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Schrödinger Operators with an Electric Field and Random or Deterministic Potentials

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Abstract. We prove that the Schrödinger operator $H=-\frac{d^2}{dx^2}+V(x)+F\cdot x$ has purely absolutely continuous spectrum for arbitrary constant external field F, for a large class of potentials; this result applies to many periodic, almost periodic and random potentials and in particular to random wells of independent depth for which we prove that when $F=0$, the spectrum is almost surely pure point with exponentially decaying eigenfunctions.

I. Introduction

This paper presents exact results on the behaviour of electrons in the presence of an electric field. We discuss below the physical aspects of the problem and of our results and then we present the mathematical aspects and the organisation of our paper.

The Physics of the Problem and of the Results

The problem of an electron in a random potential has been receiving a great deal of attention for quite a while, both from the physical and the mathematical point of view. The case of almost periodic potentials has also recently attracted a lot of workers in the field. A challenging question is the following: what is the behaviour of such systems when a constant electric field is turned on? The more the states are localized for zero field the more interesting is the problem: the most extreme case deals with one-dimensional systems for which an arbitrarily small degree of disorder implies the exponential localization of all states in the absence of electric field. Mathematically this corresponds to the fact that the associate Schrödinger operator $-\frac{d^2}{dx^2}+V(x)$ has almost surely only pure point spectrum with exponen-

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tially decaying eigenfunctions. When the electric potential $F \cdot x$ is added our results tell that the spectrum become absolutely continuous, that is all states become extended! Although it could seem of common physical wisdom (after all, the potential $F \cdot x$ goes to $-\infty$ when x goes to $-\infty$). This result is not at all trivial as can be seen from the finite difference analogue for which the spectrum, as explained below is always pure point!

It turned out that our proof is very general and applies to a large class of potentials including periodic, almost periodic and random potentials. In fact it appears as a deterministic result. We want to mention the previous work of Herbst and Howland [7] in which they proved for a class of random potentials that for $F \neq 0$, almost surely certain matrix elements of the resolvent of $H(w)$ possess meromorphic continuations to a strip below the energy axis. However they use translation analytic techniques which force the potential to be analytic; it also makes unknown in their situation whether or not the spectrum is pure point for $F=0$. In our work we do not have such restrictions because we use the powerful method due to Mourre [11].

We prove that for $F \neq 0$, the spectrum is absolutely continuous; this implies that all the states are extended, i.e. they are not square integrable. Nevertheless it is easy to check that they do decay exponentially fast in one direction.

Let us mention that the analogue finite difference operator presents a completely different situation. The operator discrete laplacian plus discrete electric field has a compact resolvent; then adding a bounded potential gives an operator with always pure point and even discrete spectrum!

The Mathematical Aspects and the Organisation of the Paper

In Sect. II we prove a deterministic result (Theorem 4), ensuring for a large class of potentials that the spectrum of the Schrödinger operator $-\frac{d^2}{dx^2} + V(x) + F \cdot x$ is purely absolutely continuous; it applies to many periodic, almost periodic and random potentials including random wells of independent depth. This result is crucially based on the powerful Mourre's theorem [11] and on an ODE trick to remove the eigenvalues. Section II is hence purely analytic and independent of Sect. III.

In Sect. III, we provide the reader with a class of random potentials to which the results of Sect. II apply a.s. and for which we prove (Theorem 6) the expected localization result for one dimensional disordered systems in the case $F=0$. The proof follows the line of [3]. It is not shorter but it is definitely more elementary in the sense that the hypoellipticity assumption which was crucial in [6, 10, 3] is advantageously replaced by a simple assumption on the potential: it is assumed to have a continuous density with bounded support. The fact that this assumption is the one that had to be made in the lattice case (see [9]) should shed some light on the very nature of the probabilistic aspects of the localization problem and should help to convince workers in this area that there is so far some unity in the existing lattice case as continuous case proofs which are usually regarded as technically completely different.

II. The Analytic Result

The following lemma is designed to prevent the technicalities of various approximation arguments from obscuring the proof of our Mourre type estimate (see Proposition 3 below) which is the essential ingredient in the proof of the main result of this section (see Theorem 4 below).

Lemma 1. *Let W be a bounded uniformly continuous function which satisfies* (#) $\lim_{r\to+\infty}\left[\operatorname{Sup}_{x\in\mathbb{R}}\left|\frac{1}{2r}\int_{x-r}^{x+r} W(y)dy\right|\right]=0$. *Then, for each $\varepsilon>0$, $\alpha\in\mathbb{R}$ and $j\in\mathbb{N}$, one can find a function $\bar{W}$ which satisfies* (#) *and a sequence $\{W_n; n\in\mathbb{Z}\}$ in $C_0^\infty(\mathbb{R})$ such that:*

i) $\|W-\bar{W}\|_\infty<\varepsilon$,

ii) $\forall n\in\mathbb{Z}$, $\operatorname{supp}\bar{W}_n\subset[n-a,n+a]$ *for some* $a>0$ *and* $\bar{W}=\sum_{n\in\mathbb{Z}}\bar{W}_n$,

iii) $\sup_{t\in\mathbb{R},\, n\in\mathbb{Z}} |t|^\alpha\left|\frac{d^j}{dt^j}(e^{-itn}\hat{\bar{W}}_n(t))\right|<+\infty$,

where ^ denotes the Fourier transform.

Remark 2. It is easy to check that if W satisfies (#) and μ is a bounded (signed) measure on $\mathbb{R}$, then the convolution $W*\mu$ satisfies also (#). This fact will be repeatedly used in the proof below. Moreover it is obvious that (#) holds whenever W is the derivative of a bounded function.

Proof. For each $n\in\mathbb{Z}$ we set $W_n=W1_{[n-1/2,n+1/2]}$, where 1_A denotes the characteristic function of the set A. Let $\varrho\in C_0^\infty(\mathbb{R})$ be nonnegative, supported in $[-1,1]$ and normalized to have integral 1. Then we define the approximate identity $(\varrho^{\varepsilon'})_{\varepsilon'>0}$ by:

$$\varrho^{\varepsilon'}(x)=\varepsilon'^{-1}\varrho(\varepsilon^{-1}x),\qquad \varepsilon>0, x\in\mathbb{R}.$$

Now, if we set $W^{\varepsilon'}=W*\varrho^{\varepsilon'}$ and $W_n^{\varepsilon'}=W_n*\varrho^{\varepsilon'}$ we know that $W^{\varepsilon'}$ satisfies (#) for each $\varepsilon'>0$ (recall Remark 2) and, since W is bounded and uniformly continuous, we can pick $\varepsilon'>0$ such that:

$$\|W-W^{\varepsilon'}\|_\infty<\varepsilon/2. \tag{2.1}$$

Let $\ell\geqq 0$ be an integer to be chosen later on. The next step is to approximate the $W_n^{\varepsilon'}$ by functions whose Fourier transforms are $O(|t|^\ell)$ at the origin uniformly in n. For each $r>0$ we define $\chi_r=\frac{1}{2r}1_{[-r,+r]}$, and we remark that W satisfies (#) means $\lim_{r\to+\infty}\|W*\chi_r\|_\infty=0$.

Then we set:

$$W^{(\varepsilon',r)}=W^{\varepsilon'}*\underbrace{(\delta-\chi_r)*\ldots*(\delta-\chi_r)}_{\ell\text{-times}}, \tag{2.2}$$

where δ denotes Dirac's measure at the origin and $f*(\delta-h)$ stands for $f-f*h$. We define similarly $\bar{W}_n^{(\varepsilon',r)}$ for each $n\in\mathbb{Z}$. In fact

$$\bar{W}_n^{(\varepsilon',r)}=\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}W_n^{\varepsilon'}*\chi_r^{*k},$$

where h^{*k} stands for $h*\ldots*h$ k-times. Now:

$$\begin{aligned}\|W^{\varepsilon'}-\bar{W}^{(\varepsilon',r)}\|_\infty &= \left\|\sum_{k=1}^{\ell}(-1)^k\binom{\ell}{k}W^{\varepsilon'}*\chi_r^{*k}\right\|_\infty \\ &\leqq \sum_{k=1}^{\ell}\binom{\ell}{k}\|(W^{\varepsilon'}*\chi_r)*\chi_r^{*(k-1)}\|_\infty \\ &\leqq \sum_{k=1}^{\ell}\binom{\ell}{k}\|W^{\varepsilon'}*\chi_r\|_\infty,\end{aligned}$$

which goes to zero when r goes to infinity. Hence we can fix $r>0$ large enough in order to have:

$$\|W^{\varepsilon'}-\bar{W}^{(\varepsilon',r)}\|_\infty<\varepsilon/2. \tag{2.3}$$

Once $\varepsilon'>0$ and $r>0$ are chosen we set $\bar{W}=W^{(\varepsilon',r)}$ and $\bar{W}_n=\bar{W}_n^{(\varepsilon',r)}$ to drop the dependence on ε' and r from the notations. Obviously, $\bar{W}_n\in C_0^\infty(\mathbb{R})$ for each n, $\bar{W}$ satisfies (#) [recall (2.2) and Remark 2], i) is a consequence of (2.1) and (2.3), and ii) is a consequence of our construction. Thus we concentrate on the proof of iii). Since by (2.2) we have:

$$\hat{\bar{W}}_n(t)=\hat{W}_n^{\varepsilon'}(t)\,[1-\hat{\chi}_r(t)]^\ell,$$

we need only check that:

$$|t|^\alpha\left|\frac{d^j}{dt^j}(e^{-itn}\hat{W}_n^{\varepsilon'}(t)\,[1-\hat{\chi}_r(t)]^\ell)\right|$$

is uniformly bounded in $t\in\mathbb{R}$ and $n\in\mathbb{Z}$. This is clear once we remark that first, for $|t|$ large

$$\begin{aligned}|t|^\alpha\left|\frac{d^k}{dt^k}(e^{-itn}\hat{W}_n^{\varepsilon'}(t))\right| &= \left|\sum_{k_1=0}^{k}\binom{k}{k_1}\frac{d^{k_1}}{dt^{k_1}}\widehat{W_n(\cdot-n)}(t)\,|t|^\alpha\frac{d^{k-k_1}}{dt^{k-k_1}}\widehat{\varrho^{\varepsilon'}}(t)\right| \\ &\leqq \text{const}\sum_{k_1=0}^{k}\binom{k}{k_1}\left\|\frac{d^{k_1}}{dt^{k_1}}\widehat{W_n(\cdot-n)}\right\|_1,\end{aligned}$$

(where $\|f\|_p$ stands for the L^p-norm of f) which is uniformly bounded in $n\in\mathbb{Z}$, and second, for $|t|$ small, expressions of the form

$$\left|\frac{d^k}{dt^k}(e^{-itn}\widehat{W_n^{\varepsilon'}}(t))\,|t|^\alpha\frac{d^{j-k}}{dt^{j-k}}[1-\hat{\chi}_r(t)]^\ell\right|$$

are bounded above by

$$\|\,|x|^kW_n^{\varepsilon'}(x-n)\|_1\left\||t|^\alpha\frac{d^{j-k}}{dt^{j-k}}[1-\hat{\chi}_r(t)]^\ell\right\|_\infty,$$

the second factor being finite provided ℓ is large enough, the first one being uniformly bounded in $n\in\mathbb{Z}$ by construction. □

From now on for each real F, H_F will denote the unique self adjoint extension of the symmetric operator $-\dfrac{d^2}{dx^2}+Fx$ defined on the space $C_0^\infty(\mathbb{R})$ (see for example the unitary equivalence used in the proof below in the case $F \neq 0$). Moreover for any self adjoint operator A, we will use the notation $E_\Delta(A)$ for the corresponding spectral projection on the Borel subset Δ of $\mathbb{R}$.

The following result will play a crucial role in checking the assumptions of Mourre's theorem.

Proposition 3. *Let W be a bounded uniformly continuous function which satisfies:*

$$\lim_{r \to +\infty} \left[\sup_{x \in \mathbb{R}} \left| \frac{1}{2r} \int_{x-r}^{x+r} W(y)\, dy \right| \right] = 0. \tag{\#}$$

Then $E_\Delta(H_F) W E_\Delta(H_F)$ is compact for each bounded Borel subset Δ of $\mathbb{R}$.

Proof. First we note that, without any loss of generality, we can assume that W has all properties of the approximation function $\bar{W}$ given by Lemma 1. The Fourier transform maps H_F into a first order differential operator and the latter is unitarily equivalent to its principal part (see [12, p. 425]). Combining these two facts in the present situation we obtain that H_F is unitarily equivalent to the multiplication operator by Fx on $L^2(\mathbb{R})$ via the formula $U_F^{-1} H_F U_F = Fx$, where U_F is the unitary transformation

$$[U_F \varphi](x) = (2\pi)^{1/2} F^{1/3} \int_{\mathbb{R}} Ai(F^{1/3}(x-y))\, \varphi(y)\, dy,$$

where Ai denotes the Airy function (see [1, p. 447]). This fact is well known (see for example [2]). Hence it is sufficient to prove that the operator $\chi_\Delta(Fx)\, U_F^{-1} W U_F \chi_\Delta(Fx)$ is Hilbert-Schmidt and the latter will be done by proving that the operator $U_F^{-1} W U_F$ has a locally bounded kernel. In fact we will prove that, for each bounded interval $\Delta \subset \mathbb{R}$ we have:

$$\sup_{x, y \in \Delta} |K_n(x, y)| = O(|n|^{-2}), \tag{2.4}$$

where $K_n(x, y)$ denotes the kernel of the operator $K_n = U_F^{-1} W_n U_F$. First we remark that:

$$K_n = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \hat{W}_n(t)\, [U_F^{-1} e^{itx} U_F]\, dt.$$

In our notations we have:

$$U_F^{-1} x\, U_F = -\frac{1}{F} H_{-F},$$

from which it is easy to deduce the formula (already used in [2]):

$$U_F^{-1} e^{itx} U_F = e^{-itH_{-F}/F} = e^{-it^3/3F} e^{-iAt^2/F} e^{iA^2 t/F} e^{itx},$$

where A stands for the operator $i\dfrac{d}{dx}$. Using it we obtain

$$K_n(x, y) = (8\pi^2)^{-1/2} \int_{-\infty}^{+\infty} dt\, t^{-1/2} e^{-itn} \hat{W}_n(t) e^{i\sigma(t, x, y)},$$

where we set $\sigma(t,x,y)=(x-y+t^2/F)^2/4t-t^3/3F+ty$. Now the estimate (2.4) follows from property iii) of Lemma 1 and some integrations by parts. □

Now we state and prove the main analytical result of the paper:

Theorem 4. *Let V be a bounded real-valued function whose first derivative is bounded, uniformly continuous and absolutely continuous and such that V'' is essentially bounded.*

Then for each $F \neq 0$, the spectrum of the self-adjoint operator:

$$H=-\frac{d^2}{dx^2}+Fx+V(x) \tag{2.5}$$

is $\mathbb{R}$ *and purely absolutely continuous.*

Note that H is obtained from H_F by a symmetric bounded perturbation. Consequently H defined by (2.5) is self adjoint and $C_0^\infty(\mathbb{R})$ remains a core for H. Moreover it is "mere gardener work" to check that the spectrum of H is the whole real line $\mathbb{R}$ (say by constructing bounded generalized eigenfunctions).

Proof. The strategy of the proof is very simple: first we prove that Mourre's theorem [11] applies to the present situation to rule out the possibility of having a singular continuous component in the spectrum, and then we use classical O.D.E. techniques (see Lemma 5 below) to show the emptyness of the point component of the spectrum.

Let A be defined by $\mathscr{D}(A)=\{f\in L^2(\mathbb{R});\, f$ absolutely continuous and $f'\in L^2(\mathbb{R})\}$ and $Af=if'$ whenever $f\in\mathscr{D}(A)$. $\mathscr{D}(A)\cap\mathscr{D}(H)\supset C_0^\infty(\mathbb{R})$ which is a core for H. If $\alpha\in\mathbb{R}$, it is easy to check that:

$$e^{-i\alpha A}He^{i\alpha A}=H+\alpha F+[V(x+\alpha)-V(x)] \tag{2.6}$$

on $C_0^\infty(\mathbb{R})$. By the closedness of H and the boundedness of V', (2.6) extends to the whole domain $\mathscr{D}(H)$, and from this we conclude that $e^{i\alpha A}$ leaves $\mathscr{D}(H)$ invariant and

$$\sup_{|\alpha|<1}\|He^{i\alpha A}\varphi\|\leqq\|H\varphi\|+F+\|V'\|_\infty<+\infty$$

for each $\varphi\in\mathscr{D}(H)$.

Let $\mathscr{S}=\mathscr{S}(\mathbb{R})$ be the Schwartz space of rapidly decreasing functions. Obviously $\mathscr{S}\subset\mathscr{D}(A)\cap\mathscr{D}(H)$, $e^{i\alpha A}\mathscr{S}\subset\mathscr{S}$ for all $\alpha\in\mathbb{R}$, $\mathscr{S}$ is a core for H and a simple computation involving only integration by parts shows that

$$i[H,A]=F+V'$$

as quadratic forms on $\mathscr{S}$. Hence, the quadratic form $i[H,A]$ is closable and bounded below and the corresponding s.a. operator is simply the bounded operator of multiplication by $F+V'$. Its domain [i.e. $L^2(\mathbb{R})$] obviously contains $\mathscr{D}(H)$ so that we have:

$$\forall\varphi,\psi\in\mathscr{D}(A)\cap\mathscr{D}(H),\qquad(\varphi,i[H,A]\psi)=(\varphi,(F+V')\psi)$$

by Proposition II.1 of [11]. This proves that the quadratic form $i[H,A]$ on $\mathscr{D}(A)\cap\mathscr{D}(H)$ is closable and bounded below and that the corresponding self-adjoint operator, say $i[H,A]^0$, is the bounded operator of multiplication by $F+V'$.

If $\varphi, \psi \in \mathscr{D}(A) \cap \mathscr{D}(H)$, a simple computation shows that:

$$(\varphi, [i[H, A]^0, A]\psi) = (\varphi, (-iV'')\psi),$$

so that on $\mathscr{D}(A) \cap \mathscr{D}(H)$, the quadratic form $[i[H, A]^0, A]$ coincides with the operator of multiplication by $-iV''$, which is assumed to be bounded so that $(H+i)^{-1/2} V''(H+i)^{-1/2}$ is a bounded operator on $L^2(\mathbb{R})$.

If Δ is any open interval we have:

$$E_\Delta(H)\, i[H, A]^0\, E_\Delta(H) = F E_\Delta(H) + E_\Delta(H)\, V' E_\Delta(H),$$

and since $F \neq 0$, checking condition e) of Mourre's theorem reduces to proving that $E_\Delta(H)\, V' E_\Delta(H)$ is a compact operator on $L^2(\mathbb{R})$. The latter is equivalent to the compactness of $E_\Delta(H_F)\, V' E_\Delta(H_F)$ because V is bounded [and thus $\mathscr{D}(H) = \mathscr{D}(H_F)$] and V' is bounded, and we conclude by using Proposition 3 above with $W = V'$. Since all the assumptions of Mourre's theorem are satisfied we know that the spectrum of H has no singular continuous component and we are left with the study of possible eigenvalues. This problem is solved by using the following lemma which is stated without proof because the latter is that of Corollary 22, p. 1414, of [5] up to some minor modifications. □

Lemma 5. *Let V be a bounded real valued function whose first derivative is bounded and absolutely continuous in a neighborhood of $-\infty$ on which $|V''(x)| = O(|x|^\alpha)$ for some $0 \leqq \alpha < 1/2$.*

Then, if $F > 0$, no solution of $\left(-\frac{d^2}{dx^2} + Fx + V(x)\right)\psi = E\psi$ is square integrable near $-\infty$.

Remark. Theorem 4 can be extended easily in one direction: namely if $W(x) \equiv |\partial_1 V(x)|$ goes to zero at infinity in all directions of $\mathbb{R}^\nu$ and V is C^2 with $\|V\|_\infty$ and $\|\partial_1^2 V\|_\infty$ finite, for $F \neq 0$, $-\Delta + V(x) + F \cdot x$ has only a.c. spectrum with an additional possibility of isolated eigenvalues of finite multiplicities. The applicability of Mourre's theorem follows in that case from a result of Avron-Herbst [2] that $W(-\Delta + F \cdot x + i)^{-1}$ is compact.

III. The Random Case

As explained in the introduction, we would like to provide the reader with a simple example of a random potential $\{V(x, w); x \in \mathbb{R}, w \in \Omega\}$ defined on a probability space $(\Omega, a, \mathbb{P})$ for which $\mathbb{P}$-almost surely in $w \in \Omega$, the operator

$$H(w) = -\frac{d^2}{dx^2} + Fx + V(x, w)$$

has dense pure point spectrum when $F = 0$ and purely absolutely continuous spectrum when the electric field is turned on (i.e. $F > 0$). The class of random potentials we introduce below is such that for all $w \in \Omega$, the function $x \to V(x, w)$ satisfies the assumption of Theorem 4 above so that the case $F > 0$ will not be argued.

Theorem 6. *Let* $V(x,w)=\sum_{n\in\mathbb{Z}}\xi_n(w)\chi(x-n)$ *for* $w\in\Omega$ *and* $x\in\mathbb{R}$, *where:*

i) $\{\xi_n; n\in\mathbb{Z}\}$ *is a sequence of independent identically distributed random variables having a common density* φ *(i.e.* $\mathbb{P}\{\xi_n\in dy\}=\varphi(y)\,dy$*), which is continuous and with compact support.*

ii) χ *is a nonidentically zero nonpositive function with support in* $[0,1]$.

Then, for $\mathbb{P}$*-almost all* $w\in\Omega$, *the self-adjoint operator:*

$$H(w)=-\frac{d^2}{dx^2}+V(x,w)$$

on $L^2(\mathbb{R})$ *has pure point spectrum with eigenfunctions falling off exponentially according to the upper Liapunov exponent of the Cauchy problem corresponding to the eigenvalue equation.*

Proof. Our proof will follow the lines of [3] for two reasons: first it is the only approach in the continuous case that proved to be efficient in the present situation and second, we want the exact rate of exponential decay of the eigenfunctions. At this point we should pause and remark that [3] gives only an upper bound on the fall-off of the eigenfunctions but as argued in [4], simple properties of the upper Liapunov exponents imply the analogous lower bounds essentially for free.

We recall the strategy and the notations of the proof of [3]. For each $w\in\Omega$, the operator $H(w)$ is in the Weyl limit point case, and by restricting to bounded intervals $[-L,+L]$ and imposing boundary conditions at $-L$ and $+L$ we can construct pure point spectral measures σ_L^w which converge vaguely as $L\to\infty$ to a measure σ^w, which is measurable in w and "which contains all the spectral information on $H(w)$." As explained on pp. 196–198 of the pedagogical part of [3], it is sufficient (and almost necessary) to prove that for $\mathbb{P}$-almost all $w\in\Omega$ and σ^w-almost all $\lambda\in\mathbb{R}$ there exists a unit vector in $\mathbb{R}^2$, say $\Theta^{\pm}_{w,\lambda}$, such that the amplitude $r^{\pm}(x)=[y(x)^2+y'(x)^2]^{1/2}$ of the solution of the eigenvalue problem $-y''+[V(x,w)-\lambda]y=0$ with initial condition $\begin{bmatrix}y(0)\\ y'(0)\end{bmatrix}=\Theta^{\pm}_{w,\lambda}$ is "equivalent" as $x\to\pm\infty$ to $e^{-\alpha(\lambda,w)|x|}$ for some $\alpha(\lambda,w)>0$ (which will then be identified with the upper Liapunov exponent of the eigenvalue equation). This is implemented by proving that for each bounded interval Δ contained in the spectrum of $H(w)$, and for each $\varepsilon>0$, there is $\delta>0$ such that:

$$\int_\Omega \mathbb{P}(dw)\int_\Delta \sigma^w(d\lambda)\inf_{\|\Theta\|=1}\left(\int_0^{+\infty} r_\lambda(x)^\delta e^{\delta[\alpha(\lambda)-\varepsilon]x}\,dx\right)<+\infty\,, \tag{3.1}$$

where we should recall that $r_\lambda(x)$ depends also on w and the initial condition Θ, and where we restricted to the case $x\to+\infty$, the case $x\to-\infty$ being treated similarily. In order to prove (3.1) the strategy consists in (see Lemma 3.3 of [3]) first, restricting the integration in x to a finite interval $[0,L']$, second substituting an approximate spectral measure σ_L^w to σ^w with $L>L'$, and getting rid of the inf by picking a particular initial condition Θ for each λ which is charged by σ_L^w, then proving that the left hand side of (3.1) modified in this way is bounded above by a constant independent of L' and L, and finally letting $L\to\infty$ and then $L'\to\infty$ using Fatou's lemma to conclude (3.1). Consequently we are left with the proofs of

Lemmas 3.1 and 3.2 of [3] which correspond to a refined version of a well known theorem of Furstenberg and to the crucial estimate alluded above. These proofs are carried out in [3] under hypoellipticity assumptions taken from [6, 10]. We now show how our present assumptions on the random potential V make possible an argument avoiding these deep facts from the theory of degenerate elliptic partial differential equations (see [8]). As usual, we introduce the so-called phase, say $\theta(x)$, of the solution of the eigenvalue equation:

$$-y''+[V(x,w)-\lambda]y=0 \tag{3.2}$$

by setting:

$$\begin{cases} y(x)=r(x)\sin\theta(x) \\ y'(x)=r(x)\cos\theta(x), \end{cases}$$

and (3.2) gives:

$$\theta'(x)=\cos^2\theta(x)+[\lambda-V(x,w)]\sin^2\theta(x). \tag{3.3}$$

If we let $L\to\infty$ and $L'\to\infty$ through integer values, all is needed for the proofs of Lemmas 3.1 and 3.2 of [3] is the fact that $\theta_\lambda(n)$ as a random variable (which depends on λ) has a continuous density uniformly bounded in n and λ (which we restrict to a bounded energy interval Δ). In contrast with [6, 10, 3] we do not need to work with the joint process (potential, phase) to have a Markov process. Indeed, by the independence of the ξ_n's and the definition of V, $\{\theta_\lambda(n); n\in\mathbb{N}\}$ is a Markov chain by itself: "if we know $\theta_\lambda(n)$ and if we want to predict $\theta_\lambda(n+1)$, we need only to solve (2.3) in $[n,n+1]$ with initial condition at n given by $\theta(n)$. The result is random: it depends on the values of V in the interval $[n,n+1]$, but this depends only on ξ_n which is independent of the values of $V(x)$ for $x\leqq n$."

At this point of the proof, everything reduces to proving the existence of a function $n_\lambda(k,\theta,\theta')$ which is continuous in $(\lambda,\theta,\theta')\in\Delta\times S'\times S'$ for each integer k, and which is uniformly bounded in its four variables and such that:

$$\int_\Omega f(\theta_\lambda(k))\,d\mathbb{P}=\int_{S'} f(\theta')\,n_\lambda(k,\theta,\theta')\,d\theta', \tag{3.4}$$

where $d\theta'$ denotes the normalized measure of the projective space S' of $\mathbb{R}^2$, and where $\theta_\lambda(k)$ appearing in the left hand side of (3.4) stands for the solution $\theta_\lambda(x)$ of (3.3) at $x=k$ and initial condition at 0 given by $\theta_\lambda(0)=\theta$. Since $\{\theta_\lambda(n); n\in\mathbb{N}\}$ is a Markov chain, by Chapman-Kolmogorov equation we need only to study the one step transition and prove that $n_\lambda(\theta,\theta')=n_\lambda(1,\theta,\theta')$ is jointly continuous (and thus bounded) in λ, θ, and θ'.

For each real ξ let us denote by $\tilde{\theta}_\lambda(x,\xi,\theta)$ the solution of the equation

$$\tilde{\theta}'=\cos^2\tilde{\theta}+[\lambda-\xi\chi]\sin^2\tilde{\theta}, \qquad \tilde{\theta}(0)=\theta \tag{3.5}$$

evaluated at x. For $k=1$ the left hand side of (3.4) is equal to:

$$\int_\Omega f(\tilde{\theta}_\lambda(1,\xi_1(w),\theta))\,d\mathbb{P}(w)=\int_{\mathbb{R}} f(\tilde{\theta}_\lambda(1,\xi,\theta))\,\varphi(\xi)\,d\xi, \tag{3.6}$$

and we would like to perform the change of variable $\theta' = \tilde{\theta}_\lambda(1, \xi, \theta)$ in the right hand side of (3.6). From (3.5) it follows that $\frac{\partial \tilde{\theta}}{\partial \xi}$ satisfies:

$$\left[\frac{\partial \tilde{\theta}}{\partial \xi}\right]'(x) = [\lambda - \xi\chi(x) - 1] \sin 2\tilde{\theta} \frac{\partial \tilde{\theta}}{\partial \xi} - \chi(x) \sin^2 \tilde{\theta}$$

(where prime stands for the derivation with respect to the variable x) which can be integrated to give the following implicit formula:

$$\frac{\partial \tilde{\theta}_\lambda}{\partial \xi}(x, \xi, \theta) = -\int_0^x \chi(s) \sin^2 \tilde{\theta}_\lambda(s, \xi, \theta) e^{\int_s^x [\lambda - \xi\chi(u) - 1] \sin 2\tilde{\theta}_\lambda(u, \xi, \theta) du} ds \tag{3.7}$$

because $\frac{\partial \tilde{\theta}_\lambda}{\partial \xi}(0, \xi, \theta) = 0$, since $\tilde{\theta}_\lambda(0, \xi, \theta) = \theta$ independently of ξ. Since we are only interested in the case $x = 0$ and since λ and ξ run through bounded intervals and since χ is bounded, we have:

$$e^{\int_s^1 [\lambda - \xi\chi(u) - 1] \sin 2\tilde{\theta}_\lambda(u, \xi, \theta) du} \geqq C$$

for some constant $C > 0$ independent of ξ, λ and $s \in [0, 1]$. Consequently [recall (3.7) and $\chi \leqq 0$] one obtains:

$$\frac{\partial \tilde{\theta}_\lambda}{\partial \xi}(1, \xi, \theta) \geqq -C \int_0^1 \chi(s) \sin^2 \tilde{\theta}_\lambda(s, \xi, \theta) ds. \tag{3.8}$$

We claim that there exists a constant $C_0 > 0$ such that for all λ in Δ, ξ in the support of φ and θ in S' we have:

$$-\int_0^1 \chi(s) \sin^2 \tilde{\theta}_\lambda(s, \xi, \theta) ds \geqq C_0. \tag{3.9}$$

Let us first check that the proof can be completed modulo this claim.

For each λ and θ fixed, $\tilde{\theta}_\lambda(1, \xi, \theta)$ is a monotone, strictly increasing function of ξ and the above mentioned change of variables in the right hand side of (3.6) gives:

$$n_\lambda(\theta, \theta') = \varphi(\tilde{\theta}_{\lambda,\theta}^{-1}(\theta')) \left[\frac{\partial \tilde{\theta}_\lambda}{\partial \xi}(1, \tilde{\theta}_{\lambda,\theta}^{-1}(\theta'), \theta)\right]^{-1},$$

where $\tilde{\theta}_{\lambda,\theta}^{-1}$ denote the reciprocal function of $\xi \to \tilde{\theta}_\lambda(1, \xi, \theta)$ restricted to the support of φ. From this formula we can read off the joint continuity of the density [note that the latter vanishes if θ' is not in the image of the support of φ under the mapping $\xi \to \tilde{\theta}_\lambda(1, \xi, \theta)$] and its uniform boundedness [by the conjunction of (3.7), (3.8), and the claim (3.9) it is obvious that a bound like $\|\varphi\|_\infty / CC_0$ would do].

As a solution of a first order differential equation in x whose coefficients are smooth in λ and ξ and whose initial condition is θ, $\tilde{\theta}$ is jointly continuous in λ, x, ξ, and θ, so that the claim reduces to proving that $\int_0^1 \chi(s) \sin^2 \tilde{\theta}_\lambda(s, \xi, \theta) ds < 0$ for each fixed λ, ξ, and θ. The latter is shown by noting that in view of (3.5), $\tilde{\theta}' = 1$ at any point where $\sin^2 \tilde{\theta} = 0$, so $\sin^2 \tilde{\theta}$ has isolated zeros and $\int_0^1 \chi(s) < 0 \Rightarrow \int_0^1 \chi(r) \sin^2 \tilde{\theta}$. □

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