RIGOROUS PERIMETER LAW UPPER BOUND ON WILSON LOOPS

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We present a proof that in an arbitrary lattice gauge theory, the expectation value of a Wilson loop decays at least as fast as the exponential of the perimeter.

The well-known Wilson criterion for confinement is via a transition from perimeter law decay of Wilson loop expectations to area law decay. It is therefore of some interest that one can prove in great generality that such expectations are always bounded from below by a bound with area decay and from above by a bound with perimeter decay. The lower bound is a result of Seiler [1] ; our goal here is a proof of the upper bound. Some time ago, Lüscher [2] presented a rather involved proof of the upper bound for hamiltonian lattice theories which he apparently never published. We decided to publish our proof because of the naturalness of the question and the simplicity of our proof. Nevertheless, we emphasize that the upper bound has much less physical significance than Seiler's lower bound (which yields a linear upper bound on the the static quark confining potential).

The crux of our proof will involve deriving a bound on the ratio between the largest gauge invariant and non-gauge invariant eigenvalues of the transfer matrix. The simple argument needed to prove this bound may be of interest in other contexts and will be presented first as a separate lemma.

Let G be a compact group with normalized invariant measure, $\mathrm{d}g$. Let Ω be a set of "configurations" with a normalized measure $\mathrm{d}\mu(x)$. Suppose there is an action of G on Ω, i.e. a (measurable) assignment $gx \in \Omega$ to each $g \in G$, $x \in \Omega$ so that $g(g'x) = (gg')x$ and suppose that for each fixed g, the map $x \to gx$ preserves the measure $\mathrm{d}\mu$. Under these assumptions, the basic Hilbert space $\mathcal{H} = L^2(\Omega, \mathrm{d}\mu)$ supports a unitary representation of G given by

$$(\mathcal{U}(g)f)(x) = f(g^{-1}x) , \tag{1}$$

Suppose that K is a bounded self-adjoint operator on $\mathcal{H}$ with integral kernel obeying

$$K(gx, gy) = K(x, y) , \tag{2}$$

for all $g \in G$, $x, y \in \Omega$. Then $\mathcal{U}(g) K \mathcal{U}(g)^{-1} = K$, so the eigenspaces of K are left invariant by $\mathcal{U}$ and thus can be associated with one or more irreducible representations of G which enter in the decomposition of $\mathcal{U}$ restricted to the eigenspace. Moreover, suppose that for some constant $c > 0$ and all $g \in G$, $x, y' \in \Omega$:

$$K(gx, y) \geqslant cK(x, y) \geqslant 0 \tag{3}$$

(obviously $c \leqslant 1$). The second inequality in (3) implies that the maximum modulus eigenvalue, λ_+, of K has a positive function in its eigenspace and thus, λ_+ its associated to the trivial representation.

Lemma. Let λ be any eigenvalue of K associated to a non-trivial representation of G. Then

$$|\lambda| \leqslant (1 - c)\lambda_+ . \tag{4}$$

Proof. Let $Pf = \int [\mathcal{U}(g)f]\, \mathrm{d}g$. so that P is the projection onto all vectors invariant under the action of G. By a direct calculation, PK has integral kernel

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$$(PK)(x,y) = \int K(g^{-1}x, y)\, dg \,.$$

so by (3) $(PK)(x, y) \leqslant c^{-1}K(x,y)$ and thus

$$[(1 - cP)K]\ (x,y) \geqslant 0 \,. \quad (5)$$

Therefore, as above, the maximum modulus eigenvalue of $(1 - cP)K$ is associated to the trivial representation and is thus $(1 - c)\lambda_+$. But since λ is associated to a non-trivial representation, λ is also an eigenvalue of $(1 - cP)K$, so (4) is proven.

We apply this to the lattice gauge theory as follows: As usual, the action of the lattice gauge theory is taken to be

$$S = \beta \sum_p \operatorname{Re} \chi(g_{\partial p}) \,, \quad (6)$$

and the Wilson loop around a contour C is given by

$$W_C = \chi(g_C) \,. \quad (7)$$

χ is the character of a non-trivial representation u, so that

$$\chi(gh^{-1}) = \sum_{i,j} u_{ij}(g)\overline{u_{ij}(h)} \,, \quad (8)$$

and g_C denotes the usual product of bond variables around a contour C. For notational simplicity, we assume that the characters defining the action (6) and the Wilson loop (7) are identical, however this plays no essential role. (All results may be immediately extended to the case where the action per plaquette is an arbitrary finite linear combination of different characters.) We will explicitly discuss only rectangular Wilson loops, although one can also bound more general loops. We make one technical assumption, namely that *the same infinite volume state is obtained by taking the different sides of the box defining the finite volume state to infinity in any order.*

Consider the theory defined in a box of size $L^{\nu-1} \times T$ with periodic boundary conditions in all directions. Using the standard transfer matrix formalism, we may re-express the theory as follows. Using gauge invariance, we need only integrate over configurations where all vertical bonds equal the identity except for the bonds linking the top and bottom of the box. Let H be the gauge group and $\Omega = H^{\nu L^{\nu-1}}$ the space of configurations of a spatial hyperplane. Thus $x \in \Omega$ is a set of spatial bond variables x_α. Let $s(x_\alpha)$ denote the action due to interactions in a single spatial hyperplane, and define the transfer matrix,

$$K(x,y) = \exp\left(\beta \sum_\alpha \chi\,(x_\alpha^{-1} y_\alpha)\right) \times \exp[\tfrac{1}{2}s(x) + \tfrac{1}{2}s(y)] \,. \quad (9)$$

Note that $K(x,y) > 0$ and $K(gx, gy) = K(x,y)$ for any gauge transformation g. The partition function may then be expressed as

$$Z_{L\times T} = \operatorname{tr}(\widetilde{P}K^T)$$

where $\widetilde{P}$ is the projection onto invariant vectors for the full planar gauge group, and is present because of the periodic boundary conditions. Using (8), the expectation of a rectangular Wilson loop of size $l \times t$ may be expressed as

$$\langle W_{l\times t}\rangle_{L\times T} = \sum_{i,j} \operatorname{tr}(\widetilde{P}K^{T-t}\widetilde{F}_{ij}^{(l)}K^t F_{ij}^{(l)})/\operatorname{tr}(\widetilde{P}K^{\mathrm{T}}) \,,$$

where $F_{ij}^{(l)}$ denotes a multiplication operator by $u_{ij}(x^l)$, and x^l is the product of bond variables along the spatial side of the loop of length l. Taking T to ∞ with L fixed, we find that

$$\langle W_{l\times t}\rangle_{L\times\infty} = \sum_{i,j} (\psi, \bar{F}_{ij}^{(l)}(K\lambda_+^{-1})^t F_{ij}^{(l)}\psi) \,, \quad (10)$$

where ψ is the unique gauge-invariant eigenvector of the transfer matrix with the maximum eigenvalue λ_+ (i.e. $K\psi = \lambda_+\psi$ and $\widetilde{P}\psi = \psi$). Consider the subgroup of gauge transformations, G, which only act at the site at the end of the chain x^l. Then, since ψ is invariant under G, and $F_{ij}^{(l)}$ transforms under $1 \otimes u$, which is non-trivial, we see that

$$(\psi, \widetilde{F}_{ij}^{(l)}(K\lambda_+^{-1})^t F_{ij}^{(l)}\psi) \leqslant (\lambda/\lambda_+)^{t-1}(\psi, \widetilde{F}_{ij}^{(l)}(K\lambda_+^{-1})\, F_{ij}^{(l)}\psi) \,, \quad (11)$$

where λ is the maximum eigenvalue associated to eigenvectors transforming under $1 \otimes u$. Since (a) g only transforms one site, (b) g leaves $s(x)$ invariant, (c) each site is involved in $2(\nu - 1)$ vertical plaquettes, and (d) the maximum change in χ is 2 dim u, we see that

$$K(gx, y)/K(x, y) \geqslant \exp[-4(\nu - 1)\beta \dim \mathscr{U}] \equiv c . \quad (12)$$

Thus, by (4), (10) and (11) we find that

$$\langle W_{l\times t}\rangle_{L\times\infty} \leqslant (1-c)^{(t-1)} \langle W_{l\times 1}\rangle_{L\times\infty} ,$$

where c is given by (12). Taking L to ∞,

$$\langle W_{l\times t}\rangle \leqslant (1-c)^{(t-1)} \langle W_{l\times 1}\rangle . \quad (13)$$

Now use symmetry in l and t (here we need the assumption that the infinite volume state $\langle \ \rangle$ is the same no matter which order we take the sides to ∞) and we obtain (since $\langle W_{1\times 1}\rangle \leqslant \dim \mathscr{U}$)

$$\langle W_{l\times t}\rangle \leqslant (\dim \mathscr{U})(1-c)^{(t+l-2)} , \quad (14)$$

which is the required upper bound.

We remark that, from (10), one easily finds the lower bound ($d \equiv \dim \mathscr{U}$)

$$d^{-1}\langle W_{l\times t}\rangle_{L\times\infty} \geqslant [d^{-1}\langle W_{l\times 1}\rangle_{L\times\infty}]^t$$

by writing out an eigenfunction expansion for $K\lambda_+^{-1}$ and using Holder's inequality $(\Sigma a_n \lambda_n^t) \geqslant (\Sigma a_n \lambda_n)^t$ if $\Sigma a_n = 1$. From this and symmetry, the Seiler lower bound

$$\langle W_{l\times t}\rangle \geqslant d(d^{-1}\langle W_{1\times 1}\rangle)^{lt}$$

follows.

References

[1] E. Seiler, Phys. Rev. D18 (1978) 482.
[2] M. Lüscher, Absence of spontaneous gauge symmetry breaking in hamiltonian lattice gauge theories, unpublished.