exp(L[\sqrt{|E|}+\sqrt{\|V\|_\infty}])$ *even for E complex*

(ii) $|f^{(m)}(E)|\leqq 2m!\exp(L[1+\sqrt{|E|}+\sqrt{\|V\|_\infty}])$.

Proof. (i) is standard; see e.g. Magnus and Winker [16, p. 20–21].

(ii) Follow from (1) and a Cauchy integral formula using a circle of radius 1. (If one cares about large m, one should take a circle of radius m^2/L to get a much better bound.) □

We define

$$\varrho_n(k)=\left[\left|\frac{d\varepsilon_n}{dk}\right|\right]^{-1} \tag{2.26}$$

so

$$\varrho(E)=\varrho_n(k) \tag{2.27}$$

if E, n, k are related by (2.18d).

Theorem 2.4. (i) $\varrho_n(k)\leqq\exp[L(1+\sqrt{|E_n(k)|}+\sqrt{\|V\|_\infty}]/|\sin\theta(k)|L$.

(ii) *For any N, L, C and $p<2$, there is a D with*

$$\int_0^K \varrho_n(k)^{(p-1)}dk\leqq D \tag{2.28}$$

for any $n<N$, and all V of period L with $\|V\|_\infty\leqq C$.

Proof. (i) Using $d\theta/dk=L$ and taking derivatives with respect to k in (2.13) yields

$$F'(\varepsilon_n(k))\frac{d\varepsilon_n}{dk}=2\sin(\theta(k))L. \tag{2.29}$$

Lemma 2.3 yields the required result.

(ii) (2.23) yields a bound on $\varepsilon_n(k)$ depending only on n and $\|V\|_\infty$. Since $\int_0^k|\sin\theta(k)|^{-(p-1)}dk<\infty$, (2.28) results. □

Remarks. 1. Taking another derivative in (2.29) shows that at points with $d\varepsilon_n/dk=0$ (necessarily at $k=0$ or K)

$$F''(E_n(K))\frac{d^2\varepsilon_n}{dk^2}=\pm 2L^2,$$

so that

$$\frac{d^2\varepsilon_n}{dk}\geqq\tfrac{1}{2}L^2\exp(-L(1+\sqrt{|\varepsilon_n(k)|}+\sqrt{\|V\|_a}) \tag{2.30}$$

(if $d\varepsilon_n/dk=0$). In physical terms this is an upper bound on an effective mass.

2. Integrating (2.29) shows that

$$|\varepsilon_n(k) - \varepsilon_n(k')| \geqq C(k-k')^2 \tag{2.31}$$

yielding the Hölder continuity of order $\frac{1}{2}$ of $\alpha(E)$ which Moser mentions without proof in [17]. (Actually, Moser quotes Hölder continuity in V but this then follows from the simple bound:

$$\alpha_V(E - \|W-V\|_\infty) \leqq \alpha_W(E) \leqq \alpha_V(E + \|W-V\|_\infty),$$

which we will prove in [3].)

Theorem 2.5. (i) *Let* $\varphi \in C_0^\infty(-\infty, \infty)$. *Let* (E', E) *be the nth band. Then for any* $p<2$, *there is a constant D depending only on* φ, *n*, $\|V\|_\infty$, *p*, *and L so that* (G_φ *given by* (2.18))

$$\int_{E'}^{E} |G_\varphi(E)|^p dE \leqq D. \tag{2.32}$$

(ii) (2.32) *remains true for a D depending only on* φ, *E*, $\|V\|_\infty$, *p*, *and L* (*even if* (E', E) *is not a single band*).

Proof. (i) Suppose φ is supported in an interval of length mL.
Then

$$\int_{\operatorname{supp}\varphi} |\Psi_n(k,x)|^2 dk \leqq m,$$

since Ψ is normalized on an interval of length L. Thus, by the Schwarz inequality

$$|\tilde{\varphi}(n;k)| \leqq \sqrt{m}\|\varphi\|_2.$$

Thus, by (2.18), (2.32) follows from

$$\int_{E'}^{E} [\varrho(E)]^p dE \leqq \text{const}.$$

But $E = \varepsilon_n(k)$ on a single band and $\varrho(E)\left|\frac{d\varepsilon_n}{dk}\right| = 1$ so this follows from (2.28).

(ii) By (i), we only need a bound on the number of distinct bands that can occur. This follows from

$$\varepsilon_n(k) \geqq \varepsilon_n(k, V=0) - \|V\|_\infty$$

and the form of $\varepsilon_n(k, V=0)$. □

Remarks. 1. All that was needed was $\varphi \in L^2$ with $\operatorname{supp}\varphi$ bounded, not $\varphi \in C_0^\infty$.

2. Since $d\varepsilon_n/dk = 2L\sin\theta(k)/f'(\varepsilon_n(k))$, if $d\varepsilon_n/dk = 0$ at the edge of a band (and this will happen if the gap is open) then $f'(\varepsilon_n(k)) \neq 0$ and thus $\int [d\varepsilon_n/dk]^{p-1} dk = \infty$ if $p \geqq 2$. For most φ's it follows that $\int |G_\varphi|^p dE = \infty$ if $p \geqq 2$.

3. By the Hausdorff-Yang inequality, $\int |(\varphi, e^{-ith}\varphi)|^q dt < \infty$ if $q > 2$. The divergence at $q = 2$ is a mirror of the $t^{-1/2}$ falloff that will occur for most φ.

Next we want continuity of G in L^p norm as V is varied.

Lemma 2.6. *Let $(\Omega, d\mu)$ by finite measure space. Let $r>1$ and let $f_n, f \in L^r$ with $\sup_n \|f_n\|_r < \infty$. Suppose that $f_n(w) \to f(w)$ pointwise a.e. Then $\|f_n - f\|_p \to 0$ for any $p<r$.*

Proof. Without loss, take $f=0$. Fix M and let $g_n(w)=f_n(w)$ if $|f_n(w)| \leqq M$ and zero otherwise. Then, $\|g_n\|_p \to 0$ by the dominated convergence theorem and

$$\|f_n - g_n\|_p^p \leqq \int_{|f_n|>M} |f_n(w)|^p dw \leqq M^{p-r} \|f_n\|_r^r,$$

so $\overline{\lim} \|f_n - g_n\|_p$ can be made arbitrarily small. □

Theorem 2.7. *Let V_n, V be continuous periodic functions with period L and let $G_\varphi^V, G_\varphi^{V_n}$ be the spectral densities for $H_n = -d^2/dx^2 + V_n$ and $H = -d^2/dx^2 + V$. Suppose that $\|V_n - V\|_\infty \to 0$ as $n \to \infty$. Then, for any fixed E' and $p<2$,*

$$\int_{-\infty}^{E'} |G_\varphi^{V_n}(E) - G_\varphi^V(E)|^p dE \to 0.$$

Proof. By the lemma and by Theorem 2.5, we need only prove pointwise convergence a.e. (since the H_n are uniformly bounded below $-\infty$ can be replaced by some E_0 so that the integrals are over finite regions). If E is in a gap for V, it will be in a gap for all large n (by the norm resolvent convergence), so $G_\varphi^{V_n}(E)=0 = G_\varphi^V(E)$ for all large n. If $E = \varepsilon_m(k)$ with $0<k<K$, the eigenvalue is simple and by a perturbation argument, there exist $k_n \to k$ so $\varepsilon_m^{V_m}(k_n) = E$, the corresponding Ψ's converge and by a Vitali theorem argument so does $d\varepsilon_n/dk$. Thus $G_\varphi^{V_n}(E)$ converges to $G_\varphi^V(E)$. The remaining points [i.e. E's with $E = \varepsilon_m(k)$ with $k=0$ or K] have measure zero. □

We are next interested in results of the form:

Pseudo Theorem 2.8. *Fix L and some normed space, X, of continuous functions of period L. Then for any C, there is a sequence g_n so that $\sum g_n < \infty$ and if $V \in X$ with $\|V\| \leqq C$, then*

$$\Delta_n(V) \leqq g_n,$$

where $\Delta_n(V)$ is the nth gap for $-d^2/dx + V$.

This inequality, the dominated convergence theorem for sums and the continuity of $\Delta_n(V)$ in V (which follows from eigenvalue perturbation theory and the fact that Δ_n is a difference of eigenvalues) shows that

Pseudo Corollary 2.9. *If $V_m, V \in X$ with $\|V_m - V\|_\infty \to 0$, then*

$$\sum_n \Delta_n(V_m) \to \sum_n \Delta_n(V).$$

We can prove the pseudo theorem or at least a bound on $\sum \Delta_n$ in three cases:

(1) $V \in C^3$ with C^3 norm. The argument in Theorem 2.12 in [16] (essentially due to Höchstadt [11]), shows that $\Delta_n(V) = 0(n^{-2})$ with errors only depending on the first three derivatives of V [that theorem only supposes V is C^2, but one needs some uniform control on the Fourier coefficients of V'' to get uniform control on $\Delta_n(V)$].

(2) Deift [7] has remarked that the result holds for $V \in C^1$ with the C^1 topology by the following beautiful argument. Given $V \in C^1$, consider the class of all W's with the same gaps [25]. If the nth gap is (α_n, β_n), then the eigenvalues μ_n of $\frac{-d^2}{dx^2} + W(x)$ with $u(0) = u(L) = 0$ boundary conditions obey $\alpha_n \leqq \mu_n \leqq \beta_n$ and, there is $C(W)$ (the lowest periodic eigenvalue) so that

$$C(W) + \sum_n [\alpha_n + \beta_n - 2\mu_n]$$

is independent of W and given as an integral of V and V' (a KdV conserved quantity) [25]. Among these W, there is one with $\mu_n = \alpha_n$ for all n [25], so since $C(W)$ can be bounded in terms of V, $\sum_n (\beta_n - \alpha_n)$ can be bounded in terms of V and V'. This gives a bound on $\sum_n \Delta_n$ but not the uniform type needed to get the pseudo corollary.

(3) We can handle a very general class of even periodic functions. For in that case, we have proven in [1] that the perturbation series in λ for $\beta_n(\lambda V)$ and $\alpha_n(\lambda V)$ converge for all large n and if $\alpha_n(\lambda V) = \sum a^n_m \lambda^m$; $\beta_n(\lambda V) = \sum b^n_m \lambda^m$, then $|a^n_m| + |b^n_m| \leqq C^m n^{-(m-1)}$ where C only depends on $\|V\|_\infty$. It follows that to control Δ_n we need only control $b^n_1 - a^n_1$ and $b^n_2 - a^n_2$ and these have simple explicit formulas in terms of the coefficients V_{n0} (Fourier coefficients of V). If $g_n \in \ell_1$ and X is the set of *even* V with $\sup(V_{n0}/g_n) < \infty$, this argument proves the pseudo theorem in that case.

Finally, we want to recall a result from [22] and prove an analog for $\mathcal{M}$:

Theorem 2.10. *Fix a Banach space X of periodic potentials of period L. Suppose that X is dense in $L^1[0, L]$ and that the norm on X dominates $\|\cdot\|_\infty$. Then for each n, the set of V for which $\Delta_n(V) \neq 0$ is a dense open set.*

The proof is simple: That the set is open is an easy consequence of the fact that $\|\cdot\|_X$ dominates $\|\cdot\|_\infty$. To prove density, suppose that $\Delta_n(V) = 0$. Then either $H(0)$ or $H(K)$ has a degenerate eigenvalue E (depending on whether n is even or odd) and the gap opens for $V + \lambda W$ if the degeneracy of this eigenvalue is removed for λ small. If u, w are the corresponding orthonormal eigenvectors for $H(K)$, then by degenerate perturbation theory [14] the degeneracy is removed *to first order* if and only if the two-by-two matrix

$$\begin{pmatrix} (u, Wu) & (u, Wv) \\ (v, Wu) & (v, Wv) \end{pmatrix}$$

has unequal eigenvalues. Thus, if W is not in the codimension 2 space (since X is dense in L^1) with

$$\int dx [u^2(x) - v^2(x)] W(x) = 0 = \int u(x) \overline{v(x) W}(x) dx,$$

the gap opens.

Moser [17] obtains the same condition but writes the codimension 2 space as the span of a codimension 3 space and one dimensional space so his result looks somewhat different. He uses discriminant theory in place of eigenvalue perturbation theory.

We will also need:

Theorem 2.11. *Fix N and let X be the N dimensional space of V of the form $\sum_1^N a_n \cos(x/2^n)$. Then for each m, for a dense open set of X, the mth gap associated to $L=2^N$ is open.*

Proof. The edges of the mth gap are Dirichlet and Neumann eigenvalues (at $k=0$) with fixed index. These eigenvalues are real analytic functions of the a's, so the set of V's with the mth gap closed is the zero set of a real analytic function. It is therefore, either all of X or the complement of a dense open set. Since $V(x) = \cos(x/2^n)$ has all gaps open [21], it cannot be all of X. □

3. Limit Periodic Potentials

In this section we prove Theorem 1 and 2 and some related results. Given the preliminary work in Sect. 2, the proofs are all easy.

Proof of Theorem 1. We consider the case $\mathscr{M}$; the case $\mathscr{L}$ is similar. By Lemma 1.1, density of the V's with Cantor spectrum implies genericity. Since the periodic potentials in $\mathscr{M}$ are dense in $\mathscr{M}$, it suffices to prove that given any ε_0 and periodic V, we can find $W \in \mathscr{M}$ with $\|V-W\| \leqq \varepsilon_0$ so that $-d^2/dx^2 + W$ has Cantor spectrum. Suppose $V = \sum_{n=0}^{N} a_n^{(0)} \cos(x/2^n)$. We will construct $S_j = \sum_{n=0}^{N+j} a_n^{(j)} \cos(x/2^n)$ so that

$$\|S_j\| \leqq \varepsilon_0/2^j \tag{3.1}$$

and take $W = V + \sum_1^\infty S_j$. The S_k will be chosen so that

$$W_k \equiv V + \sum_{j=1}^{k} S_j$$

has all gaps associated to $L=2^{N+k}$ open and so that all the gaps present for W_{k-1} with energy in $(-\infty, k)$ don't shrink very much. This latter fact will imply all gaps persist in the $k \to \infty$ limit and then (2.25) will imply that the spectrum is nowhere dense.

Here are the details. Suppose that $S_1, \ldots, S_{k-1}$ have been picked so that $H_{k-1} = -d^2/dx^2 + W_{k-1}$ has all $2^{N+(k-1)}$ gaps open so that in particular, [by (2.25)] for any E, there is a number $\tilde{E} \notin \sigma(H_{k-1})$ with

$$|E - \tilde{E}| \leqq c(E)/2^{N+k}, \tag{3.2}$$

where $c(E)$ is bounded as E runs through compact. $\sigma(H_{k-1})$ has a finite number of gaps in $(-\infty, k)$. Let α_k be the minimum gap size among all this finite number of gaps and let $\beta_k = \min(\alpha_1, \ldots, \alpha_k)$ (constructed inductively). We will pick S_k so that

(i) (3.1) holds.

(ii) $\|S_k\| \leqq 2^{-k}[\tfrac{1}{3}\beta_k]$.

(iii) H_k has all 2^{N+k} gaps open.

Such a choice is possible since the set of potentials will all gaps open is dense by Theorem 2.11.

Let $W = \lim_{k-\infty} W_k$. Let $(a-\delta, a+\delta) \subset (-\infty, k)$ be a gap for H_k. We claim that H has a gap, Δ, in its spectrum containing $(a-\delta', a+\delta')$ for some δ' with

$$\delta' > \delta/3\,; \qquad |\delta - \delta'| \leqq \varepsilon_0/2^{k-1}\,. \tag{3.3}$$

For, by the construction of β_k and (ii), $\|W - W_k\| \leqq \frac{1}{3}\delta$, so

$$(a-\delta/3, a+\delta/3) \cap \sigma(H) = \emptyset$$

(this uses the elementary $E \in \sigma(A)$ and $\|A - B\| \leqq \delta$ implies $\sigma(B) \cap [E-\delta, E+\delta] \neq \emptyset$). The second half of (3.3) follows the same argument and (3.1).

Given E and ε, first pick k_0 so large that $c(E)/2^{N+k_0}$ [given by (3.2)] is smaller than $\varepsilon/3$. Then pick $k_1 > k_0$ so large that any gap for H_k $(k > k_1)$ containing a point in $(E - \varepsilon/2, E + \varepsilon/2)$ must lie completely in $(-\infty, k)$ $\left(\text{i.e. take } k_1 \geqq E + \varepsilon/2 + \sup_j \|W_j\|_\infty\right)$. Finally, pick $k > k_1$ so that $\varepsilon_0/2^{k-1} \leqq \varepsilon/3$.

By (3.2), we can find $\tilde{E} \notin \sigma(H_k)$ with $|E - \tilde{E}| \leqq \varepsilon/3$. Suppose $\tilde{E}$ is in a gap $(a-\delta, a+\delta)$ for H_k. By the above, we can find $\hat{E} \notin \sigma(H)$ so that $|E - \hat{E}| \leqq 2\varepsilon/3 < \varepsilon$. Thus $(-\infty, \infty) \backslash \sigma(H)$ is dense. □

It is worth emphasizing the simple reason that these spectra are Cantor sets. There is a tendency for gaps to open up about energies $(\pi k/2^n)^2$ and these points are dense.

Proof of Theorem 2. Let $p = 3/2$ [any p in $(1,2)$ will do]. Let $\{\varphi_n\}_{n=1}^\infty$ be a countable subset of C_0^∞ which is dense in L^2. Let $G_n^k(E)$ be the spectral density for H_k for vector φ_n. In going through the above construction, pick S_k so that

$$\sup_{n \leqq k} \left[\int_{-\infty}^{k} |G_n^{k-1}(E) - G_n^k(E)|^p \right]^{1/p} \leqq 1/2^k\,.$$

This can be done by Theorem 2.7. Since each $G_n^k(E)$ lies locally in L^p (by Theorem 2.5, we conclude that for each n, E_0, there is a number $b(n, E_0) < \infty$, so that

$$\left.\sup_k \left[\int_{-\infty}^{E_0} |G_n^k(E)|^p dE \right]^{1/p} \leqq b(n, E_0)\right)\,.$$

Let A be a finite union of open sets. By general principles, if P^k are the spectral projections for H_k,

$$(\varphi_n, P_A \varphi_n) \leqq \overline{\lim_{k<\infty}}\, (\varphi_n, P_A^k \varphi_n)\,,$$

since $H_k \to H$ is norm resolvent sense (see [14]). Thus, for such A:

$$(\varphi_n, P_A \varphi_n) \leqq b(n, E_0) |A|^{1/q} \tag{3.4}$$

if $A \subset (-\infty, E)$. Taking limits and using regularity of measures, (3.4) holds for any A and thus the spectral measure for φ_n is absolutely continuous. Since the φ_n are dense, H has purely absolutely continuous spectrum.

By general results of Davies and Simon [6] any even bounded potential in one dimension has absolutely continuous spectrum of even multiplicity. By general results on eigenfunctions, and spectra [4, 15, 23], any spectra has multiplicity at most 2. □

In [6], Davies and Simon show that for a large class of V, $\mathscr{H}_{ac}=\mathscr{H}_+^\ell\oplus\mathscr{H}_+^r=\mathscr{H}_-^\ell\oplus\mathscr{H}_-^r$ where $\mathscr{H}_+^\ell$ as those vectors which as $t\to\infty$ move to the left, etc. They proved that periodic V are reflectionless in the sense that

$$\mathscr{H}_-^\ell=\mathscr{H}_+^r. \tag{3.5}$$

It seems to us likely that this holds for a.p. V. This may seem at variance with our intuition that localized wave packets suffer many reflections accounting for the anomalous long time behavior. (3.5) can only hold if the vectors in $\mathscr{H}_-^\ell$ have very slow falloff in x and are quite far from being of compact support. We think this could well happen.

By the Pseudo Corollary 2.9 one concludes the following:

Theorem 3. *Let $V\in\mathscr{M}$ be periodic. Then given any ε, we can find $W\in\mathscr{M}$ so that*

(i) *$\sigma(-d^2/dx^2+W)$ is a Cantor set.*

(ii) *$\sigma(-d^2/dx^2+W)$ is purely absolutely continuous of multiplicity* 2.

(iii) $\|V-W\|\leqq\varepsilon$.

(iv) *The Lebesgue measure of the symmetric difference of $\sigma(-d^2/dx^2+V)$ and $\sigma(-d^2/dx^2+W)$ is less than ε.*

Appendix 1. A Child's Garden of Almost Periodic Functions

In this appendix, we want to review some of the basic properties of almost periodic functions. We use a definition equivalent to Bohr's original definition but more suited for our purposes. Given f on R^ν and t in R^ν, f_t is the function $f_t(x)=f(x-t)$.

Definition (Bochner). A bounded continuous function, f, on $\mathbb{R}^\nu$ is called *almost periodic* (a.p.) if and only if $\{f_t\}_{t\in\mathbb{R}^\nu}$ is a precompact set in the $\|\cdot\|_\infty$-norm. The compact space obtained by taking the closure is called the *hull* of f, denoted Ω_f.

Any a.p. function is uniformly continuous, for if not, there exist $t_n\downarrow 0$ so that $\|f_{t_n}-f\|_\infty\geqq\varepsilon$. But, by compactness of Ω_f, the f_{t_n} have some uniform limit g. Since $f_{t_n}\to f$ pointwise, $f=g$ so $\|f_{t_n}-f_n\|_\infty$ cannot be larger than ε. This contradiction establishes the result.

Sums and products of a.p. functions are easily seen to be a.p. and periodic functions are a.p. Thus, if f is a.p. and α is real, $e^{-i\alpha x}f(x)$ is a.p.

Given any $g\in\Omega_f$, g is a function on $\mathbb{R}^\nu$, so we can define g_t. We claim $g_t\in\Omega_f$, for if $f_{t_n}\to g$, then $f_{t+t_n}\to g_t$. Thus, we can define a flow $T_t: g\to g_t$ on Ω_f. By the uniform continuity of f, $\|T_tf_s-f_s\|_\infty=\|T_tf-f\|_\infty$ goes to zero as $t\downarrow 0$ so the map $(g,t)\to T_tg$ is jointly continuous and the T_t are isometries. The image of f under the T_t are isometries, every orbit is dense.

Ω_f also has a natural abelian group structure. To avoid confusion between this structure and addition of functions, we will think of Ω_f abstractly. The identity e corresponds to the function f and if $w=f_t$, $w'=f_s$, then $w*w'$ will correspond to

f_{t+s}. To see the group structure, let w, w' be the limit of f_{s_n} and f_{t_n}. By a simple calculation

$$\|f_{s_n+t_n} - f_{s_m+t_m}\|_\infty \leqq \|f_{s_n} - f_{s_m}\|_\infty + \|f_{t_n} - f_{t_m}\|_\infty$$

is Cauchy. We call the limit $w * w'$. The uniqueness, group properties and continuity are easy to check. The reals are a dense subgroup of the group under $t \to f_t$.

Ω_f as a compact topological group has a natural normalized Haar measure $d\mu_f$. Since $\mathbb{R}^\nu$ is a dense subgroup of Ω_f, it is easy to see that T_t is an ergodic flow. In particular, if G is continuous on Ω_f and $\nu = 1$, then

$$\int G(w')d\mu_f(w') = \lim_{n\to\infty} \frac{1}{2n} \int_{-n}^{n} G(w+t)dt. \tag{A.1.1}$$

For this holds for a.p. w by the ergodic theorem and then by the uniform continuity of G and the denseness of the orbit $\{w*t\}_{t=-\infty}^{\infty}$, for all w. (A similar formula is true on $\mathbb{R}^\nu$.)

We thus see that [taking $G(g) = g(0)$] every a.p. function has an average and that this average is Haar measure on the hull. [Moreover, any continuous G on Ω_f defines an a.p. function by $g_{(G)}(t) = G(t)$ which is a.p. since $\{g_{(G),s}\}$ lies in the image of Ω_f under a continuous map; the hull of Ω_{g_G} is naturally a quotient group of Ω_f.]

The dual group, $\hat{\Omega}_f$, of characters on Ω_f is naturally a subgroup of $\hat{\mathbb{R}}^\nu$. Thinking of $\hat{\mathbb{R}}^\nu$ as $\mathbb{R}^\nu$ under $\alpha \mapsto \mathcal{X}_\alpha$ given by $\mathcal{X}_\alpha(t) = e^{i\alpha\cdot t}$, we obtain $\hat{\Omega}_f$ as a subgroup of $\mathbb{R}^\nu$, called the *frequency module of f*. $\hat{\Omega}_f$ is countable since Ω_f has a countable dense set. Since $\hat{\Omega}_f$ is a subgroup of $\mathbb{R}^\nu$, given $\alpha, \beta \in \hat{\Omega}_f$, and integers n, m, we have that $n\alpha + m\beta \in \hat{\Omega}_f$, i.e. $\hat{\Omega}_f$ is a module over $\mathbb{Z}$.

The Peter-Weyl theorem assures us that any f is a uniform limit of finite sums of the form $\sum_{i=1}^{n} c_i e^{i\alpha_i t}$ with $\alpha_i \in \hat{\Omega}_f$. From this it follows that (taking $\nu = 1$ for simplicity).

Theorem A.1.1. *The frequency module, $\hat{\Omega}_f$, is the module generated by*

$$\left\{\alpha \,\middle|\, \lim_{n\to\infty} \frac{1}{2n} \int_{-n}^{n} e^{-i\alpha x} f(x)dx \neq 0\right\}.$$

Thus, every countable module in $\mathbb{Z}$ is the frequency module of some a.p. f.

Two classes of a.p. functions on $(-\infty, \infty)$ are of especial interest:

Definition. We say that f is *limit periodic* if and only if f is a uniform limit of periodic functions. We say that f is *quasiperiodic* if and only if there exist numbers $\alpha_1, \ldots, \alpha_n$ and a function F on T^n (the n-torus) with

$$f(x) = F([[\alpha_i x]]), \tag{A.1.2}$$

with $[[\alpha_i x]]$ the fractional part of $\alpha_i x$ (thinking of T as $\mathbb{R}/\mathbb{Z}$).

It is easy to see that both classes are a.p. Indeed, if the α_i are rationally independent, then Ω_f is T^n under a bicontinuous homomorphism, i.e. $w=(\theta_1,\ldots,\theta_n)$ is associated to $F([[\alpha_i x+\theta_i]])$. One can tell these special classes by looking at the frequency module:

Theorem A.1.2. *Let f be a.p. Then f is quasiperiodic if and only if $\hat{\Omega}_f$ is finitely generated.*

Proof. If f is quasiperiodic given by (A.1.2), then $\hat{\Omega}_f=\left\{2\pi\sum_{i=1}^{n} m_i\alpha_i|m_i \text{ integral}\right\}$ is generated by $\alpha_1,\ldots,\alpha_n$. Conversely, let $\hat{\Omega}_f$ be finitely generated. If the discrete topology is put on $\hat{\Omega}_f$ it is then just $\mathbb{Z}^n$ for some n so $\hat{\hat{\Omega}}_f=\Omega_f$ (by Pontryagin duality) is just T^n. Since $\mathbb{R}$ is densely imbedded in T^n only as $x\to([\alpha_i x])$ A.1.2 holds. (One can use Peter-Weyl in place of Pontryagin duality.) □

Theorem A.1.3. *Let f be a.p. Then f is limit periodic if and only if $\hat{\Omega}_f$ has the property that any $\alpha,\beta\in\hat{\Omega}_f$ have a common divisor in $\hat{\Omega}_f$.*

Proof. Let f be l.p. Pick α,β with $a=\lim\limits_{n\to\infty}\frac{1}{2n}\int_{-n}^{n} dxe^{-i\alpha x}f(x)$ and $b=\lim\limits_{n\to\infty}\frac{1}{2n}\int_{-n}^{n} e^{-i\beta x}f(x)dx$ both non-zero. Choose g periodic with $\|f-g\|_\infty \leqq \frac{1}{2}\min(|a|,|b|)$. It follows that

$$\lim(2n)^{-1}\int dxe^{-i\gamma x}g(x)dx\neq 0$$

for $\gamma=\alpha,\beta$ and thus if g has period L, $2\pi/L$ divides both α and β. Similarly any finite subset of the generating set $\left\{\alpha|\lim(2n)^{-1}\int_{-n}^{n} dxe^{-i\alpha x}f(x)\neq 0\right\}$ have a common divisor so $\hat{\Omega}_f$ has the required property.

Conversely, if $\hat{\Omega}_f$ has the property any finite sum $\sum_{j=1}^{n} a_j e^{i\alpha_j x}$ with $\alpha_j\in\hat{\Omega}_f$ is periodic since the α_j have a common divisor. By the Peter-Weyl theorem, f is a limit of periodic functions. □

Corollary A.1.4. *If f is both limit periodic and quasiperiodic, then f is periodic.*

Proof. Any finite generated module with the divisor property is single generated. □

Corollary A.1.5. *If f is limit periodic, there exist periodic functions W_j of fixed L_1 and integers $n_1=1,n_2,\ldots,$ so that $\|W_j\|_\infty\leqq 2^{-(j-2)}\|f\|_\infty$ and*

$$f(x)=\sum_j W_j(x/n_1\ldots n_j).$$

Proof. By Peter-Weyl, find f_j, finite combinations of the $\{e^{i\alpha x}\}_{\alpha\in\hat{\Omega}_f}$ with $\|f_j-f\|_\infty \leqq 2^{-(j-1)}\|f\|_\infty$ with $f_0=0$. Then $\|f_j-f_{j+1}\|_\infty\leqq 2^{-(j-2)}\|f\|_\infty$ and taking

$g_j = f_{j+1} - f_j$, we see that $f = \sum_1^\infty g_j$. By the divisor property for $\hat{\Omega}_f$, each g_{j+1} is periodic with a period which is a multiple of the period f_j. □

We will need the following in a later paper; it is an immediate consequence of the Peter-Weyl theorems on Ω_f.

Theorem A.1.6. *Let f be a.p. For any ε, there is a quasiperiodic g with $\|f-g\|_\infty < \varepsilon$ and $\hat{\Omega}_g \subset \hat{\Omega}_f$.*

Appendix 2. Some General Features of Almost Periodic Schrödinger Operators

In this appendix, we will follow two simple theorems concerning $-\Delta + V(x)$ on $L^2(\mathbb{R}^\nu)$ with V a.p.: (i) that the spectrum is constant over the hull; (ii) that the spectrum is purely essential. Virtually identical results (in one sense stronger) for $\nu = 1$ were obtained by Johnson at the same time [12]. In any event, these results are only mildly stronger than old results of the Russian group studying random potentials (see e.g. Pastur [19]): a.p. potentials can be viewed as a special case of random potentials determined by an ergodic process – thus e.g. it is known that the spectrum is a.e. constant on the hull [19]. We prove pointwise constancy.

Theorem A.2.1. *Let Ω be the hull of an a.p. function on $\mathbb{R}^\nu$ and given $w \in \Omega$, let $V_w(x)$ be the corresponding a.p. function. Then* spec$(-\Delta + V_w)$ *is independent of w.*

Proof. Let $w, w' \in \Omega$. By symmetry, we need only show that $E \notin \text{spec}(-\Delta + V_w)$ implies $E \notin \text{spec}(-\Delta + V_{w'})$. Find x_n so that $V_w(x + x_n) \to V_{w'}(x)$ uniformly. Then $-\Delta + V_w(\cdot + x_n)$ converges to $-\Delta + V_{w'}$ in norm resolvent sense so that if Γ is an open set in $\mathbb{R}$, $\Gamma \cap \bigcap_n \text{spec}\{-\Delta + V_w(\cdot + x_n)\} = \emptyset$ implies $\Gamma \cap \text{spec}\{-\Delta + V_{w'}\}$ is empty by general principles [14, 20]. Since

$$\text{spec}\{-\Delta + V_w(\cdot + x_n)\} = \text{spec}(-\Delta + V_w\}$$

and the resolvent set is open, we are done. □

Theorem A.2.2. *Let V be an a.p. function on $\mathbb{R}^\nu$. Then the spectrum of $-\Delta + V$ is purely essential, i.e. there are no isolated eigenvalues of finite multiplicity.*

Proof. Let E be an isolated eigenvalue of finite multiplicity. Let P be the corresponding spectral projection. Pick $x_n \to \infty$ so that $V(x + x_n) \to W(x)$ uniformly in x. Let $P(x_n)$ be the spectral projection for $\{E\}$ associated to $-\Delta + V(\cdot + x_n)$ and let P_∞ be the corresponding projection for $-\Delta + W$. By the norm resolvent converge $P(x_n) \to P_\infty$ in norm. But $P(x_n) = U_n^{-1} P(0) U_n$, where the U_n go strongly to zero. Since $P(0)$ is compact, we conclude that $P(x_n) \to 0$ strongly so $P_\infty = 0$. This contradicts the fact that spec$(-\Delta + V) = $ spec$(-\Delta + W)$ by Theorem A.2.1. □

Corollary A.2.3. *Let V be an a.p. function on $(-\infty, \infty)$. Then the spectrum of $-d^2/dx^2 + V(x)$ has no isolated points.*

Proof. Any isolated point is an eigenvalue of finite multiplicity. In one dimension eigenvalues must have multiplicity one so there cannot be isolated points by Theorem A.2.2. □

Remarks. 1. In the ν dimensional periodic case, it is known there are no eigenvalues [24]. Is this true in the a.p. case in the sense that there are no *isolated* eigenvalue even in case $\nu > 1$?

2. Our belief (2) in Sect. 1 says that even if $\nu = 1$, there can be non-isolated eigenvalues [but by the above, $\{w| -d^2/dx^2 + V_w(t)$ has a fixed eigenvalue $E\}$ will have measure 0]. In fact Johnson [13] and Moser [18] have constructed a.p. functions V, so that $-d^2/dx^2 + V(x)$ has at least one eigenvalue.

3. Johnson [12] proves a stronger version of Theorem A.2.1 in that for any boundary condition at 0, the operator $-d^2/dx^2 + V(x)$ on $[0, \infty)$ has essential spectrum equal to that on the whole line. This is easy to prove, for one can always find $t_n \to \infty$ so that $\| V(\cdot + t_n) - V(\cdot)\|_\infty \to 0$. Given any sequence $\varphi_m \in C_0^\infty(-\infty, \infty)$ and E so that $\|(H_0 + V - E)\varphi_m\|_\infty \to 0$, we can find a subsequence $t_{n(m)}$ so that $\operatorname{supp}[\varphi_m(\cdot + t_{n(m)})] \subset [m, \infty)$. Then

$$\|(H_0 + V - E)\varphi_m(\cdot + t_{n(m)})\| \to 0$$

so

$$E \in \sigma_{\mathrm{ess}}(-d^2/dx^2 + V \upharpoonright L^2(0, \infty)).$$

Similarly

$$E \in \sigma_{\mathrm{ess}}(-d^2/dx + V \upharpoonright L^2(-\infty, 0)).$$

Since $\sigma_{\mathrm{ess}}(d^2/dx^2 + V)$ is the union of the two half line essential spectra the equality of all three is true.

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References

1. Avron, J., Simon, B.: The asymptotics of the gap in the Mathieu equation. Ann. Phys. **134**, 76–84 (1981)
2. Avron, J., Simon, B.: Transient and recurrent spectrum. J. Funct. Anal. (to appear)
3. Avron, J., Simon, B.: In preparation
4. Berezanski, J.M.: Expansion in eigenfunctions of self-adjoint operators. Am. Math. Soc. Providence (1968)
5. Choquet, G.: Lectures in analysis. I. New York: Benjamin 1969
6. Davis, E.B., Simon, B.: Commun. Math. Phys. **63**, 277–301 (1978)
7. Deift, P.: Private communication
8. Dubrovin, B.: The periodic problem for KdV equation in the class of finite zone potentials. Funkt. Anal. Pril. **9** (3), 215–223 (1975)
9. Eastham, M.: The spectral theory of periodic differential equations. Edinburgh: Scottish Academic Press 1973
10. Gel'fand, I.: Dokl. Akad. Nauk **73**, 117–1120 (1950)
11. Hochstadt, H.: Proc. Am. Math. Soc. **14**, 930–932 (1963)
12. Johnson, R.: The recurrent Hill's equation. U.S.C. (preprint)
13. Johnson, R.: In preparation
14. Kato, T.: Perturbation theory for linear operators. Berlin, Heidelberg, New York: Springer 1966
15. Kovanlenko, V., Semenov, Yu.: Russ. Math. Surv. **33**, 119–157 (1978)
16. Magnus, W., Winkler, S.: Hill's equation. New York: Interscience 1966

17. Moser, J.: An example of a Schrödinger equation with almost periodic potential and nowhere dense spectrum. ETH (preprint)
18. Moser, J.: Private communication
19. Pastur, L.: Commun. Math. Phys. **75**, 179–186 (1980)
20. Reed, M., Simon, B.: Methods of modern mathematical physics. I. Functional analysis. London, New York: Academic Press 1972
21. Reed, M., Simon, B.: Methods of modern mathematical physics. IV. Analysis of operators. London, New York: Academic Press 1978
22. Simon, B.: Ann. Inst. H. Poincaré A **24**, 91–93 (1976)
23. Simon, B.: Schrödinger semigroups (in preparation)
24. Thomas, L.: Commun. Math. Phys. **33**, 335–343 (1973)
25. Trubowitz, E.: Commun. Pure Appl. Math. **30**, 321–337 (1977)
26. Grossmann, A.: Statistical mechanics and field theory. Sen, R.N., Weil, C. (eds.). Jerusalem: Israel University Press 1972
27. Blount, E.I.: Solid State Physics **13**. London, New York: Academic Press 1961

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