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SHORTER NOTES

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SCHRÖDINGER OPERATOR METHODS IN THE STUDY OF A CERTAIN NONLINEAR P.D.E.

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ABSTRACT. We prove that $\Delta u + hu^{\alpha} = 0$ has no positive solutions for certain h, α by studying the linearized equation $(\Delta + hu^{\alpha-1})\psi = e\psi$.

In this note we show that some of the work of Gidas and Spruck [2] on the absence of positive solutions of

$$\Delta u(x) + h(x)u^{\alpha}(x) = 0 \tag{1}$$

can be recovered with simple arguments about the number of eigenvalues of linear operators.

THEOREM. *Let D be the domain $\{|x| > r_0\}$ of $\mathbf{R}^n$, $n \geqslant 3$, and h a locally L^{∞} positive function satisfying $h(x) \geqslant \text{const}\,|x|^{\sigma}$, $\sigma > -2$. If $1 < \alpha < (n+\sigma)/(n-2)$, then no positive C^2 function u satisfies* (1).

PROOF. It has been shown by Allegretto [1] and Piepenbrink [3] (see also §C8 of [6]) that the existence of positive solutions of an equation $(-\Delta + q(x))u = 0$ (conventionally with a different sign from (1)) on an exterior domain implies that $\dim P_{(-\infty,0)}(-\Delta + q(x)) < \infty$, where $P_{(-\infty,0)}(-\Delta + q(x))$ is the spectral projection for the open interval $(-\infty, 0)$ associated with any selfadjoint realization of $-\Delta + q$. In other words, there are only a finite number of negative bound states. On the other hand, suppose that u is a positive solution of (1) on D. Then, since $\Delta u < 0$, the subharmonic comparison argument of [4] shows that $u(x) > cg_R(x)$ for some constant c and $r_0 < |x| \leqslant R$, where $g_R(x)$ is the Green function for $\Delta g_R = -\delta(x)$ on $\{|x| \leqslant R\}$ with Dirichlet B.C. at $|x| = R$. The constant c is such that $u(x) > cg_R(x)$ when $|x| = r_0$. Since $g_R(x)$ increases monotonically to $1/\omega_n |x|^{n-2}$ as $R \uparrow \infty$ on D,

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it follows that $u(x) \geqslant \text{const}\,|x|^{-(n-2)}$. Therefore

$$-h(x)u^{\alpha-1} < -\text{const}\,|x|^{\sigma-(\alpha-1)(n-2)} < -\text{const}\,|x|^{-2+\varepsilon},$$

where $\varepsilon > 0$. But any potential $q(x) < -\text{const}\,|x|^{-2+\varepsilon}$ gives rise to an *infinite* number of negative bound states by the min-max principle [5], so there is a contradiction when we identify $q(x) = -hu^{\alpha-1}$. □

REMARKS. 1. Δ may be replaced with a more general elliptic operator $\partial_i a_{ij}(x)\partial_j$ such that $a_{ij} \in C^2$ and positive definite for each x and satisfying a growth condition [3].

2. h need only be locally L^p with $p > n/2$. Also one only needs a weak solution with $hu^{\alpha-1} \in L^p_{\text{loc}}$.

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