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The Mathematical Theory of Resonances Whose Widths Are Exponentially Small, II

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It is shown how the rigorous justification of resonance widths in Paper I [5] can be simplified by exploiting Langer's trick of expanding the independent variable rather than the dependent variable [9].

1. INTRODUCTION

In an earlier paper [5] (henceforth I), two of us analyzed a number of basic "resonance" problems coming from ordinary differential equations and separable partial differential equations of quantum physics. Typical of the problems is the study of the kth eigenvalue, $\Lambda_k(\beta)$, of

$$\frac{-d^2}{dx^2} + x^2 + \beta x^4 \tag{1.1}$$

which has a natural selfadjoint realization on $L^2(R)$ when $\beta > 0$. These eigenvalues can be analytically continued to a neighborhood of $\beta = 0$, cut along the negative axis [10] (indeed to the whole cut plane [8]) and we proved in Paper I that as $\mu \downarrow 0$

$$\operatorname{Im} \Lambda_k(-\mu + i0) = 2^{3k-1}[(k-1)!\sqrt{\pi}]^{-1}\mu^{-k+1/2}\exp(-2/3\mu)(1 + O(\mu^\alpha)) \tag{1.2}$$

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for some $\alpha > 0$. Putting (1.2) into a suitable Cauchy integral formula yields the asymptotics of the coefficients in the asymptotic series for Λ_k: specifically,

$$\Lambda_1(\beta) \sim \sum_{n=0}^{\infty} a_n \beta^n \tag{1.3}$$

in the classical sense and

$$a_n = (4\pi)^{-3/2}(-1)^{n+1}(\tfrac{3}{2})^{n+1/2}\Gamma(n+\tfrac{1}{2})(1+O(n^{-\gamma})) \tag{1.4}$$

for some $\gamma > 0$. Equations (1.2) and (1.4) are known as the Bender–Wu formulae after the work of C. Bender and T. T. Wu [2] who first established them on a nonrigorous basis.

The difficulty in deriving formulae like (1.2) is that $\operatorname{Im} \Lambda_k$ is exponentially small, while many quantities like $\operatorname{Re} \Lambda_k$ have divergent power series with coefficients at all orders. Thus a delicate balancing is required to prevent the unavoidable $O(\mu^n)$ approximations from overwhelming the exponential terms which one wants to find.

The strategy of Paper I has two parts, the second being particularly difficult to implement. Our goal in this paper is to sketch an alternative way of implementing the same strategy but with different and considerably simpler tactics. We will illustrate this in the study of problem (1.1); a related argument would apply to the other problems treated in Paper I, notably the hydrogen Stark problem. Some remarks on this case can be found in [4].

The first step of the strategy was to derive an equation for the imaginary part of the resonance in terms of the eigenfunction and its derivatives at a variable point, x, which is carefully chosen in the second part of the proof. Here we shall simplify with respect to Paper I, by using a form which does not involve the differentiation of wave functions with respect to a parameter.

The second step in the strategy of Paper I was to develop asymptotics of the eigenfunctions exploiting variation of parameters. Explicitly, we expanded solutions of

$$-\psi''(x) + V(x)\,\psi(x) = \Lambda\psi(x) \tag{1.5}$$

in terms of solutions of

$$-\psi_0''(x) + V_0(x)\,\psi_0(x) = \Lambda_0\,\psi_0(x), \tag{1.6}$$

writing $\psi(x) = \psi_0(x) + \delta\psi(x)$, and analyzed $\delta\psi$ as a perturbation.

Because of the long-range nature of various potentials and corrections, it was necessary to divide the region of analysis into five subregions (!) and to choose distinct V_0 (and ψ_0) in each region matching at the borders of adjacent regions.

Fortunately, it has been discovered by N. Corngold [3] in an analysis of a Fokker–Planck equation which led to an ODE close to (1.1) with $\beta = -\mu < 0$, that a single functional form sufficed if one used Langer's idea [9] of relating (1.5) and (1.6) by trying $\psi(x) = c(x)\,\psi_0(x + \delta x)$ instead of $\psi = \psi_0 + \delta\psi$ and solving perturbatively for δx.

The ψ_0 we will use (following [3]) are parabolic cylinder functions. They are *not* those associated to the unperturbed equation $-\psi_0'' + x^2\psi_0 = \Lambda\psi_0$, but rather the "upside-down" oscillator

$$\left(\frac{-d^2}{d\xi^2} + a - \frac{\xi^2}{4}\right) g(\xi) = 0 \tag{1.7}$$

obtained by trying to map the polynomial $x^2 - \mu x^4 - \Lambda$ conformally onto $a - \xi^2/4$ in a certain region of the complex plane [6]. The relationship between x and ξ on real axis is shown in Fig. 1. Upon reflection it is not surprising that (1.7) enters. Matching at $x = 0$ is not very hard; the critical issue is to treat the entire tunnelling region properly and this is accomplished by (1.7).

We do not use Langer's approach strictly. As we will see, this would lead to a complicated nonlinear second-order equation and it would not be so easy to estimate errors in approximate solutions to it. Rather, we make an approximate change of variable $x \to \xi$ and write the resulting eigenfunction equation for $\psi(x)$ as an equation for $g(\xi) = (x_\xi)^{-1/2}\psi(x(\xi))$ which we estimate by variation of parameters thinking of g as $g_0 + \delta g_0$. The point is that by using this modified approach we get the best of both worlds: We have the ease of estimates of the variation of parameters while the use of Langer's idea avoids the need for more than one set of estimates.

We should emphasize that when we finally get down to estimates in Section 4, we will not put in all the details of a complete mathematical proof; indeed we have not felt it necessary to check several obvious details. What is lacking is an analysis that the mapping ξ takes the relevant x-space contour into a region where the asymptotic formulae we use for parabolic cylinder functions are valid. This could be rectified in part by a straightforward analysis of the map $x \to \xi$; the other part is made more difficult by the fact that the standard tables do not always list the full region of validity of formulae in the complex plane and one has to go back to original papers on these asymptotics. Our guiding philosophy has been that since a proof with all i's dotted and t's crossed exists in Paper I, there is a point to describing alternative tactics with the calculation of leading order complete but with the estimates of higher orders only sketched with some details left to the diligent reader (an author's favorite person!).

In Section 2, we derive the necessary formula for $\operatorname{Im}\Lambda$; in Section 3, we discuss the necessary comparison functions for the variation of parameters which we sketch in Section 4.

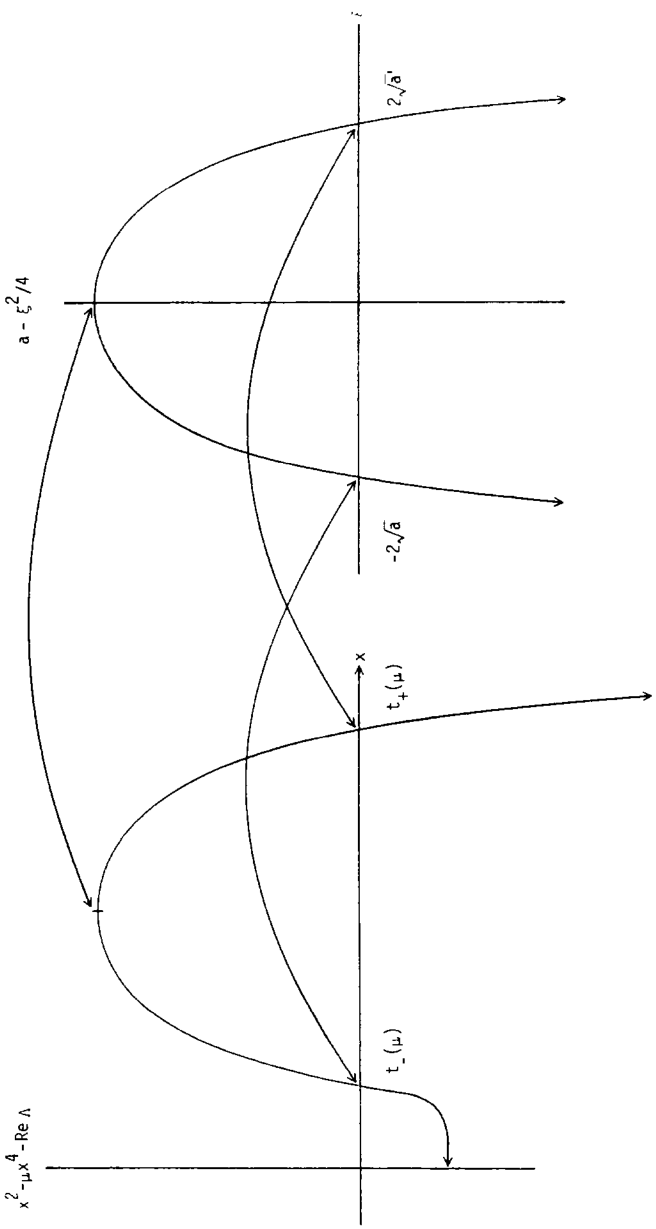

FIG. 1. The relationship between x and ξ on the real axis is shown. The roots of the polynomials correspond exactly, while the maxima only correspond approximately.

2. The Anharmonic Oscillator: Formula for Im Λ

The analytic continuation of eigenvlues of (1.1) to the region $|\operatorname{Arg}\beta| < \pi$ produces eigenvalues associated with eigenfunctions which are L^2 along the real axis [10]. There is not an L^2 solution along the axis for $\beta = -\mu + i0$, rather, as realized already by Bender and Wu [2] (see also [10]), there is an eigenfunction decaying in a suitable region of the lower half complex x plane. Explicitly, there is a function $\psi(x, \mu)$ solving

$$\left(\frac{-d^2}{dx^2} + x^2 - \mu x^4\right) \psi(x, \mu) = \Lambda \psi(x, \mu) \tag{2.1}$$

which

(a) obeys either $\psi'(0) = 0$ (if k is odd) or $\psi(0) = 0$ (if k is even),

(b) is subdominant (exponentially decaying) as $|x| \to \infty$ in the sector $0 > \arg x > -\pi/3$ of the complex x plane.

The existence and uniqueness of such solutions is a standard part of ODE theory; see Hille [6], Hsieh and Sibuya [7] or the appendix of Dicke in [10].

Given a point x_0 on the positive real axis, suppose we normalize ψ by

$$\int_0^{x_0} |\psi(x)|^2 \, dx = 1. \tag{2.2}$$

Then multiplying (2.1) by $\bar{\psi}$ and integrating, we find

$$\begin{aligned} \operatorname{Im} \Lambda &= \operatorname{Im} \int_0^{x_0} \bar{\psi} \left(\frac{-d^2}{dx^2} + x^2 - \mu x^4\right) \psi \, dx \\ &= \operatorname{Im} \int_0^{x_0} \bar{\psi} \left(\frac{-d^2}{dx^2}\right) \psi \, dx \\ &= \frac{1}{2} \operatorname{Im} \int_0^{x_0} (\psi \bar{\psi}'' - \bar{\psi} \psi''). \end{aligned}$$

So, since the boundary term at 0 vanishes, an integration by parts yields

$$\operatorname{Im} \Lambda = \operatorname{Im}(\psi(x_0) \, \bar{\psi}'(x_0)) \tag{2.3}$$

if ψ is normalized by (2.2). x_0 is an arbitrary point. Since we know we have a tunnelling problem, we will choose it to be the larger turning point (solution of $x^2 - \mu x^2 - \operatorname{Re} \Lambda = 0$) and will find that with the normalization (2.2), ψ and ψ' can be estimated well enough to get out the exponential form of $\psi(x_0)$.

3. The Comparison Functions

The transformation $x \to \xi(x)$ with inverse $\xi \to x(\xi)$ will take a solution $\psi(x)$ into a solution of (1.7)

$$g(\xi) \equiv (x_\xi)^{-1/2}\psi(x(\xi))$$

if and only if

$$\xi_x^2 (a - \tfrac{1}{4}\xi^2) = x^2 - \mu x^4 - \Lambda - \xi_x^{1/2} \frac{d^2}{dx^2} (\xi_x^{-1/2}).$$

We will approximate this by dropping the last term (Schwarzian derivative) and replacing Λ by $\operatorname{Re}\Lambda$. We will then discover that g, instead of solving (1.7), will solve a modified equation close to (1.7). We will solve this modified equations by using variation of parameters based on the solutions of (1.7). Explicitly, we will define $\xi(x)$ by requiring

$$(a - \xi^2/4)^{1/2}\, d\xi = (x^2 - \mu x^4 - \operatorname{Re}\Lambda)^{1/2}\, dx. \tag{3.1}$$

We want to choose a so that we can consistently pick branches of square roots with $\xi(x)$ real for all real x in $(0, \infty)$. Thus a must be chosen so that

$$\int_{-2\sqrt{a}}^{2\sqrt{a}} (a - \tfrac{1}{4}\xi^2)^{1/2}\, d\xi = \int_{t_-(\mu)}^{t_+(\mu)} (x^2 - \mu x^4 + \operatorname{Re}\Lambda)^{1/2}\, dx, \tag{3.2}$$

where $t_\pm(\mu)$ are the two solutions of $x^2 - \mu x^4 + \operatorname{Re}\Lambda = 0$, i.e.,

$$t_\pm(\mu) = \{(2\mu)^{-1}[1 \pm \sqrt{1 - 4\mu(\operatorname{Re}\Lambda)}]\}^{1/2}.$$

Then for $t_-(\mu) \leqslant x \leqslant t_+(\mu)$ we defined $\xi(x)$ by

$$\int_{-\sqrt{2a}}^{\xi(x)} (a - \tfrac{1}{4}\xi^2)^{1/2}\, d\xi = \int_{t_-(\mu)}^{x} (z^2 - \mu z^4 - \operatorname{Re}\Lambda)^{1/2}\, dz \tag{3.3a}$$

for $0 \leqslant x \leqslant t_-(\mu)$, we define $\xi(x)$ by

$$\int_{\xi(x)}^{-\sqrt{2a}} (\tfrac{1}{4}\xi^2 - a)^{1/2}\, d\xi = \int_{x}^{t_-(\mu)} (\operatorname{Re}\Lambda + \mu z^4 - z^2)^{1/2}\, dz \tag{3.3b}$$

and for $t_+(\mu) \leqslant x \leqslant \infty$,

$$\int_{2\sqrt{a}}^{\xi(x)} (\tfrac{1}{4}\xi^2 - a)^{1/2}\, d\xi = \int_{t_+(\Lambda)}^{x} (\operatorname{Re}\Lambda + \mu z^4 - z^2)^{1/2}\, dz. \tag{3.3c}$$

In terms of standard elliptic functions E and K, one can rewrite (3.2) as

$$a = (3\pi)^{-1}\mu^{-1/2} t_+(\mu)\{E(q) - 2\mu t_-(\mu)^2 K(q)\},$$

where

$$q = \sqrt{\frac{2\sqrt{(1-4\mu \operatorname{Re} \Lambda)}}{1+\sqrt{(1-4\mu \operatorname{Re} \Lambda)}}} = 1 + \frac{\mu}{2} \operatorname{Re} \Lambda + O(\mu^2).$$

By either using the asymptotics of elliptic integrals, or more simply by directly analysing (3.2) one finds that

$$a = \frac{1}{3\pi\mu} + \frac{\operatorname{Re} \Lambda}{4\pi} \ln \mu + \frac{\operatorname{Re} \Lambda}{4\pi} \left[\ln \left(\frac{\operatorname{Re} \Lambda}{16} \right) - 1 \right] + O(\mu \ln \mu). \quad (3.4)$$

By construction $\xi(x)$ is strictly monotone on $[0, \infty)$ and so there exists an inverse function $x(\xi)$. Define

$$g(\xi) = (x_\xi)^{-1/2} \psi(x(\xi)). \quad (3.5)$$

Then ψ satisfies (2.1) if and only if

$$\frac{d^2}{d\xi^2} g(\xi) = \{a - \tfrac{1}{4}\xi^2 + V(\xi)\} g(\xi), \quad (3.6a)$$

where

$$V(\xi) = -ix_\xi^2 \operatorname{Im} \Lambda + \frac{3}{4} \frac{(x_{\xi\xi})^2}{(x_\xi)^2} - \frac{1}{2} \frac{x_{\xi\xi\xi}}{x_\xi}. \quad (3.6b)$$

The above considerations set up a correspondence between solutions on the real axis of (3.6) and solutions of (2.1). To complete the relation to resonance problems we must turn to consideration in the complex plane, extending (3.3c) to complex x. It is not hard to see that if $|x| \to \infty$ with $\arg x \to \theta$, then $|\xi| \to \infty$ with $\arg \xi \to \frac{3}{2}\theta$. Thus the region of decay of ψ (i.e., $\arg x \in (\pi/3, 0)$) goes into $\arg \xi \in (-\pi/2, 0)$. We want to first discuss the existence of subdominant solutions in this sector and then establish the identity of the two subdominant solutions. Relevant to the analysis of (3.6) are the asymptotics in the region $\arg \xi \in (-\pi/2, 0)$, $|\xi| \gg \sqrt{2a}$:

$$x \sim \left(\frac{3}{4}\right)^{1/3} \mu^{-1/6} \xi^{2/3}; \qquad x_\xi = \frac{2}{3} \left(\frac{3}{4}\right)^{1/3} \mu^{-1/6} \xi^{-1/3},$$

$$x_{\xi\xi} \sim -\frac{2}{9} \left(\frac{3}{4}\right)^{1/3} \mu^{-1/6} \xi^{-4/3}; \qquad x_{\xi\xi\xi} \sim \frac{8}{27} \left(\frac{3}{4}\right)^{1/3} \mu^{-1/6} \xi^{-7/3},$$

so $V = O(\xi^{-2}) + O(\xi^{-2/3})$ when μ is fixed, and it is standard ODE theory to obtain unique subdominant solutions of (3.6) in the region $-\pi/2 < \arg \xi < 0$ [6].

To finish the identification of the two solutions, it suffices to show that a single contour in the x plane goes to infinity in the sector $-\pi/3+\varepsilon < \arg(x-t_+(\mu)) < -\varepsilon$ maps into the sector $-\pi/2+\varepsilon' < \arg(\xi - 2\sqrt{a}) < -\varepsilon'$ because of the uniqueness of subdominant solutions. This follows by letting the x contour begin at a point on the real axis $x_0 \gg t_+(\mu)$ and moving directly down to $\arg x = -\pi/4$ (say) and going then directly along the line $\arg x = -\pi/4$. If x_0 is large, it is easy to see the ξ image is suitable.

We summarize with:

PROPOSITION 3.1. *If* $g(\xi)$ *satisfies* (3.6) *with* $g(\xi(0))=0$ (*resp.* $g'(\xi(0))=0$) *and* $\lim_{|\xi|\to\infty, \lim \arg\xi\in(-\pi/2,0)} g(\xi)=0$, *then* $\psi(x) = (\xi_x(x))^{-1/2} g(\xi(x))$ *is well-defined on the real axis, satisfies* (2.1), $\psi(0)=0$ (*resp.* $\psi'(0)=0$), *and its analytic continuation satisfies* $\lim_{|x|\to\infty, \lim \arg x\in(-\pi/3,0)} \psi(x)=0$.

As for the boundary conditions at $\xi(0)$, we need only note that $\xi(x)$, $x_\xi(\xi)$, $\xi_x(x)$ *are all nonvanishing at* $\xi(0)$ *or* $x=0$.

4. The Estimates: A Sketch

In this section we want to sketch two things. First, we want to show that if $g(\xi)$ is the subdominant solution of (3.6), obeying the b.c. at the origin and the transform of (2.2), then, uniformly on $-2\sqrt{a} \leqslant \xi \leqslant 2\sqrt{a}$,

$$g(\xi) = NE^*(a,\xi)(1+O(\mu^\alpha)), \tag{4.1a}$$

$$g'(\xi) = N\frac{d}{d\xi}E^*(a,\xi)(1+O(\mu^\alpha)), \tag{4.1b}$$

where N is a normalization factor (to be evaluated below), $\alpha > 0$ and E^* is a parabolic cylinder function as defined on p. 693 of Abramowitz and Stegun [1] (which we henceforth call A-S). Second, we will show how to use (4.1) to derive the Bender–Wu formula. In our analysis, we exploit formulae of A–S which are claimed there only for real ξ; the complete asymptotics in the complex plane for the equivalent Whittaker functions have been worked out by Taylor [11]. Actually, Taylor uses Langer's method to reduce Whittaker functions to Airy functions. It might seem more logical for us to directly transform (2.1) to Airy's equation, but we regard it to some extent as an historical accident that Airy function asymptotics were worked out first, and find the reduction to parabolic cylinder functions more natural since there are two turning points.

We first show how (4.1) implies the Bender–Wu formula (1.2). By Picard's method, one easily sees that on any finite interval, $\psi(x)$ goes uniformly to a harmonic oscillator wave function a priori up to a constant.

Since, by A–S 19.17, 6–7, 19.20, 3–4, 10.4.9 and 10.4.59, $g(\xi)$ decays through the barrier from $-2\sqrt{a}$ to $2\sqrt{a}$, we have that $\psi(x)\to\phi_k(x)=\pi^{-1/4}2^{-(k-2)/2}(\sqrt{(k-1)!})^{-1}H_{k-1}(x)\,e^{-x^2/2}$ and $\psi'\to\phi_k'$ uniformly on compact intervals when ψ is normalized by (2.2). The constant N is then determined by matching $\phi_k(x)$ and $E^*(a,\xi(x))$ at any point $0\ll x\ll\mu^{-1/8}$ (say). Using the above A–S formulae, one finds that

$$N=\frac{2^{-k/2}}{\pi^{1/4}\sqrt{(k-1)!}}\left(\sqrt{\frac{2k-1}{e}}\right)^{(2k-1)/2}e^{-\pi a} \tag{4.2}$$

To compute $\operatorname{Im}\Lambda$, we calculate

$$\operatorname{Im}\left[\psi(x_0)\frac{d}{dx}\bar{\psi}(x_0)\right]=\operatorname{Im}\left[x_\xi^{1/2}\,g(\xi)\frac{d\xi}{dx}\frac{d}{d\xi}x_\xi^{1/2}\,\bar{g}(\xi)\right]\bigg|_{\xi=\xi(x_0)}$$

$$=\operatorname{Im}\,g(\xi)\frac{d}{d\xi}\bar{g}(\xi)$$

(since $|g|^2(x_\xi^{-1/2}\,dx_\xi^{1/2}/d\xi)$ is real)

$$=(2i)^{-1}\mathscr{W}(NE^*(a,\xi),NE(a,\xi))=N^2$$

by A–S (19.18.2); here $\mathscr{W}$ is a Wronskian. Thus, using (3.4) and (4.2) we find that (1.2) holds.

That leaves the verification of (4.1). This we do following the analysis in Paper I. Let $\eta_-(\xi)$ denote the continuation of $NE^*(a,\xi)$ into the lower right quadrant and let $\eta_+=N^{-1}E^*(a,\xi)$. (Henceforth we suppress explicit dependence upon μ.) As in Paper I, if $b_\pm(\xi)$ are defined by

$$g(\xi)=b_+(\xi)\,\eta_+(\xi)+b_-(\xi)\,\eta_-(\xi),$$

$$g'(\xi)=b_+(\xi)\,\eta_+'(\xi)+b_-(\xi)\,\eta_-'(\xi),$$

then

$$\begin{pmatrix}b_+\\ b_-\end{pmatrix}'=A(\xi)\begin{pmatrix}b_+\\ b_-\end{pmatrix},$$

where

$$A(\xi)=\frac{V(\xi)}{2i}\begin{bmatrix}-\eta_-\eta_+ & -\eta_-^2\\ \eta_+^2 & \eta_-\eta_+\end{bmatrix} \tag{4.3}$$

with V given in (3.6b). Thus, if we define M_ξ on pairs of function $f(\xi)=\binom{f_+}{f_-}$ by $(M_\xi f)(\xi)=\int_{\xi,\arg\bar{\xi}\to-\pi/4\,(\text{say})}^{\infty}A(\bar{\xi})f(\bar{\xi})\,d\bar{\xi}$, formally $b_\pm$ are given

$$\begin{pmatrix}b_+\\ b_-\end{pmatrix}(\xi)=\sum_{j=0}^{\infty}M_\xi^j\begin{pmatrix}1\\ 0\end{pmatrix} \tag{4.4}$$

Estimating absolutely the terms in this sum, we will prove that

$$\begin{aligned} b_+ &= O(\mu^\alpha \eta_-/\eta_+), \\ b_- &= 1 + O(\mu^\alpha) \end{aligned} \tag{4.5}$$

for some $\alpha > 0$, uniformly in $\xi \in [0, 2\sqrt{a}) \cup C$, where C is a contour in the sector $-\varepsilon > \arg \xi > -\pi/2 + \varepsilon$ connecting x_0 to ∞, (4.5) immediately yields (4.1).

In analogy with Taylor [11], we write η_- in terms of Airy functions:

$$\eta_-(\xi) = \sqrt{\pi}\, u^{1/4}(\mathrm{Bi}(u) - i\,\mathrm{Ai}(u)) \left(\frac{\xi^2}{4a} - 1\right)^{-1/4},$$

where u is the function

$$\begin{aligned} u(\xi) &= \left(3a \int_1^{\xi/2\sqrt{a}} (s^2 - 1)^{1/2}\, ds\right)^{2/3} \\ &= \left[-\frac{3a}{2} \operatorname{arc\,cosh}\left(\frac{\xi}{2\sqrt{a}}\right) + \frac{3\xi}{4}\left(\frac{\xi^2}{4} - a\right)^{1/2}\right]^{2/3}. \end{aligned}$$

Then using facts about Airy functions in A–S, we estimate the matrix elements in (4.3) by

$$\begin{aligned} |\eta_- \eta_+(\xi)| &\leqslant Cf(a, \xi), \\ |\eta_-^2(\xi)| &\leqslant CN^2 f(a, \xi)\, |\exp(-\tfrac{2}{3}u^{3/2})|, \\ |\eta_+^2(\xi)| &\leqslant CN^{-2} f(a, \xi)\, |\exp(+\tfrac{2}{3}u^{3/2})| \end{aligned}$$

with $f(a, \xi) = \min(a^{1/6}, |\xi^2/4a - 1|^{-1/2})$.

As in Paper I, we estimate the terms in (4.4) using a weighted norm for functions with the weight $\exp(\frac{4}{3}u^{3/2})\, N^{-2}$ on the plus component. Using that idea and the above estimates on $\eta_+ \eta_-$, etc., we find that (4.5) will follow from

$$\int_{\xi(0)\ \text{on the contour}}^{\infty} |d\xi|\, |V(\xi)| \min\left(a^{1/6}, \left|\frac{\xi^2}{4a} - 1\right|^{-1/2}\right) = O(\mu^c). \tag{4.6}$$

Since we know that $|\operatorname{Im} \Lambda| = O(\mu^{-n})$ for all n we obtain (4.6) with $c = \frac{1}{2}$ using the estimates in the table below:

Function	Near Turning Point	Other Real Values	Large Complex Values
X_ξ	$O(\mu^{1/3})$	$O(1)$	$O((\mu/\xi)^{1/3})$
$(X_\xi)^{-1}$	$O(\mu^{-1/3})$	$O(1)$	$O((\mu/\xi)^{-1/3})$
$X_{\xi\xi}$	$O(\mu^{5/6})$	$O(\mu^{1/2})$	$O(\mu^{1/3}\xi^{-4/3})$
$X_{\xi\xi\xi}$	$O(\mu^{4/3})$	$O(\mu)$	$O(\mu^{1/3}\xi^{-7/3})$

This completes the sketch of the proof of (4.1).

References

1. M Abramowitz and I. Stegun, "Handbook of Mathematical Functions," Dover, New York, 1965.
2. C. Bender and T. T. Wu, Anharmonic oscillator, I, II, *Phys. Rev.* **184** (1969), 1231–1260; *Phys. Rev. D.* **7** (1973), 1620–1636.
3. N. Corngold, Kinetic equation for a weakly coupled test particle, II, Approach to equilibrium, *Phys. Rev. A* **24** (1981), 656–666.
4. E. M. Harrell, Estimating tunnelling phenomena, *Internat. J. Quant. Chem.* **21** (1982), 199–208.
5. E. M. Harrell and B. Simon, The mathematical theory of resonances whose widths are exponentially small, *Duke Math. J.* **47** (1980), 845–902.
6. E. Hille, "Ordinary Differential Equations in the Complex Domain," Wiley, New York, 1976.
7. P. F. Hsieh and Y. Sibuya, On the asymptotic integration of second order linear ordinary differential equations with polynomial coefficients, *J. Math. Anal. Appl.* **16** (1966), 84–103.
8. J. J. Loeffel and A. Martin, Propriétés Analytique des Niveaux de l'Oscillateur Anharmonique et Convergence des Approximants de Padé, Proc. RCP and CERN – Ref. – Th. 1167, 25 May 1970.
9. R. E. Langer, On the connection formulas and the solutions of the wave equation, *Phys. Rev.* **51** (1937), 669–676.
10. B. Simon, Coupling constant analyticity for the anharmonic oscillator, *Ann. Physics* **58** (1970), 76–136.
11. W. C. Taylor, A complete set of asymptotic formulae for the Whittaker function and the Laguerre polynomials, *J. Math. Phys.* **38** (1939), 34–49.