JOURNAL OF FUNCTIONAL ANALYSIS 53, 84–98 (1983)

Nonclassical Eigenvalue Asymptotics*

BARRY SIMON

Departments of Mathematics and Physics, California Institute of Technology, Pasadena, California 91125

Communicated by P. Lax

The leading asymptotics for the growth of the number of eigenvalues of the two-dimensional Dirichlet Laplacian in the regions $\{(x,y) \mid |x|^{\mu} \, |y| \leqslant 1\}$ and for $-\Delta + |x|^{\alpha} \, |y|^{\beta}$ all of which are non-Weyl because of infinite phase space volumes are computed. Along the way, a general inequality on quantum partition functions computed in a kind of Born–Oppenheimer approximation is proved.

1. INTRODUCTION

A celebrated theorem of Weyl [20] asserts that if A is the Dirichlet Laplacian in a bounded region Ω in R^2, the number of eigenvalues $N(E)$ of A less than E is asymptotically equal to $\frac{1}{2} |\Omega| \, E$, where $|\Omega|$ is the volume of Ω. It is not hard to extend this to unbounded regions of finite volume.

In a recent paper [15], we gave several proofs that the class of infinite volume regions $\{(x,y) \mid |x|^{\mu} \, |y| \leqslant 1\}$ yield Dirichlet Laplacians with discrete spectrum despite the fact that $|\Omega| = \infty$. Our main goal is to determine the leading order divergence of $N(E)$ for such Dirichlet Laplacians and for the closely related operators $-\Delta + x^{\alpha} y^{\beta}$. By leading order, we mean both the power of E and the appropriate constant. Some of the methods in [15] (in particular, the Fefferman–Phong theorem [5]) can obtain at least the correct leading power.

We recall the Karamata–Tauberian theorem which reduces the large E asymptotics of $N_A(E)$ to the small t divergence of $\mathrm{Tr}(e^{-tA})$.

THEOREM 1.1. *$\lim_{E \to \infty} E^{-\gamma} N_A(E) = c$ if and only if $\lim_{t \downarrow 0} t^{\gamma} \, \mathrm{Tr}(e^{-tA}) = c\Gamma(\gamma + 1)$. Also $\lim_{E \to \infty} E^{-\gamma} (\ln E)^{-1} \, N_A(E) = c$ if and only if $\lim_{t \downarrow 0} t^{\gamma} (\ln t^{-1})^{-1} \, \mathrm{Tr}(e^{-tA}) = c\Gamma(\gamma + 1)$.*

For a proof of the first statement, see, e.g., [16, Theorem 10.3]. The second statement has a similar proof. Thus we concentrate on the small t behavior of $\mathrm{Tr}(e^{-tA})$ (a strategy of Kac [8]). In Section 3, we will prove

* Research partially supported by USNSF Grant MCS-81-20833.

0022-1236/83 $3.00

THEOREM 1.2. *Let* $A = -\Delta + |x|^\alpha |y|^\beta$ *with* $\alpha < \beta$. *Let* $\nu = (\beta + 2)/2\alpha$ *and let* $a = \mathrm{Tr}((-(d^2/dy^2) + |y|^\beta)^{-\nu}) < \infty$ *the trace being on* $L^2(R)$. *Then*

$$\lim_{t \downarrow 0} t^{(\nu + 1/2)} \mathrm{Tr}(e^{-tA}) = a\pi^{-1/2}\Gamma(\nu + 1).$$

Remark. By symmetry if $\beta < \alpha$, we need only interchange β and α.
In Section 4, we will prove

THEOREM 1.3. *Let* A *be the Dirichlet Laplacian for the region* $\{(x, y) \mid |x|^\mu |y| \leqslant 1\}$ *with* $\mu > 1$. *Then*

$$\lim_{t \downarrow 0} t^{(1/2)(\mu + 1)} \mathrm{Tr}(e^{-tA}) = \pi^{-1/2} \left(\frac{\pi}{2}\right)^{-\mu} \zeta(\mu) \Gamma\left(\frac{\mu}{2} + 1\right),$$

where $\zeta(\mu) = \sum_1^\infty n^{-\mu}$ *is the usual zeta function.*

Remark 1. Again, by symmetry, if $\mu < 1$, the above formula holds if everywhere μ is replaced by $1/\mu$.

Remark 2. The A of this theorem is (up to $x \leftrightarrow y$ interchange) the limit of the A of Theorem 1.2 if $\alpha, \beta \to \infty$ with $\beta/\alpha = \mu$. Then $\nu \to \mu/2$ and the limit of the a of Theorem 1.2 is

$$\mathrm{Tr}((\tilde{p}^2)^{-\nu}), \qquad \text{where } \tilde{p}^2 \text{ is the Dirichlet Laplacian}$$

for $[-1, 1]$. Thus a has a limit $\sum_1^\infty [(\pi n/2)^2]^{-\mu/2}$ and the theorem says that in some sense, the $t \downarrow 0$ and the $\alpha \to \infty$ can be interchanged.

In Section 5, we will handle the somewhat more subtle cases where $\alpha = \beta$ or $\mu = 1$. We will prove the following pair of results:

THEOREM 1.4. *Let* $A = -\Delta + |xy|^\alpha$. *Then*

$$\lim_{t \downarrow 0} t^{(1 + \alpha^{-1})} [\ln(t^{-1})]^{-1} \mathrm{Tr}(e^{-tA}) = \pi^{-1} \Gamma\left(2 + \frac{1}{\alpha}\right).$$

THEOREM 1.5. *Let* A *be the Dirichlet Laplacian for the region* $|xy| \leqslant 1$. *Then* $\lim_{t \downarrow 0} t(\ln t^{-1})^{-1} \mathrm{Tr}(e^{-tA}) = 1/\pi$.

Thus for the operators of Theorems 1.2–1.5 we have

$$\lim_{E \to \infty} E^{-(\nu + 1/2)} N_A(E) = a\Gamma(\nu + 1)/\pi^{1/2}\Gamma(\nu + \tfrac{3}{2}) \qquad \text{(Theorem 1.2)},$$

$$\lim_{E \to \infty} E^{-1/2(\mu + 1)} N_A(E) = \zeta(\mu) \left(\frac{\pi}{2}\right)^{-\mu} \Gamma(\tfrac{1}{2}\mu + 1)/\pi^{1/2}\Gamma(\tfrac{1}{2}\mu + \tfrac{3}{2}) \qquad \text{(Theorem 1.3)},$$

$$\lim_{E\to\infty} E^{-(\nu+1/2)}(\ln E)^{-1} N_A(E) = \pi^{-1} \qquad \text{(Theorem 1.4)},$$

$$\lim_{E\to\infty} E^{-1}(\ln E)^{-1} N_A(E) = 1/\pi \qquad \text{(Theorem 1.5)}.$$

While we have succeeded in computing the leading asymptotic behavior of $N(E)$, we are lacking a general geometric interpretation of the answer. This is an important open question. We note the remarkable simplicity of π^{-1} as the high energy constant for the case of Theorems 1.4, 1.5.

For any of the above operators, a useful tool is a general inequality we prove in Section 2. If $H = -\Delta + V$ on $L^2(R^\nu)$, we let $Z_Q(t) = \mathrm{Tr}(e^{-tH})$ and $Z_{\mathrm{cl}}(t) = \int (d^\nu \mathbf{r}\, d^\nu p/(2\pi)^\nu)\, e^{-t(p^2+V(\mathbf{r}))}$. Golden [6] and Thompson [18], using an abstract operator indequality, proved that

$$Z_Q(t) \leqslant Z_{\mathrm{cl}}(t). \tag{1.1}$$

For the operators here, $Z_{\mathrm{cl}} \equiv \infty$ while $Z_Q < \infty$, so this inequality is not so useful! Suppose that we write $\nu = \alpha + \beta$ and $r \in R^\nu$ as (x, y) with $x \in R^\alpha$, $y \in R^\beta$. Let $\varepsilon_1(x) \leqslant \varepsilon_2(x) \leqslant \cdots$ be the eigenvalues of $-\Delta_y + V(x, y)$ as an operator on $L^2(R^\beta)$, listed in order, counting multiplicity. Define

$$Z_{\mathrm{SB}}(t) \equiv \sum_k \mathrm{Tr}_{L^2(R^\alpha)}(e^{-t(-\Delta_x + \epsilon_k(x))}).$$

We will prove in Section 2 that

$$Z_Q(t) \leqslant Z_{\mathrm{SB}}(t). \tag{1.2}$$

Applying (1.1) to $\mathrm{Tr}(e^{-t(-\Delta_x+\varepsilon_k(x))})$ we obtain, by using $\sum_k e^{-t\varepsilon_k(x)} = \mathrm{Tr}_{L^2(R^\beta}(e^{-t(-\Delta_y+V(x,y))})$,

$$Z_{\mathrm{SB}}(t) \leqslant Z_{\mathrm{SGT}}(t), \tag{1.3}$$

where

$$Z_{\mathrm{SGT}}(t) = \int \frac{d^\alpha p\, d^\alpha x}{(2\pi)^\alpha}\, e^{-tp^2}\, \mathrm{Tr}_{L^2(R^\beta)}(e^{-t(-\Delta_y+V(x,y))}). \tag{1.4}$$

Applying (1.1) to the trace in (1.4), one finds the last of the string of inequalities

$$Z_Q(t) \leqslant Z_{\mathrm{SB}}(t) \leqslant Z_{\mathrm{SGT}}(t) \leqslant Z_{\mathrm{cl}}(t). \tag{1.5}$$

The point of (1.5) in the context of the Theorems 1.2, 1.3 is that $Z_{\mathrm{SB}}(t) < \infty$ and, if one slices in the right direction (one can clearly try to take slices in x as above, or alternately in y), then, as we will prove

$\lim_{t \downarrow 0} Z_Q(t)/Z_{\mathrm{SGT}}(t) = 1$. Then we will be able to compute the small t behavior of $Z_{\mathrm{SGT}}(t)$ explicitly because of the scaling properties of the regions or potentials.

Equation (1.2) says that one can obtain an upper bound on Z_Q by slicing and putting the slices together. We thus call (1.2) the "sliced bread inequalities" and Z_{SB} the sliced bread partition function, SGT stands for "sliced Golden–Thompson."

We remark that since $Z_{\mathrm{SGT}}(t) = \mathrm{Tr}(e^{+t\Delta_x} e^{-t(-\Delta_y + V)})$, the inequality $Z_Q \leqslant Z_{\mathrm{SGT}}$ (which suffices for Theorems 1.2, 1.3 but not for Theorems 1.4, 1.5) folows from the abstract Golden–Thompson inequality.

2. Sliced Bread Inequalities

On $R^\nu = R^\alpha \times R^\beta$ write $r \in R^\nu$ as (x, y) with $x \in R^\alpha$, $y \in R^\beta$. We will suppose that V is a continuous function on R^ν bounded from below, although it is clear one can get away with much less regularity and still obtain the inequalities here. We want information about $H = -\Delta + V$ on $L^2(R^\nu)$. For each fixed x, we can define $H_x = -\Delta_y + V(x, y)$ on $L^2(R^\beta)$. We let $\varepsilon_1(x) \leqslant \varepsilon_2(x) \leqslant \cdots$ be the eigenvalues of H_x counting multiplicity with the convention that if $\sum_x = \inf \sigma_{\mathrm{ess}}(H_x) < \infty$ and H_x has exactly k_0 eigenvalues below $\sum_x$ (counting multiplicity), then $\varepsilon_l(x) = \sum_x$ for $l > k_0$. If the y's are electron coordinates and the x's are nuclear coordinates, the $\varepsilon_k(x)$ are the familiar Born–Oppenheimer curves (see, e.g., [2]). In that context, it is an old true folk theorem, that $\inf \sigma(H) \geqslant \inf \sigma(-\Delta_x + \varepsilon_1(x))$. Some thought suggests it might be true in some sense that H is larger than $\bigoplus_{k=1}^{\infty} [-\Delta_x + \varepsilon_k(x)] = H_{\mathrm{SB}}$. For instance, one might hope that the nth eigenvalue of H is larger than the nth eigenvalue of H_{SB}. This is false as seen by the following:

Example. Let $H(\lambda) = -\Delta + x^2 + (y + \lambda x)^2$. H_{SB} is independent of λ and unitarily equivalent to $H(0)$. Thus, if $\varepsilon_n(H) \geqslant \varepsilon_n(H_{\mathrm{SB}})$, we must have that $\varepsilon_n(H(\lambda)) \geqslant \varepsilon_n(H(0))$. We claim that for λ small, $\varepsilon_2(H(\lambda)) < \varepsilon_2(H(0))$ invalidating the above possibility. For a direct calculation of harmonic oscillators eigenvalues show that

$$\varepsilon_2(H(\lambda)) = 4 - \left(\frac{\sqrt{5} - 2}{2}\right) \lambda^2 + O(\lambda^4)$$

However, we will prove here

Theorem 2.1 (Sliced bread inequalities). $\mathrm{Tr}(e^{-tH}) \leqslant \mathrm{Tr}(e^{-tH_{\mathrm{SB}}}) \equiv \sum_k \mathrm{Tr}(e^{-t(-\Delta_x + \varepsilon_k(x))})$.

Remark. This means that if $e^{-tH_{SB}}$ is trace class, so is e^{-tH} and the inequality holds.

The above situation, namely, that in the passage from H_{SB} to H, ground state energies and free energies ($\equiv -\ln \mathrm{Tr}(\cdots)$) go up, but excited states may not, is very reminiscent of what happens when a magnetic field is turned on in nonrelativistic quantum mechanics: In that case, ground state energies [11] and free energies go up [12] but excited states may not. This is not mere coincidence. There is a formal connection between the two sets of ideas. Let $W(x)$ be the diagonal matrix on l_2 with entries $\varepsilon_k(x)$, so H_{SB} is $-\Delta + W$ on $L^2(R^\alpha; l_2)$. Let $U(x): l_2 \to L^2(R^\beta)$ be unitary so that $H_x = U(x)\, W(x)\, U(x)^{-1}$ (we suppose that H_x has no essential spectrum). Let $V: L^2(R^\alpha; l_2) \to L^2(R^\nu)$ by $(Vf)(x, y) = U(x) f(x)$. Then a *formal* calculation shows that

$$V^{-1} H V = (-i\nabla_x - A(x))^2 + W(x),$$

where

$$A(x) = iU(x)^{-1}\, \nabla U(x)$$

is formally selfadjoint. Thus, Theorem 2.1 is a kind of extended diamagnetic inequality except that A is vector-valued and has zero curvature (this should be distinguished from the vector-valued diamagnetic inequalities of Hess *et al.* [7], who require that each $W(x)$ be a multiple $V(x)$ of the identity). We remark that by combining our technique here and the method of [13], one can easily prove that:

THEOREM 2.2. *Let $A(x)$ be a matrix valued function from R^α to the Hermitian $n \times n$ matrices. Let $W(x)$ be a function taking values in the diagonal self-adjoint matrices whose eigenvalues are increasing. Then*

$$\mathrm{Tr}_{C^n}(e^{-tH(A)})(x, x') \leqslant \mathrm{Tr}_{C^n}(e^{-tH(0)})(x, x'),$$

where $H(A) = (-i\nabla - A)^2 + W$.

While the above intuition is useful to understand why Theorem 2.1 should be true, our proof does not use this. Rather, what is basic is the following theorem of Ky Fan [4] (see also Marcus and Moyls [9], Mirsky [10], DeBruyn [3] and Capel and Tindemans [1, 19]).

LEMMA 2.3. *Let $A_1, \ldots, A_n$ be trace class operators on some Hilbert space $\mathscr{H}$ and let $A_1^*, \ldots, A_n^*$ be the diagonal matrices on l_2 whose eigenvalues are the singular values of $A_1, \ldots, A_n, \ldots$. Then*

$$|\mathrm{Tr}(A_1 \cdots A_n)| \leqslant \mathrm{Tr}(A_1^* \cdots A_n^*). \tag{2.1}$$

Proof. We sketch a proof for the reader's convenience. We begin by noting that

$$\|\Lambda^k(A_1 \cdots A_n)\| \leqslant \|\Lambda^k(A_1)\| \cdots \|\Lambda^k(A_n)\|,$$

where $\Lambda^k(\cdot)$ is the k-fold alternating product. Since $\|\Lambda^k(C)\| = \mu_1(C) \cdots \mu_k(C)$ with $\mu_j(\cdot)$ the jth singular value, we have that

$$\prod_{j=1}^{k} \mu_j(A_1 \cdots A_n) \leqslant \prod_{j=1}^{k} [\mu_j(A_1) \cdots \mu_j(A_n)]$$

for each k. General symmetric rearrangement results (see, e.g., [1], Corollary 1.10]) imply that

$$\operatorname{Tr}(|A_1 \cdots A_n|) = \sum_1^{\infty} \mu_j(A_1 \cdots A_n) \leqslant \sum_1^{\infty} \mu_j(A_1) \cdots \mu_j(A_n) = \operatorname{Tr}(A_1^* \cdots A_n^*)$$

and (2.1) follows from $|\operatorname{Tr}(C)| \leqslant \operatorname{Tr}(|C|)$. ∎

From this lemma, we obtain a general result which shows that the fact that the H_x's are Schrodinger operators is irrelevant to the truth of sliced bread.

THEOREM 2.4. *Suppose that $W(x)$ is a continuous $n \times n$ symmetric matrix-valued function on R^α with eigenvalues $\varepsilon_1(x) \leqslant \cdots \leqslant \varepsilon_n(x)$ and $\inf_x \varepsilon_1(x) > -\infty$. Let H be the operator $-\Delta + W$ on $L^2(R^\alpha; \mathbb{C}^n)$. Then*

$$\operatorname{tr}_{\mathbb{C}^n}(e^{-tH})(x, x') \leqslant \sum_j (e^{-tH_j})(x, x'),$$

where $H_j = -\Delta_x + \varepsilon_j(x)$ on $L^2(R^\alpha; \mathbb{C})$ and $C(x, x')$ is the integral kernel of C.

Remarks. By general principles [14], e^{-tH_j} has a continuous integral kernel, and by similar arguments, so does the *partial* trace $\operatorname{tr}_{\mathbb{C}^n}(e^{-tH})$. Below, when we use the Trotter product formula, in principle we only get an equality a.e. but then continuity of the integral kernel yields a pointwise inequality.

Proof. By the above remark, it suffices to prove

$$\operatorname{tr}_{\mathbb{C}^n}[e^{+t\Delta/n} e^{-tW/n})^n](x, x') \leqslant \sum_j (e^{+t\Delta/n} e^{-t\varepsilon_j/n})^n(x, x')$$

and then appeal to the Troter formula. Since $e^{t\Delta}$ acts as the identity on $\mathbb{C}^n$, it comes out of the trace and writing out the explicit *positive* integral kernel, we only need that

$$\mathrm{Tr}(e^{-tW(x_1)/n} \cdots e^{-tW(x_n)/n}) \leqslant \sum_j e^{-t\varepsilon_j(x_1)/n} \cdots e^{-t\varepsilon_j(x_n)/n}.$$

Since $[e^{-tW(x)/n}]^* = e^{-t\varepsilon(x)/n}$, this inequality is precisely lemma 2.3. ∎

Proof of Theorem 2.1. (a) First we note that since e^{-tH_j} and $\mathrm{tr}_{\mathbb{C}^n}(e^{-tH})$ have continuous positive integral kernels, we can compute traces by setting $x = x'$ and integrating (even if the trace is infinity); see, e.g., [17, Theorem 3.9]. Thus, in the context of Theorem 2.4

$$\mathrm{Tr}(e^{-tH}) \leqslant \sum_j \mathrm{Tr}(e^{-tH_j}). \tag{2.2}$$

(b) We claim that (2.2) extends to the situation, where $\mathbb{C}^n$ is replaced by l_2 and each $W(x)$ has discrete spectrum and common domain with $\lim_{j\to\infty} \varepsilon_j(x) = \infty$. For let $\{P_n\}$ be a sequence of rank n projections in l_2 with range in the common domain, and $P_n \to I$ in strong graph norm. Let $H_n \equiv [I \otimes P_n](H)[I \otimes P_n]$ and $\varepsilon_j^{(n)}(x)$ the eigenvalues of $P_n W(x) P_n$ on $\mathrm{Ran}\, P_n$. Then $\mathrm{Tr}_{L^2(R^n;\, \mathbb{C}^n)}(e^{-tH_n}) \nearrow \mathrm{Tr}(e^{-tH})$ since $H_n \to^w H$ and one has Fatou's lemma (see [17, Theorem 2.7]). Similarly $\sum_{j=1}^n \mathrm{Tr}(e^{-tH_j^{(n)}}) \nearrow \sum_1^\infty \mathrm{Tr}(e^{-tH_j})$, so (2.2) extends. ($a_n \nearrow a$ here means $a_n \to a$ and $a_n \leqslant a$.)

(c) Let $H_{(l,\varepsilon)}$ be the operator obtained by setting $V(x, y) = \infty$ if $|y| \geqslant l$ (i.e., putting Dirichlet B.C. there) and adding εx^2 to V otherwise. Then, by (b),

$$\mathrm{Tr}(e^{-tH_{l,\varepsilon}}) \leqslant \sum_1^\infty \mathrm{Tr}(e^{-tH_{j,(l,\varepsilon)}}).$$

But everything is monotone increasing if $l \to \infty$, $\varepsilon \downarrow 0$, so as above, we obtain the general result. ∎

3. Eigenvalue Asymptotics, $\alpha \neq \beta$, Finite

Our goal in this section will be to prove Theorem 1.2. Let

$$F(x, t) = \mathrm{Tr}_{L^2(dy)}\left(\exp\left[-t\left(\frac{-d^2}{dy^2} + |x|^\alpha\, |y|^\beta\right)\right]\right).$$

Then, the sliced Golden–Thompson inequality implies (doing the p integral explicitly)

$$Z_Q(t) \leqslant (\pi t)^{-1/2} \int_0^\infty F(x, t)\, dx \tag{3.1}$$

On the other hand, the Feynman–Kac formula [16], reads

$$Z_Q(t) = (4\pi t)^{-1} \int dx\, dy\, E_{(x,y)(x,y);2t} \left(\exp\left(-\int_0^{2t} \tfrac{1}{2}\, |b_1(s)|^\alpha\, |b_2(s)|^\beta\, ds \right) \right),$$

where $E_{\mathbf{r},\mathbf{r},w}$ is expectation with respect to conventional Brownian motion conditioned to start and end at $\mathbf{r}$ in time w. The funny factors of 2 are caused by the fact that we have taken $-\Delta$ not $-\frac{1}{2}\Delta$. Clearly, we can get a lower bound on Z_Q by only taking b_1 paths with $\sup_{0 \leqslant s \leqslant 2t} |b_1(s) - x| \leqslant 1$ and then replacing $|b_1(s)|^\alpha$ by its upper bound $(|x| + 1)^\alpha$. Since

$$\operatorname*{Prob}_{0 \leqslant s \leqslant 2t} (|b_1(s) - x| \leqslant 1) \geqslant 1 - \rho(t) \tag{3.2}$$

with $\rho(t) \to 0$ (as $e^{-(1-\varepsilon)/4t}$), we find that

$$\begin{aligned} Z_Q(t) &\geqslant (\pi t)^{-1/2} [1 - \rho(t)] \int_0^\infty F(|x| + 1, t)\, dx \\ &= (\pi t)^{-1/2} [1 - \rho(t)] \int_1^\infty F(x, t)\, dx. \end{aligned} \tag{3.3}$$

Thus, the theorem follows from $\rho(t) \to 0$ as $t \downarrow 0$ and

$$\lim_{t \downarrow 0} t^\nu \int_0^1 F(x, t)\, dx = 0 \tag{3.4}$$

$$\lim_{t \downarrow 0} t^\nu \int_0^\infty F(x, t)\, dx = a\Gamma(\nu + 1). \tag{3.5}$$

We begin our study with the use of scaling. Since $y \to \lambda y$, $d/dy \to \lambda^{-1}(d/dy)$ is unitary implementable

$$F(x\lambda^{2\nu}, t\lambda^{-2}) = F(x, t) \tag{3.6}$$

(which is why $\nu = (\beta + 2)/2\alpha$ enters naturally). Next, we note that for x fixed, the small t behavior of $F(x, t)$ is given by classical phase space (see [16, p. 110]), so

$$F(1, t) \sim (2\pi)^{-1} \int dy\, dq\, e^{-t(q^2 + |y|^\beta)} = Dt^{-a}$$

with $a = \frac{1}{2} + \beta^{-1}$ and D a suitable constant. Here $\sim$ means the ratio goes to 1, and so for t small

$$F(1, t) \leqslant D' t^{-\alpha\nu/\beta} \tag{3.7}$$

since $a = (\beta + 2)/2\beta = \alpha\nu/\beta$. By scaling (3.6)

$$F(x, t) = F(1, tx^{1/\nu}) \leqslant D' x^{-\alpha/\beta} t^{-\alpha\nu/\beta}$$

for t small and $|x| \leqslant 1$. Since $\alpha/\beta < 1$, (3.4) is immediate.

We note here that, since (3.7) is asymptotically exact, if $\beta \leqslant \alpha$, $\int_0^1 F(x, t)\, dx = \infty$ and we see that there is a right way and wrong way to slice in sliced Golden–Thompson if $\alpha \neq \beta$ (indeed, the part of the phase space integral which is divergent comes from x large if $\alpha < \beta$ and it is useful to slice that transversely). In particular, if $\alpha = \beta$, sliced Golden–Thompson does not prove finiteness, and sliced bread will be needed.

Now we use scaling again, to get

$$\int_0^\infty F(x, t)\, dx = \int_0^\infty F(xt^\nu, 1)\, dx = t^{-\nu} \int_0^\infty F(x, 1)\, dx$$

so (3.5) is equivalent to

$$\int_0^\infty F(x, 1)\, dx = a\Gamma(\nu + 1). \tag{3.8}$$

Let $A = (-d^2/dy^2) + |y|^\beta$. Then, using scaling once more (!)

$$\begin{aligned} \int_0^\infty F(x, 1)\, dx &= \int_0^\infty F(1, x^{1/\nu})\, dx \\ &= \nu \int_0^\infty s^{\nu-1} F(1, s)\, ds \\ &= \nu \operatorname{Tr}\left(\int_0^\infty s^{\nu-1} e^{-sA}\, ds\right) \\ &= \operatorname{Tr}(A^{-\nu})\, \nu\Gamma(\nu) \end{aligned}$$

as required.

Finally, we remark that the finiteness of $\operatorname{Tr}(A^{-\nu})$ is immediate from the finiteness of $\int_0^\infty F(x, 1)\, dx$ which follows, since we saw above that $F(x, 1) \sim x^{-\alpha/\beta}$ for x small (and decays as $\exp(-cx^{1/\nu})$ for x large).

4. Eigenvalue Asymptotics, $\alpha \neq \beta$, Infinite

Our goal in this section will be to prove Theorem 1.3. Let $A(x)$ be the Dirichlet "Laplacian," $-d^2/dy^2$ on the interval $[-x^{-\mu}, x^{-\mu}]$, and let

$$F(x, t) = \mathrm{Tr}(\exp(-tA(x))).$$

By taking suitable infinite potential limits in (3.1), the proof is reduced to the analog of (3.4), (3.5), where ν is now replaced by $\mu/2$. The same scaling relation

$$F(x, t) = F(\lambda^{\mu} x, \lambda^{-2} t)$$

still holds, so by just following the proof, we see that all one needs is that $\mathrm{Tr}(A(1)^{-1/2)}\mu) = (\pi/2)^{-2}\zeta(\mu)$. Since $A(1)$ has eigenvalues $((\pi/2)k)^2$, $(k = 1, 2, \ldots)$ this is immediate.

5. Eigenvalue Asymptotics, $\alpha = \beta$

In this section, we prove Theorems 1.4 and 1.5. As we have already seen, sliced Golden–Thompson is useless here, so we use sliced bread. In fact, a miracle not guaranteed to happen does, and sliced bread, which cuts 2 of the four "horns" of the potential transversely and 2 nontransversely, is asymptotically exact. (If it were not, one could use the local version of sliced bread to locally slice each horn in a transverse way. We note another miracle involving sliced bread: If $\mu > 1$, we saw in Section 3 that $Z_Q/Z_{\mathrm{SGT}} \to 1$ so by (1.2, 3) $Z_Q/Z_{\mathrm{SB}} \to 1$ if one makes x slices which are transverse to the significant horns; if we y slice, $Z_Q/Z_{\mathrm{SB}} \not\to 1$ but miraculously it does get the right asymptotic power, and the ratio goes to a finite constant!) Another simplifying feature which makes up for the failure of sliced Golden–Thompson is the presence of a log and the fact that the value of a logarithmically divergent integral is rather insensitive to cutoffs.

We begin with the lower bound and take $\alpha = \beta$ finite. As before, we go to a Feynman Kac formula, but this time we throw away some points in the x, y integral, keeping only pairs x, y with $|x| \geqslant t^{1/2}(\ln t)^2$, $|y| \geqslant t^{1/2}(\ln t)^2$. We only consider paths with $\sup_{0 \leqslant s \leqslant 2t} |b_1(s) - x| \leqslant t^{1/2} |\ln t|$ and $\sup_{0 \leqslant s \leqslant 2t} |b_2(s) - x| \leqslant t^{1/2} |\ln t|$. The measure of such paths is again $1 - \rho(t)$with $\rho(t) \to 0$ (but this time only as $e^{-D(\ln t)^2}$). Let $z(x, y) \equiv xy$. Then $d \ln z/dx = 1/x$, $d \ln z/dy = 1/y$ so everywhere along the path $|\ln z(b_1(s), b_2(s)) - \ln z(x, y)| \leqslant c/|\ln t|$ and thus if $\kappa(t) = \exp(+2/\ln t)$, we have $z(b(s)) \leqslant \kappa(t)\, z(0)$ for all s. Thus

$$\operatorname{Tr}(e^{-tH}) \geqslant (4\pi t)^{-1}(1-\rho(t)) \int_{|x|,|y| \geqslant t^{1/2}(\ln t)^2} dx\, dy\, e^{-t\kappa^\alpha z^\alpha}$$

$$= (4\pi t)^{-1}(1-\rho(t))\, 4 \int_{t(\ln t)^4} dz\, e^{-t\kappa^\alpha z^\alpha} \left[\int_{t^{1/2}(\ln t)^2}^{z/t^{1/2}(\ln t)^2} \frac{dy}{y} \right]$$

$$= (\pi t)^{-1}(1-\rho(t)) \int_{t(\ln t)^4} \{dz \ln[z/t(\ln t)^4]\, e^{-t\kappa^\alpha z^\alpha}\}$$

$$= (\pi t)^{-1}(1-\rho(t))\, \kappa^{-1}\, t^{-1/\alpha}$$

$$\times \int_{t^{1+\alpha^{-1}}(\ln t)^4 \kappa} dw\, e^{-w^\alpha} \ln[w/\kappa t^{1+1/\alpha}(\ln t)^4]$$

where the 4 in the second line comes from the four quadrants, and in the last step we set $w = t^{1/\alpha}\kappa z$. Since $\kappa \to 1$ and $\rho \to 0$, we see that

$$\underline{\lim}\, t^{1+\alpha^{-1}} \{\ln(t^{-1-\alpha^{-1}})\}^{-1} \operatorname{Tr}(e^{-tH}) \geqslant \pi^{-1} \int_0^\infty e^{-w^\alpha}\, dw.$$

The integral is easily seen so equal $(1/\alpha)\,\Gamma(1/\alpha) = \Gamma(1+(1/\alpha))$ so since $\ln(t^{-1-\alpha^{-1}}) = (1+\alpha^{-1}) \ln t^{-1}$ and $(1+(1/\alpha))\,\Gamma(1+(1/\alpha)) = \Gamma(2+(1/\alpha))$ we find hat

$$\underline{\lim}\, t^{1+\alpha^{-1}} |\ln t|^{-1} \operatorname{Tr}(e^{-tH}) \geqslant \pi^{-1}\Gamma\left(2+\frac{1}{\alpha}\right). \tag{5.1}$$

Essentially identical arguments (but now we only take those x, y with $\kappa z < 1$ and $x, y \geqslant t^{1/2}(\ln t)^2$) yield the lower bound on the Dirichlet Laplacian in the region $|xy| \leqslant 1$

$$\underline{\lim}\, t^{-1} |\ln t|^{-1} \operatorname{Tr}(e^{-tH}) \geqslant \pi^{-1}. \tag{5.2}$$

We will get the upper bound using sliced bread inequalities. To compute the small s behavior of $Z_{SB}(s)$, we need a preliminary lemma:

LEMMA 5.1. *Let $A_g = (-d^2/dx^2) + g\,|x|^\gamma$. Let $F_g(s) = \operatorname{Tr}(\exp(-sA_g))$ and $N_g(E)$ the dimension of the spectral projection ofb A_g for the interval $[0, E]$. Then (A, N, F without subscript mean $g = 1$)*

(a) $F_g(s) = F(g^\tau s)$, *where* $\tau = 2/(\gamma+2)$,

(b) $N_g(E) = N(g^{-\tau}E)$,

(c) $\lim_{s\downarrow 0} s^{+\mu} F(s) = \pi^{-1/2}\Gamma(1+\gamma^{-1})$, *where* $\mu = (\gamma+2)/2\gamma$,

(d) $\lim_{E\to\infty} E^{-\mu} N(E) = \pi^{-1/2}\Gamma(1+\gamma^{-1})/\Gamma(\mu+1)$,

(e) $\lim_{s\downarrow 0} s^{+(\mu+1)} F'(s) = -\mu\pi^{-1/2}\Gamma(1+\gamma^{-1})$.

Proof. Parts (a) and (b) follow just by scaling as we used above. For (c), one notes that it can be proven that $F(s)/Z_{\text{cl}}(s) \to 1$ as $s \downarrow 0$ (see, e.g., [16]) and

$$Z_{\text{cl}}(s) = (2\pi)^{-1} \int dx\, dp\, e^{-s(p^2+|x|^\gamma)} = s^{-\mu}\pi^{-1/2}\Gamma(1+\gamma^{-1}).$$

Part (d) then follows from the Tauberian theorem (Theorem 1.1). Part (e) which just says the asmptotics in (c) can be formally differentiated follows from (d) if we note that

$$\begin{aligned} -F'(s) &= \int_0^\infty xe^{-xs}\, dN(x) \\ &= \int_0^\infty e^{-xs}(xs-1)\, N(x)\, dx \\ &= s^{-1}\int_0^\infty e^{-y}(y-1)\, N\left(\frac{y}{s}\right) dy. \quad \blacksquare \end{aligned}$$

Below, we will always use scaling to rewrite things for $g = 1$ without comment but distinct values of γ will enter, so we will write $N^{(\gamma)}$, etc. Let $\varepsilon_j(x)$ be the jth eigenvalue of $(-d^2/dy^2) + |xy|^\alpha$ so $\varepsilon_j(x) = |x|^{1/\nu}\, \varepsilon_j(1) \equiv |x|^{1/\nu}\, \varepsilon_j$ with $\nu = \frac{1}{2} + \alpha^{-1}$ as in the statement of Theorem 1.2. Thus

$$\begin{aligned} Z_{\text{SB}}(s) &= \sum_j \operatorname{Tr}\left(\exp\left[-s\left(\frac{-d^2}{dx^2} + \varepsilon_j(x)\right)\right]\right) \\ &= \sum_j \operatorname{Tr}\left(\exp\left(-s\left[\frac{-d^2}{dx^2} + \varepsilon_j|x|^{1/\nu}\right]\right)\right) \\ &= \sum_j \operatorname{Tr}\left(\exp\left(-s\varepsilon_j^b\left[\frac{-d^2}{dx^2} + |x|^{1/\nu}\right]\right)\right), \end{aligned}$$

where we have used the scaling relation so

$$b = \frac{2}{2+\nu^{-1}} = \frac{\alpha+2}{2\alpha+2} = \frac{\nu}{\nu+(1/2)}$$

Thus

$$\begin{aligned} Z_{\text{SB}}(s) &= \sum_j F^{1/\nu)}(s\varepsilon_j^b) \\ &= \int F^{(1/\nu)}(sE^b)\, dN^{(\alpha)}(E) \\ &= -\int_0^\infty sbE^{b-1}F^{(1/\nu)\prime}(sE^b)\, N^{(\alpha)}(E)\, dE, \end{aligned} \tag{5.3}$$

where we integrate by parts using $N=0$ for E small and $FN \to 0$ at ∞ for s fixed to see no boundary terms are present. We will now find the small s asymptotic behavior of $Z_{\mathrm{SB}}(s)$ from the integral representation (5.3) and the known small s behavior of $F^{(1/\nu)\prime}(s)$ and large E behavior of $N^{\alpha}(E)$.

In (5.3), we begin by noting that since $N(E)=0$ for E small, the integral goes from $E_0>0$ to ∞. We will pick $E_1<E_2$ s-dependent and separarately analyze the integral one the intervals (E_0, E_1), (E_1, E_2), (E_2, ∞). We will take E and E_2 so that

$$E_1^b s = |\ln s|^{-1}; \qquad E_2^b s = 1.$$

We will show that on the level of $s^{-1-\alpha^{-1}}(|\ln s|)^{-1}$ the integrals on (E_1, E_2) and (E_2, ∞) contribute zero and the (E_0, E_1) contributes exactly an amount on this scale identical to the lower bound we have.

On (E_2, ∞), we note that $N^{(\alpha)}(E) \leqslant cE^{\nu}$ for *all* E because of the asymptotic result, Lemma 5.1(d). Moreover, in the region $y \geqslant 1$, $-F'(y) \equiv \sum \tilde{\varepsilon}_j e^{-y\tilde{\varepsilon}_j} \leqslant De^{-cy}$, where $\tilde{\varepsilon}_j$ are the eigenvalues of $(-d^2/dy^2) + |y|^{1/\nu}$. Thus (where c is a constant which changes from equation to equation)

$$-\int_{E_2}^{\infty} (sb)\, E^{b-1} N^{(\alpha)}(E)\, F^{(1/\nu)\prime}(sE^b)\, dE \leqslant c\int_{E_2}^{\infty} (s)\, E^{b-1+\nu} \exp(-csE^b)\, dE$$

$$= css^{-(b+\nu)/b} \int_1^{\infty} y^{\nu/b} e^{-cy}\, dy$$

is bounded by $s^{-\nu/b} = s^{-(\nu+1/2)}$ since $b=\nu/(\nu+\frac{1}{2})$. As a result on the $s^{-(\alpha^{-1}+1)}|\ln s|$ level (recall $\alpha^{-1}+1=\nu+\frac{1}{2}$) this integral does not count.

On (E_1, E_2), we bound $N^{\alpha}(E)$ as above and $-F'(y)$ by $Cy^{-\mu-1}$, where $\mu=(\nu^{-1}+2)/2\nu^{-1}=\nu+\frac{1}{2}$. Then

$$-\int_{E_1}^{E_2} (sb)\, E^{b-1} N^{(\alpha)}(E)\, F^{(1/\nu)\prime}(sE^b)\, dE$$

$$\leqslant c\int_{E_1}^{E_2} sE^{b-1+\nu}(sE^b)^{-\nu-3/2}\, dE$$

$$= \leqslant cs^{-(\nu+1/2)} \int_{E_1}^{E_2} E^{-1}\, dE = cs^{-(\nu+1/2)} \ln(E_2/E_1),$$

where we have used $b-1+\nu-b(\nu+\frac{3}{2}) = -1+\nu-b(\nu+\frac{1}{2}) = -1$ since $b(\nu+\frac{1}{2})=\nu$. Since $\ln(E_2/E_1) = b^{-1}\ln(|\ln s|)$ this integral is $O(s^{-1-\alpha^{-1}}\ln_2(s^{-1}))$ which is small on the $s^{-1-\alpha^{-1}}\ln(s^{-1})$ level.

Finally, for the integral from E_0 to E_1, we first claim that since the arguments of F' (namely, sE^b) are bounded above by $|\ln(s)|^{-1}$, we can replace F' by its asymptotic value making a multiplicative error of the form $1+o(1)$, i.e., we can bound F' above and below by $(1 \pm \varepsilon(s))$ [Asym. form] with $\varepsilon(s) \downarrow 0$. Thus, if $\sim$ means the ratio goes to 1, we see that

$$A \equiv -\int_{E_0}^{E_1} (sb)\, E^{b-1} N^{(\alpha)}(E)\, F^{(1/\nu)\prime}(sE^b)\, dE$$

$$\sim \int_{E_0}^{E_1} (sb)\, E^{-1} s^{-\nu-3/2} [(\nu+1/2)\, \pi^{-1/2}]\, \Gamma(1+\nu)\, a \left[\frac{N(E)}{aE^{\nu}} \right] dE,$$

where $a = \pi^{-1/2}\Gamma(1+\alpha^{-1})/\Gamma(\nu+1$, so

$$A \sim \pi^{-1}\nu\Gamma(1+\alpha^{-1})\, s^{-1-\alpha^{-1}} \int_{E_0}^{E_1} E^{-1} \left[\frac{N(E)}{aE^{\nu}} \right] dE$$

$$\sim \pi^{-1}\nu\Gamma(1+\alpha^{-1})\, s^{-1-\alpha^{-1}} \ln(E_1/E_0)$$

since $N(E)/aE^{\nu} \to 1$ at ∞. But

$$\ln(E_1/E_0) = \ln[cs^{-1/b}\, |\ln s|^{-1/b}] \sim \frac{1}{b} \ln(s^{-1})$$

so multiplied by $s^{1+\alpha^{-1}} |\ln s|^{-1}$, the integral goes to

$$\pi^{-1}\nu b^{-1}\Gamma(1+\alpha^{-1}) = \pi^{-1}(\nu+\tfrac{1}{2})\, \Gamma(1+\alpha^{-1})$$
$$= \pi^{-1}(1+\alpha^{-1})\, \Gamma(1+\alpha^{-1}) = \pi^{-1}\Gamma(2+\alpha^{-1}).$$

Thus, our upper and lower bounds are the same and the theorems are proven!

Acknowledgments

It is a pleasure to thank B. Siu for doing some instructive calculations, M. Aizenman for the critical remark that there was a right way and a wrong way to slice bread (in the context of Theorem 1.3 and Eq. (1.2)), J. Avron, H. Capel, C. Fefferman, and E. Lieb for valuable discussions or correspondence, and P. Deift for a careful reading of the manuscript.

Note added in proof. D. Robert, in Comportement asymptotique des valeurs propres d'opérateurs du type Schrodinger a potentiel "dégénéré", *J. Math. Pures Appl.* **61** (1982), 275–300, has obtained the asymptotics of the eigenvalues of a class of operators closely related to the ones studied in these papers. Robert's work, which precedes ours by roughly two years, uses rather different methods.

References

1. H. Capel and P. Tindemans, An inequality for the trace of matrix products, *Rep. Math. Phys.* **6** (1974), 225–235.
2. J. M. Combes, P. Duclos, and R. Seiler, The Born–Oppenheimer approximation, *in* "Rigorous Atomic and Molecular Physics" (G. Velo and A. Wightman, Eds.), Plenum, New York, 1981.
3. N. G De Bruyn, Inequaliteis concerning minors and eigenvalues, *Nieuw Arch. Wisk.* **4** (1956), 18–35.
4. K. Fan, Maximum properties and inequalities for the eigenvalues of completely continuous operators, *Proc. Nat. Acad. Sci.* **37** (1951), 760–761.
5. C. Fefferman and D. Phong, The uncertainty principle and sharp Garding inequalities, *Comm. Pure Appl. Math.* **34** (1981), 285–331.
6. S. Golden, Lower bounds for Helmholtz functions, *Phys. Rev.* **137B**(1965), 1127–1128.
7. H. Hess, R. Schrader, and D. Uhlenbrock, Domination of semigroups and generalization of Kato's inequality, *Duke Math. J.* **44** (1977), 893–904.
8. M. Kac, Can you hear the shape of a drum, *Amer. Math. Monthly* **73** (1966), 1–23.
9. M. Marcus and B. Moyls, On the maximum principle of Ky Fan, *Canad. J. Math.* **9** (1957), 313–320.
10. L. Mirsky, Maximum principles in matrix theory, *Proc. Glasgow Math. Assoc.* **4** (1954), 34–37.
11. B. Simon, Universal diamagnetism of spinless Bose systems, *Phys. Rev. Lett.* **36** (1976), 1083–1084.
12. B. Simon, An abstract Kato's inequality for generators of positivity preserving semigroups, *Indag. Math. J.* **26** (1977), 1067–1073.
13. B. Simon, Maximal and minimal Schrodinger forms, *J. Operator Theory* **1** (1979), 37–47.
14. B. Simon, Schrodinger semigroups, *Bull. Amer. Math. Soc.* **7** (1982), 447–526.
15. B. Simon, Some quantum operators with discrete spectrum but classically continuous spectrum, *Ann. Phyics* **146** (1983), 209–220.
16. B. Simon, "Functional Integration and Quantum Physics," Academic Press, New York, 1979
17. B. Simon, "Trace Ideals and Their Applications," Cambridge Univ. Press, London, 1979.
18. C. Thompson, Inequality with applications in statistical mechanics, *J. Math. Phys.* **6** (1965), 1812–1813.
19. P. Tindemans and H. Capel, On the free energy in systems with separable interactions, *Physica* **79A** (1975), 478–502.
20. H. Weyl, The asympotic behavior of eigenvalues of linear partial differential equations, *Math Ann.* **71** (1911), 441–469.

Printed in Belgium