

Equality of the density of states in a wide class of tight-binding Lorentzian random models

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(Received 5 November 1982)

We prove directly the equality of the density of states in a wide class of tight-binding Lorentzian random models, including the Lloyd model, the  $\tan(2\pi\alpha n + \theta)$  model of Grepel, Fishman, and Prange, and a model with potential  $\sum_i \psi_i \tan(2\pi\alpha_i n)$ , where  $\sum_i \psi_i = 1$  and the  $\alpha_i$  are rationally independent.

In a recent paper,<sup>1</sup> Grepel *et al.* found exact solutions for the class of tight-binding Schrödinger operators  $[(H_0 u)(n) = u(n+1) + u(n-1)]$ .

$$(Hu)(n) = (H_0 u)(n) + \lambda \tan(2\pi\alpha n + \theta)u(n) \quad (1)$$

at least when  $\alpha$  is an irrational not well approximated by rationals. They noted that the density of states for the model was identical to that for the Lloyd model,<sup>2</sup> that is, a model of the form

$$H = H_0 + \lambda V \quad (2)$$

where  $V(n)$  are independent identically distributed random variables with distribution

$$\frac{1}{\pi} (1+x^2)^{-1} dx \quad (3)$$

Our goal in this Brief Report is to give a new proof of this equality which, at the same time, shows that other objects such as the off-diagonal-averaged Green's function and the finite-volume-averaged density of states are equal. More interestingly, we will show a large number of other models having the same density of states, e.g., (2) with  $V(n) = \sum_{i=1}^k \psi_i \tan(2\pi\alpha_i n)$ , where  $\sum_{i=1}^k \psi_i = \lambda$  and the  $\alpha_i$  are rationally independent, or the model of Ref. 3 where  $V(n) = \lambda \tan(\alpha n^2 + \theta)$  so long as  $\theta$  is averaged over. Our results are not restricted to one dimension.

The key indications of the general phenomena found here are twofold: (a) The probability distribution of  $x = \tan\theta$  if  $\theta$  is uniformly distributed on  $[0, \pi)$  is given by Eq. (3), suggesting the equality in Ref. 1 is not coincidental. (b) The solution of the Lloyd model can be expressed by saying that its density of states is the same as that in a third model, namely, the average of the density of states over the ensemble of operators where  $V$  is a constant  $\lambda c$ , with  $c$  a random variable with distribution given by Eq. (3).<sup>4</sup> This striking fact about the original Lloyd model, that constant Lorentzian and completely independent Lorentzian have the same density of

states, suggests a special property of Lorentzians is at work. This is seen in the following:

*Lemma:* Fix arbitrary reals  $\alpha_1, \dots, \alpha_k$  and positive numbers  $\psi_1, \dots, \psi_k$  with  $\sum_{j=1}^k \psi_j = 1$ . Let

$$x(\theta) = \sum_{j=1}^k \psi_j \tan(\alpha_j + \theta) \quad .$$

Then

$$\frac{1}{2\pi} \int_0^{2\pi} e^{itx(\theta)} d\theta = e^{-|t|} \quad .$$

*Proof:* By changing the sign of  $\alpha_i$  and  $\theta$ , we can suppose  $t > 0$ . Since  $\text{Im} \tan(z) > 0$  if  $\text{Im} z > 0$ , we see that<sup>5</sup>

$$\lim_{\epsilon \downarrow 0} \frac{1}{2\pi} \int_0^{2\pi} e^{itx(\theta+i\epsilon)} d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{itx(\theta)} d\theta \quad .$$

In the integral, we change variables to  $z = e^{i\theta}$  and observe that  $x(\theta+i\epsilon)$  has no singularities in  $|z| \leq 1$ , so the entire integral comes from the pole at  $z=0$  due to the change of variables  $d\theta = dz/(iz)^{-1}$ . At  $z=0$ ,  $x$  is  $i$ , which proves the result.

The  $\alpha$ 's (and for that matter the  $\psi$ 's, so long as  $\sum_{j=1}^k \psi_j = 1$ ) can be random variables and the expectation value of  $e^{itx}$  is still  $e^{-|t|}$  so long as  $\theta$  is uniformly distributed. Therefore, for large classes of random  $V$ 's, including the Lloyd model,<sup>2</sup> the Maryland model,<sup>1</sup> the above random-constant model, and the  $\sum_i \psi_i \tan(2\pi\alpha_i n + \theta_i)$  model discussed above,<sup>6</sup> one has it that any positive<sup>7</sup> combination of the potential at different sites,  $\sum_k a_k V(k)$ , has a Lorentz distribution with half-width  $\sum_k a_k$ .

Next, suppose that  $M_0$  is a finite matrix and  $V(n)$  is a random diagonal matrix and we want to evaluate

$$\text{Exp}[e^{it(M_0+V)}(n,m)] = P_{nm}(t) \quad ,$$

where Exp is the expectation value over the ensemble of  $V$ 's. Since  $M_0$  and  $V$  are finite matrices, we

can expand (and obtain a convergent series)

$$e^{it(M_0+V)} = e^{itV} + i \int_0^t e^{isV} M_0 e^{i(t-s)V} + \dots$$

Taking expectations, we see that  $P_{nm}(t)$  is a sum (over  $j$  and intermediate sites  $n_1, \dots, n_j$ ) and an integral (over  $s$  variables) of

$$\text{Exp}(e^{is_1 V(n_1)} \dots e^{is_j V(n_j)})$$

with all  $s_j > 0$  and  $\sum_i s_i = t$ . Thus  $P_{nm}(t)$  only depends on the distribution of convex combinations of  $V$ .<sup>8</sup> In particular, if we now specialize to a situation where for  $s_j \geq 0$ ,  $\sum_j s_j = t$ ,

$$\text{Exp} \left[ \exp \left( i \sum_{k=1}^j s_k V(k) \right) \right] = e^{-\lambda|t|}, \quad (4)$$

and we see that

$$P_{nm}(t) = e^{-\lambda|t|} P_{nm}^{(0)}(t), \quad (5)$$

where  $P^{(0)}$  is the value of  $P$  when  $V \equiv 0$ , i.e.,  $e^{itM_0}(n, m)$ .

To summarize, if  $V$  obeys (4), then (5) will hold, and by the lemma, (4) holds for many distinct choices of  $V$ , in particular, both for the Lloyd model and for the Maryland model, Eq. (1) averaged over  $\theta$ .

Once (5) holds in finite volume, it remains true for infinite volume (if  $P^{(0)}$  has a limit as it does if  $M_0 = H_0$ ).  $P_{00}(t)$  is just the Fourier transform of the averaged<sup>9</sup> density of states, so the density of states only depends on  $\lambda$  and is just the convolution of the density of states for  $M_0$  and the Lorentz distribution  $\pi^{-1}(x^2 + \lambda^2)^{-1}(\lambda dx)$ . The same argument works for the Fourier transform of  $P_{nm}(t)$ , which is the averaged Green's function which one can thereby see decays exponentially when  $\lambda > 0$  and  $M_0 = H_0$ . In one dimension, once the density of states only depends

on  $V$  through (4), the same is true of the Lyapunov exponent by the Thouless formula.<sup>10</sup>

We close with a series of remarks about further examples. (a) While the  $\alpha$  independence of the density of states in (1) was only proven in Ref. 1 for  $\alpha$ , which are not well approximated by rationals, our argument does not depend on Diophantine properties. If  $\alpha$  is a Liouville number, one can show that the spectrum is purely singular continuous,<sup>11</sup> so we have examples of dense point and singular continuous spectrum with the same density of states. This illustrates once again<sup>12</sup> that the density of states and the spectrum are often crude indications of the true physics. (b) The method works in any dimension: Thus the  $\nu$ -dimensional Lloyd model has the same density of states as  $\sum_i \psi_i \tan(2\pi \bar{\alpha}_i \cdot \bar{n})$ , where the  $\bar{\alpha}$  are rationally independent  $\nu$ -component vectors. (c)  $M_0$  need not be translation invariant. Thus, if  $V$  is any potential for which  $H_0 + V$  has a density of states  $d\tilde{k}$ , then the average  $dk$  of the density of states of  $H_0 + V + \epsilon \tan(\alpha n + \theta)$  over  $\theta$  is just the convolution of  $d\tilde{k}$  and  $\pi^{-1} \epsilon dx / [x^2 + \epsilon^2]$ . In particular,  $dk$  is supported everywhere and  $\gamma$ , the average over  $\theta$  of the Lyapunov exponent, is strictly positive for all  $E$ . In the case where the density of states is independent of  $\theta$ , we see the perturbed operator has spectrum  $(-\infty, \infty)$  with *no* absolutely continuous component.<sup>13</sup> In particular, for *any* almost periodic function  $V$ , if  $\alpha$  is rationally independent of the frequencies of  $V$  and  $\epsilon$  is *any* nonzero number,  $H_0 + V + \epsilon \tan(\alpha n)$  has these properties.

#### ACKNOWLEDGMENTS

It is a pleasure to thank J. Avron and T. Spencer for useful discussions. This research was supported in part by the National Science Foundation (U.S.) under Grant No. MCS-81-20833.

<sup>1</sup>D. Grempel, S. Fishman, and R. Prange, Phys. Rev. Lett. **49**, 833 (1982).

<sup>2</sup>P. Lloyd, J. Phys. C **2**, 1717 (1969).

<sup>3</sup>S. Fishman, D. Grempel, and R. Prange, Phys. Rev. Lett. **49**, 509 (1982).

<sup>4</sup>That is, the density of states is just the convolution of the free density of states and the distribution of  $\lambda c$ .

<sup>5</sup>This is somewhat subtle, e.g., it is false if  $t < 0$  (where we have to replace  $\epsilon \downarrow 0$  by  $\epsilon \uparrow 0$ ).

<sup>6</sup>Since the  $\alpha_i$  are rationally independent, limits of translates of  $V$  are precisely the class of potentials with  $\theta_i$  arbitrary and independently distributed.

<sup>7</sup>Of course, other combinations have a distribution which is dependent on which  $V$  is chosen. Interesting physics like localization which is model dependent must then depend on more than such positive combinations.

<sup>8</sup>This realization and the argument in this paragraph were arrived at in discussions with Tom Spencer.

<sup>9</sup>In many cases, e.g., rationally independent  $\alpha$ , one can show

that the density of states is equal for almost all potentials in the class, so that averaging is not necessary.

<sup>10</sup>D. Herbert and R. Jones, J. Phys. C **4**, 1145 (1971); D. Thouless, *ibid.* **5**, 77 (1972).

<sup>11</sup>One does this by using the analysis of J. Avron and B. Simon, Bull. Am. Math. Soc. **6**, 81 (1982). One can show that for almost all  $\theta$ , the argument of A. Gordon, Usp. Mat. Nauk **31**, 257 (1976), applies to show there can be no normalizable eigenfunctions if  $\alpha$  is a Liouville number.

<sup>12</sup>Aubry duality for the density of states [see G. Andre and S. Aubry, Ann. Israel Phys. Soc. **3**, 133 (1980)] shows that in the  $\cos(2\pi\alpha n)$  model the density of states are up to a scale factor the same in the extended and localized region.

<sup>13</sup>That a strictly positive Lyapunov exponent implies that there is no absolutely continuous spectrum is a result of L. Pastur, Commun. Math. Phys. **75**, 179 (1980).