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Determination of Eigenvalues by Divergent Perturbation Series*

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We study eigenvalue problems for operators $H_0 + \beta V$, where the perturbation series is finite order by order but divergent for any β. We prove that, under suitable conditions, the series nevertheless determines the level *uniquely* [and is not merely asymptotic] because some control of the remainder term, R_N, uniform in N is present; in fact, for β real, positive, and small, the perturbation series is actually Borel summable. We discuss applications to finite-dimensional oscillators and spatially cutoff field theories already announced plus some additional examples.

1. INTRODUCTION

One is faced often, particularly in physical applications, with a perturbation problem of finding eigenvalues of operator sums $H_0 + \beta V$, where the perturbation series are finite term by term but divergent for any value of β. The usual interpretation of such a series is that it is asymptotic[1], but we are interested here in answering the more ambitious question of whether the eigenvalues might possess some additional property which allows one to determine them uniquely from the Rayleigh–Schrödinger perturbation series. We have in mind using the following criterion of Carleman [2]:

CARLEMAN'S THEOREM. *Let $g(z)$ be a function analytic in a sector, $S = \{z \mid 0 < |z| < B; |\arg z| < \pi/2 + \epsilon\}$, for some B, $\epsilon > 0$. Suppose, for all $z \in S$ and all n, $|g(z)| \leqslant b_n |z|^n$ where $\sum b_n^{-1/n} = \infty$. Then*

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[1] For cases where this can be proven, see Kato [9, 10].

$g(z) \equiv 0$. *In particular, if f is analytic in S and obeys the strong asymptotic condition*

$$|f(z) - \sum_{n=0}^{N} a_n z^n| < C\sigma^{N+1}(N+1)! \, |z|^{N+1} \tag{1}$$

for all N and all $z \in S$, for some formal series $\sum a_n z^n$, then f is uniquely determined by the fact that it obeys (1) *for some C, σ.*

Actually, if f obeys condition (1), it can be recovered explicitly in $\{z \mid 0 < |z| < B; |\arg z| < \epsilon\}$ by the method of Borel summability (Watson's theorem; see, e.g., Hardy [6]). However, we regard this as of secondary importance to the fact that when an energy level obeys (1) [with f = the level; $z = \beta$ and a_n = the Rayleigh–Schrödinger coefficients], the level is "determined" uniquely by the Rayleigh–Schrödinger series.

Recently, Graffi, Grecchi, and Simon [5] have announced generalized Borel summability for finite-dimensional anharmonic oscillators [14][2], and Simon [15] has announced a similar result for the ground state energy in spatially cutoff $(\phi^4)_2$ field theories [4]; in both these cases, the perturbation series is known to diverge [1, 8, 14]. It seems worth while to present detailed proofs of the results sketched in [5] and [15] in the context of abstract Hilbert space operators, H_0 and V, and this is our goal here. To give the flavor of the type of results we shall prove, let us quote one of them:

THEOREM. *Let H_0 and V be positive self-adjoint operators with*[3] $C^\infty(H_0) \subset D(V)$ *and* $V[C^\infty(H_0)] \subset C^\infty(H_0)$. *If there are constants, C, n so that*

$$\|(H_0 + 1)^m V\psi\| \leqslant C \|(H_0 + n)^{m+2} \psi\|$$

for all $m = 0, 1, 2,\ldots$ and all $\psi \in C^\infty(H_0)$, then any isolated, nondegenerate eigenvalue of H_0 has a Borel summable Rayleigh–Schrödinger series, summing (for $0 < \beta$ small) to an eigenvalue of $H_0 + \beta V$ (defined by a Friedrich's extension). In particular, the eigenvalue is "determined" by the perturbation series and a strong asymptotic condition of type (1).

[2] It was earlier proven by Loeffel *et al.* [11], that the levels of one-dimensional x^4 and x^6 oscillators had convergent diagonal Padé approximants; so, in particular, it was known in this case that the levels were determined by their power series.

[3] $D(A)$ = domain of the operator A; $C^\infty(A) = \bigcap_{n=1}^{\infty} D(A^n)$.

2. The Main Theorem

Our main theorem has an embarrassingly simple proof but is also something of a cheat since the conditions are picked precisely to allow a simple proof. In Sections 3 and 4, our goal will be to find conditions on operators H_0 and V which are more easily verifiable and which imply the conditions of our main theorem.

The Main Theorem (Theorem 1). *Let $H_0 > 0$ and V be self-adjoint operators so that V leaves $C^\infty(H_0)$ invariant. Let Ω_0 be the eigenvector for an isolated nondegenerate eigenvalue, E_0 of H_0. Suppose:*

(a) *For β in the cut plane (i.e., $\beta \neq$ a negative real), $H_0 + \beta V$ defined as a quadratic form on $C^\infty(H_0)$ has a form closure which is an analytic family in the sense of Kato* [9][4].

(b) *For any $E \notin \operatorname{spec}(H_0)$, $\|(H_0 + \beta V - E)^{-1} - (H_0 - E)^{-1}\| \to 0$ uniformly as $|\beta| \downarrow 0$ in some sector $\{\beta \mid |\arg \beta| < \pi/2 + \epsilon\}$ for some $\epsilon > 0$.*

(c) *For some C, σ_0, and a and all N*

$$\|[V(H_0 - E)^{-1}]^N \Omega_0\| < C\sigma_0^N N!$$

for all E with $|E - E_0| = a$, where E_0 is the only point of $\operatorname{spec}(H_0)$ in $\{E \mid |E - E_0| \leqslant a\}$. Then the Rayleigh–Schrödinger series for the perturbed level $E(\beta)$ with $E(0) = E_0$ and for the normalized perturbed eigenvector $\Omega(\beta)$ with $\Omega(0) = \Omega_0$ are finite term by term and are Borel summable to $E(\beta)$, $\Omega(\beta)$ for $|\beta|$ small; $|\arg \beta| < \epsilon$. In particular, $E(\beta)$ is uniquely determined by the Rayleigh–Schrödinger series and a strong asymptotic condition of type (1).

Note. We intend Borel summability of $\Omega(\beta)$ to mean Borel summability of the numerical function $\langle \phi, \Omega(\beta) \rangle$ for any ϕ.

Proof. Because of the *norm* resolvant convergence (b), the isolated nondegenerate eigenvalues of H_0 are *stable* in the sense of Kato [9, pp. 437–439], i.e., for $|\beta|$ sufficiently small with $|\arg \beta| < \pi/2 + \epsilon$, $H_0 + \beta V$ has only one point of its spectrum in the disc $\{E \mid |E - E_0| \leqslant a\}$, and the projection

$$P(\beta) = -(2\pi i)^{-1} \int_{|E-E_0|=a} (H_0 + \beta V - E)^{-1}\, dE$$

[4] That is for $\lambda \notin \operatorname{spec}(H_0 + \beta_0 V)$, $(H_0 + \beta V - \lambda)^{-1}$ is a bounded-operator-valued analytic function for β near β_0.

onto the corresponding eigenspace is one dimensional. To prove a strong asymptotic condition on $E(\beta)$ and $\langle\phi, \Omega(\beta)\rangle$, we need only prove a bound on the remainder to the asymptotic series for $P(\beta)\Omega_0$ of the form

$$\| P(\beta)\,\Omega_0 - \sum_{n=0}^{N} \phi_n\beta^n \| \leqslant C_0\sigma_0^{N+1}(N+1)!\, |\,\beta\,|^{N+1} \tag{2}$$

and then use

$$E(\beta) = E_0 + \beta\langle V\Omega_0\,, P(\beta)\,\Omega_0\rangle/\langle\Omega_0\,, P(\beta)\,\Omega_0\rangle$$

and

$$\langle\phi, \Omega(\beta)\rangle = \langle\phi, P(\beta)\,\Omega_0\rangle/\langle\Omega_0\,, P(\beta)\,\Omega_0\rangle^{1/2}.$$

To prove a bound on the remainder of the asymptotic series for $P(\beta)\Omega_0$, we need only obtain a bound on the remainder of the series for $(H_0 + \beta V - E)^{-1}\Omega_0$ uniform in E with $|\,E - E_0\,| = a$ and then integrate. This is the critical simplification, for the asymptotic series, for the resolvent is just a geometric series and we have thereby eliminated all the complicated additional terms in the perturbation series for E [9, pp. 83–84]. We thus write:

$$\begin{aligned}(H_0 + \beta V - E)^{-1}\,\Omega_0 = &\sum_{n=0}^{N} (H_0 - E)^{-1}[-V(H_0 - E)^{-1}]^n\,\beta^n\Omega_0\\ &+ \beta^{N+1}(H_0 + \beta V - E)^{-1}[-V(H_0 - E)^{-1}]^{N+1}\,\Omega_0\,.\end{aligned} \tag{3}$$

Since $V : C^\infty(H_0) \to C^\infty(H_0)$; $(H_0 - E)^{-1} : C^\infty(H_0) \to C^\infty(H_0)$, the geometric series with remainder is valid if both sides are considered as maps of $C^\infty(H_0)$ into $\mathscr{H}$. Since $\Omega_0 \in C^\infty(H_0)$, (3) is valid. Thus, we need only prove for $|\,\beta\,|$ small, $|\arg\beta\,| < \pi/2 + \epsilon$, that

$$\|(H_0 + \beta V - E)^{-1}[V(H_0 - E)^{-1}]^{N+1}\,\Omega_0\,\| \leqslant D\sigma_0^{N+1}(N+1)!$$

uniformly for E with $|\,E - E_0\,| = a$. But by (b), $\|(H_0 + \beta V - E)^{-1}\| \leqslant F$, some constant independent of β as long as $|\,\beta\,|$ is small and in the sector. Thus using (c) and letting $D = CF$, we obtain the required bound on the remainder of the asymptotic series.

3. Conditions (a) and (b)

We give in this section more manageable conditions on the operator H_0 and V which imply conditions (a) and (b) of the main theorem. The theorem for condition (a) is simple:

THEOREM 2. *Let* H_0 *and* V *be self-adjoint operators with quadratic form domains* $Q(H_0)$, $Q(V)$. *Suppose* $H_0 > 0$, *and* V_+, V_- *are the positive and negative parts of* V, *i.e.*, $V = V_+ - V_-$; $V_+, V_- \geqslant 0$; $V_+V_- = 0$. *Finally suppose*

(i) $Q(H_0) \cap Q(V)$ *is dense.*

(ii) *For any* $\beta > 0$, $H_0 + \beta V_-$ *is bounded below as a quadratic form*; *i.e.*, V_- *is a small form perturbation of* H_0.

Then $H_0 + \beta V$ defined as a quadratic form on $Q(H_0) \cap Q(V) = Q(H_0) \cap Q(V_+)$ is a closed sectorial quadratic form[5] for any β in the cut plane and $H_0 + \beta V$ is an analytic family of type (B); in particular [9, pp. 393–403], condition (a) of Theorem 1 holds.

Remarks. 1. The estimates $C_\beta \leqslant H_0 + \beta V_-$ for all $\beta > 0$ imply estimates $H_0 \leqslant 2(H_0 + \beta V) + a_\beta$; $|V| \leqslant 2 |\beta|^{-1} (H_0 + \beta V) + e_\beta$ for $\beta > 0$. These estimates are essentially the reason the form is closed.

2. Once the form is closed, the analyticity is immediate since $H_0 + \beta V$ clearly has analytic expectation values.

3. This result is proven by Simon and Hoegh–Krohn [16]; so we only sketch the proof below.

4. It is an open question whether (ii) can be weakened to merely require $H_0 + \beta V$ bounded below.

Sketch.[6] Let

$$\beta = |\beta| \, e^{i\theta}(|\theta| < \pi).$$

One proves $\mathrm{Re}(e^{-i\theta/2}\langle\psi, (H_0 + \beta V)\psi\rangle)$ is bounded below as ψ runs through all $\psi \in Q(H_0) \cap Q(V)$ with $|\psi| = 1$, and the argument of $b + e^{-i\theta/2}\langle\psi, (H_0 + \beta V)\psi\rangle$ can be made to lie in $[-|\theta|/2 - \delta, |\theta|/2 + \delta]$ for arbitrary $\delta > 0$ by taking b large. The idea is that $\arg(e^{i\theta/2}\langle\psi, |\beta| V_+\psi\rangle)$ is $\theta/2$ and, by taking b large and using (ii), $\arg(e^{-i\theta/2}\langle\psi, (H_0+\beta V_-)\psi\rangle + b)$

[5] Our definition of sectorial is slightly more general than Kato's; we say a closed form, $\langle\cdot, H \cdot\rangle$, is sectorial if there is a real ϕ, a $\theta < \pi$ and a $Z \in C$ with

$$|\arg(e^{i\phi}\langle\Omega, (H + Z)\Omega\rangle)| < \theta \qquad \text{for all} \quad \Omega \in \mathscr{H}.$$

Kato requires ϕ, $Z = 0$.

[6] For details, see [16].

is in $(-\theta/2 - \delta, \theta/2 + \delta)$. Thus, $H_0 + \beta V$ is sectorial; to prove it is closed, one must only prove if

$$\psi_m \in Q(H_0) \cap Q(V) \quad \text{and} \quad \langle(\psi_m - \psi_n), H_0(\psi_m - \psi_n)\rangle \to 0;$$
$$\langle(\psi_m - \psi_n), V(\psi_m - \psi_n)\rangle \to 0 \quad \text{and} \quad \| \psi_m - \psi_n \| \to 0;$$

then $\psi = \lim \psi_n \in Q(H_0) \cap Q(V)$. This follows since $\psi \in Q(H_0)$ by the H_0-convergence and the fact that H_0 is closed; since V_- is a small perturbation of H_0, $\psi \in Q(V_-)$ so $\langle(\psi_m - \psi_n), V_+(\psi_m - \psi_n)\rangle \to 0$ so that $\psi \in Q(V_+)$. Thus, $\psi \in Q(| V |) = Q(V)$.

For condition (b) of the main theorem, we have two distinct subsidiary methods of verification: the first is a generalization of the condition used in [15] for $(\phi^4)_2$, and the second is a generalization of the method used for x^{2m} perturbations in [14].

THEOREM 3. *Let H_0 and V obey the conditions of Theorem 2 and suppose in addition $| V | \leqslant \alpha H_0^2 + \gamma$ for some α, γ (equivalently, $| V |^{1/2}$ is a small operator perturbation of H_0). Then condition* (b) *of the main theorem holds (for any sector, $| \arg \beta | \leqslant \phi < \pi$).*

We first prove a lemma:

LEMMA 3.1. *Let the condition of Theorem 2 hold, and let $\theta < \pi$ be given. Then there are B, E and a, so that $| \arg \beta | < \theta$, $0 < | \beta | < B$ imply*:

(i) $E \notin \operatorname{spec}(H_0 + \beta V)$,

(ii) $\| | V |^{1/2}(H_0 + \beta V - E)^{-1} \| \leqslant a | \beta |^{-1/2}$.

Proof. Pick $0 < \eta < \min(\pi/2, \pi - \theta)$. Then geometry (Fig. 1) shows,

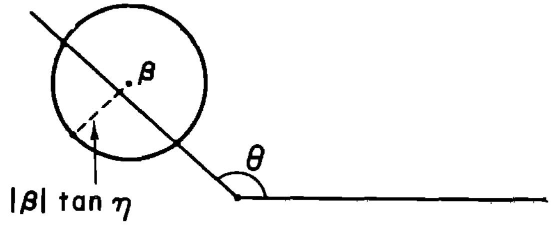

FIGURE 1

for any β with $|\arg \beta| < \theta$ and any ϕ real, $|\arg(\beta + |\beta| \tan \eta e^{i\phi})| < \pi$. Let $\gamma = |\beta| \tan \eta e^{i\phi}$. Then, by writing

$$H_0 + \beta V + \gamma |V| = H_0 + (\beta - \gamma) V_- + (\beta + \gamma) V_+$$

and using the technique of the proof of Theorem 2, we see that this operator is uniformly sectorial if we keep $|\beta|$ small with β in the sector $|\arg \beta| < \theta$. Thus for some b and some ϕ_0 independent of β in the region and of ϕ, $\mathrm{Re}(\langle\psi, (H_0 + \beta V + \gamma |V|)\psi\rangle e^{i\phi_0} + b\langle\psi, \psi\rangle) \geqslant 0$. Equivalently:

$$|\beta| \tan \eta \langle\psi, |V| \psi\rangle \leqslant \mathrm{Re}(\langle\psi, H_0 + \beta V\psi\rangle e^{i\sigma_0} + b\langle\psi, \psi\rangle).$$

In Theorem 2, we saw $H_0 + \beta V$ is uniformly sectorial for β in the sector; so $(H_0 + \beta V - E)^{-1}$ is uniformly bounded for suitable E; thus, letting $\psi = (H_0 + \beta V - E)^{-1}\phi$, we see

$$\begin{aligned} |\beta| (\tan \eta) \| |V|^{1/2}(H_0 + \beta V - E)^{-1} \phi \| \\ \leqslant \|\phi\|^2 \{\|(H_0 + \beta V - E)^{-1*}(H_0 + \beta V)(H_0 + \beta V - E)^{-1}\| \\ + b \|(H_0 + \beta V - E)^{-1}\|^2\} \leqslant C \|\phi\|^2 \end{aligned}$$

independent of β in the sector.

Proof of Theorem 3. By standard arguments [9, pp. 173–174], we need only prove $\|(H_0 + \beta V - E)^{-1} - (H_0 - E)^{-1}\| \to 0$ uniformly as $|\beta| \downarrow 0$ in the sector $|\arg \beta| < \theta$ for one fixed E, say the E of the lemma. Write $V = U |V|$ with U unitary and write: $(H_0 + \beta V - E)^{-1} - (H_0 - E)^{-1} = \beta(H_0 - E)^{-1} |V|^{1/2} U |V|^{1/2} (H_0 + \beta V - E)^{-1}$. By assumption, $(H_0 - E)^{-1} |V|^{1/2}$ is bounded and $|\beta|^{1/2} |V|^{1/2}(H_0 + \beta V - E)^{-1}$ is bounded by the lemma; so the norm of the difference goes to zero at least as fast as $|\beta|^{1/2}$.

There is a second result allowing worse growth of V relative to H_0 [7] but requiring more linking of H_0 and V in the sense of bounds on the commutations:

THEOREM 4. *Suppose H_0 and V obey the conditions of Theorem 2 and that $V : C^\infty(H_0) \to C^\infty(H_0)$. Suppose also $(\forall a)$ $(\exists b, C)$ so that on $C^\infty(H_0) \times C^\infty(H_0)$ and for all $0 \leqslant \epsilon \leqslant C$,*

[7] E.g., $V = x^{2m}$; $H_0 = p^2 + x^2$.

(i) $\pm\epsilon[H_0^{1/2}, [H_0^{1/2}, V]] \leqslant a(H_0^2 + \epsilon^2 V^2) + b,$

(ii) $\pm i\epsilon[H_0, V] \leqslant a(H_0^2 + \epsilon^2 V^2) + b,$

(iii) $| V |^{2/n} \leqslant \alpha(H_0^2 + 1)$ *for some integer n.*

Then condition (b) of the main theorem holds in any sector $| \arg \beta | \leqslant \phi < \pi$.

We prove this theorem by first proving the lemmas:

LEMMA 4.1. *If* (ii) *holds, then for any* $\phi > 0$, *there is a* B *and a* d *such that for* $| \beta | < B$ *and* $\phi < | \arg \beta | < \pi - \phi$:

$$H_0^2 + | \beta |^2 V^2 \leqslant d[(H_0 + \beta V)^+(H_0 + \beta V) + 1].$$

Remark. This is an abstraction of a result in [14, Section II.9].

Proof.

$$\begin{aligned}(H_0 + \beta V)^\dagger(H_0 + \beta V) &= H_0^2 + | \beta |^2 V^2 + (\operatorname{Re}\beta)(H_0 V + VH_0) + \operatorname{Im}\beta[H_0, V] \\ &= \frac{| \operatorname{Re}\beta |}{| \beta |}[H_0 \pm | \beta | V]^2 + \left(1 - \frac{| \operatorname{Re}\beta |}{| \beta |}\right) \\ &\quad \times (H_0^2 + | \beta |^2 V^2) \pm \frac{| \operatorname{Im}\beta |}{| \beta |}\{| \beta | [H_0, V]\} \\ &\geqslant 1/2\left(1 - \frac{| \operatorname{Re}\beta |}{| \beta |}\right)(H_0 + | \beta |^2 V^2) + e\end{aligned}$$

if $\phi < | \arg \beta | < \pi - \phi$ and $| \beta |$ is small, where, in the last step, we used (ii) with $a = (2 \sin \phi)^{-1} (1 - \cos \phi)$.

LEMMA 4.2. *If* (i) *and* (ii) *hold, then for any* $\phi < \pi/2$, *there is a* B *and* d *such that for* $| \beta | < B$ *and* $| \arg \beta | < \phi$:

$$H_0^2 + | \beta |^2 V^2 \leqslant d[(H_0 + \beta V)^\dagger(H_0 + \beta V) + 1).$$

Remark. The double commutator technique we use is due to Jaffe [7].

Proof.

$$\begin{aligned}(H_0 + \beta V)^\dagger(H_0 + \beta V) &= H_0^2 + | \beta |^2 V^2 + \operatorname{Re}\beta(H_0 V + VH_0) + \operatorname{Im}\beta[H_0, V] \\ &= H_0^2 + | \beta |^2 V^2 + \operatorname{Re}\beta[H_0^{1/2}, [H_0^{1/2}, V]] \\ &\neq \operatorname{Im}\beta[H_0, V] + 2 \operatorname{Re}\beta H_0^{1/2} V H_0^{1/2} \\ &\geqslant \tfrac{1}{2}H_0^2 + \tfrac{1}{2} | \beta |^2 V^2 - 2b\end{aligned}$$

using (i) and (ii) with $a = 1/4 \sin \phi$.

Proof of Theorem 4. Write

$$\|(H_0 + \beta V - E)^{-1} - (H_0 - E)^{-1}\|$$
$$= |\beta|^{1/n} (\|(H_0 + \beta V - E)^{-1} |\beta V|^{1-1/n}\|)(\| |V|^{1/n} (H_0 - E)^{-1}\|).$$

By the quadratic estimates of Lemmas 4.1 and 4.2 and by (iii) each $(\|\cdots\|)$ factor is bounded; so the norm of the difference goes to 0.

Remarks. 1. In case $V \leqslant \alpha(H_0{}^2 + 1)$ and (i) and (ii) hold, the difference of the resolvants goes to zero as β.

2. In [15], the method of Theorem 3 was used for spatially cutoff $(\phi^4)_2$ theories. Alternately, one can prove the commutator estimates (i) (see [4]) and (ii) of Theorem 4 and conclude

$$\|(H_0 + \beta V - E)^{-1} - (H_0 - E)^{-1}\| = O(\beta),$$

following Remark 1.[8]

4. Condition (c)

Before turning to proving (c) for weaker requirements, we should like to give arguments as to why (c) is not only sufficient for a strong asymptotic condition to hold, but *almost* necessary. For, if a strong asymptotic condition holds, the Rayleigh–Schrödinger coefficients obey $|a_n| < D\sigma^n n!$ automatically. These a_n are obtained from the geometric series for the resolvant by integrating in a circle and then dividing a series for $\langle \Omega_0, VP\Omega_0 \rangle$ by a series for $\langle \Omega_0, P\Omega_0 \rangle$. It is, of course, possible to have condition (c) fail but still have $|a_n| < D\sigma^n n!$ but only if there are considerable concellations, either in the integration or the division—thus we conclude the usual situation is to have (c) when a strong asymptotic condition holds.

The simplest theorem about condition (c) is:

THEOREM 5. *Let the conditions of Theorem 2 hold and suppose an operator, C, exists so that:*

(1) *C commutes with H_0 and* $0 \leqslant C \leqslant H_0$,

[8] It is a pleasure to thank Lon Rosen for a discussion of estimate (ii) in that case.

(2) *$V : C^\infty(C) \to C^\infty(C)$ and for some constants m and d:*

$$\|(C+1)^n V\psi\| \leqslant d\,\|(C+m+1)^{n+2}\psi\| \tag{4}$$

for $n = 0, 1, 2,\ldots$ and all $\psi \in C^\infty(C)$.

Then condition (c) of the main theorem holds for any nondegenerate, isolated, unperturbed level, Ω_0.

LEMMA 5.1. *If (4) holds for $n = a, b$ two real numbers, it holds for all $a \leqslant n \leqslant b$.*

Proof. This is a simple application of the standard interpolation trick of Thorin and Stein. (4) is equivalent to

$$f(n) \equiv (C+1)^n V(C+m+1)^{-n-2}$$

being bounded with $|f(n)| \leqslant d$. Now $f(z)$ is analytic in the strip $a < \operatorname{Re} z < b$ with $|f(z)| \leqslant d$ if $\operatorname{Re} z = a, b$. It thus follows that $|f(z)| \leqslant d$ for all z in the strip (by the maximum modulus principle).

Proof of Theorem 5. We first prove for any k and $n = 0, 1, 2,\ldots$

$$\|(C+1+k)^{n/2} V\psi\| \leqslant d\,\|(C+k+m+1)^{n/2}(C+m+1)^2\psi\|. \tag{5}$$

This is equivalent to an operator inequality:

$$V(C+1+k)^n V \leqslant d^2(C+k+m+1)^n(C+m+1)^{\cdot}. \tag{6}$$

To prove (6), we compute:

$$V(C+1+k)^n V = \sum_{j=0}^{n} \binom{n}{j} k^j V(C+1)^{n-j} V$$

$$\leqslant d^2 \sum_{j=0}^{n} \binom{n}{j} k^j (C+m+1)^{n+4-j} \qquad \text{(by (4) and Lemma 5.1)}$$

$$= d^2(C+m+1)^4(C+m+k+1)^n.$$

Now write

$$[V(H_0-E)^{-1}]^n = \left[\prod_{j=1}^{n} W_j Y_j Z_j\right][C+nm+1]^n,$$

where

$$W_j = [C + (j-1)\, m + 1]^{j-1}\, V[C + jm + 1]^{-j+1}[C + m + 1]^{-2},$$
$$Y_j = (A - E)^{-1}(C + m + 1),$$
$$Z_j = (C + m + 1)(C + jm + 1)^{-1}.$$

The inequality (5) says $\| W_j \| \leqslant d$, and we obviously have $\| Y_j \| = y$ for some y independent of E with $| E - E_0 | = \epsilon$. Finally $\| Z_j \| \leqslant 1$; so:

$$\begin{aligned} \|[B(A - E)^{-1}]^n\, \Omega_0 \| &< (dy)^n \, \|(C + nm + 1)^n\, \Omega_0 \| \\ &= (dy)^n (nm + C + 1)^n \\ &< A(dy)^n (mx)^n\, n! \end{aligned}$$

for suitable constants A and x.

Finally, let us state two more specialized criteria for condition (c) to hold which may be useful in certain circumstances:

THEOREM 6. *Suppose H_0 has purely discrete spectrum with eigenvectors $\Omega_0, \Omega_1, \ldots$ and eigenvalues $0 < E_0 \leqslant E_1 \leqslant \cdots$ and suppose there are integers P, Q so that:*

(i) $V^2 \leqslant \alpha^2 H_0^{2P}$,

(ii) $\langle Q_n, V\Omega_m\rangle = 0$ *if* $| n - m | > Q$,

(iii) $| E_{nQ} |^{n(P-1)} \leqslant \hat{C}\hat{\sigma}^n n!$.

Then condition (i) of the main theorem holds.

Proof. Let us write $v_{nm} = \langle \psi_n, V\psi_m \rangle$. Then

$$| v_{nm} |^2 \leqslant \sum_k | v_{km} |^2 = \langle \psi_m, V^2 \psi_m \rangle \leqslant \alpha^2 \langle \psi_m, H_0^{2P} \psi_m \rangle = d^2 E_m^{2P};$$

so $| v_{nm} | \leqslant \alpha E_m{}^P$.

Next, pick σ_0 so $| E - E_0 | = \epsilon$ implies $| E - E_m |^{-1} \leqslant \sigma_0 \, | E_m |^{-1}$, $m = 0, 1, \ldots$.

Now consider

$$\begin{aligned} \|[V(H_0 - E)^{-1}]^N\, \Omega_0 \| &\leqslant \sum_m |\langle \Omega_m, [V(H_0 - E)^{-1}]^N\, \Omega \rangle| \\ &\leqslant \sum_{\substack{m_1, \ldots, m_N \\ m_{N+1} = 0}} \prod_{i=1}^{N} | v_{m_i m_{i+t}} | \, (E_{m_{i+1}} - E)^{-1} \end{aligned}$$

Because of condition (ii), there are fewer than $(2Q + 1)^N$ terms in the sum; so it is enough to establish a $C\sigma^N N!$ bound on every term of the form $\prod_{i=1}^N | v_{m_i m_{i+1}} | \, | E_{m_{i+1}} - E |^{-1}$ with $m_{N+1} = 0$. If this is to be nonzero, each $m_i \leqslant nQ$ so:

$$\prod_{i=1}^{N} | v_{m_i m_{i+1}} | \, | E_{m_{i+1}} - E |^{-1} \leqslant \prod_{i=1}^{N} (E^P_{m_{i+1}}) \, | E_{m_{i+1}} - E |^{-1}$$

$$\leqslant \sigma_0{}^N \prod_{i=1}^{N} (E^P_{m_{i+1}}) \, | E_{m_{i+t}} |^{-1}$$

$$\leqslant | E_{NQ} |^{N(P-1)} \sigma^N \leqslant C(\tilde{\sigma}\sigma_0)^N N!.$$

THEOREM 7. *Suppose there exist a positive self-adjoint operator C, an integer k, operators $V_1, \ldots, V_m$, and numbers $\alpha_1, \ldots, \alpha_m$; $d_1, \ldots, d_m$ so that:*

(i) $V = \sum_{i=1}^m V_i$,

(ii) $\| V_i(C + 1)^{-k-1} \| \leqslant \alpha_i$,

(iii) $(C + 1)^k \leqslant (H_0 + 1)$,

(iv) $[C, V_i] = d_i V_i$ *on* $C^\infty(H_0)$,

(v) $[C, H_0] = 0$ *in the sense of self-adjoint operators commuting.*

Then condition (c) holds.

Proof. We need only prove a bound of the form

$$\| V_{i_1}(H_0 - E)^{-1} \cdots V_{i_n}(H_0 - E)^{-1} \Omega_0 \| \leqslant C\sigma^n n! \qquad \text{for} \quad \| [V(H_0 - E)^{-1}]^n \Omega_0 \|$$

is bounded by m^n such terms. Let $C\Omega_0 = c_0 \Omega_0$. Write

$$V_{i_1}(H_0 - E)^{-1} \cdots V_{i_n}(H_0 - E)^{-1} \Omega_0$$

$$= \prod_{j=1}^{n} \{[V_{i_j}(C + 1)^{-k-1}][(C + 1)^k (H_0 - E)^{-1}][C + 1]\} \Omega_0$$

$$= \prod_{j=1}^{n} \left\{[V_{i_j}(C + 1)^{-k-1}][(C + 1)^k (H_0 - E)^{-1}] \left[\sum_{k=j+1}^{n} d_i + c_0\right]\right\} \Omega_0$$

using (IV). Thus

$$\|(V_{i_t} \cdots (H_0 - E)^{-1} \Omega_0 \|$$

$$\leqslant \max_{i=1,\ldots,m} \alpha_i)^n (\sup_{|E| \to \epsilon} \|(C + 1)^k (H_0 - E)^{-1} \|)^n (\max_{i=1,\ldots,m} | d_i | + c_0)^n n!.$$

5. Examples

(a) *x^4-oscillators* [5, 14, 15]. These are the motivating examples. For the theory with finitely many degrees of freedom or for the spatially cutoff $(\phi^4)_2$ field theory of Glimm–Jaffe, all the conditions of Theorem 1 hold. (a) follows from Theorem 2; in the x^4-case, $V \geqslant 0$ and in the $(\phi^4)_2$ case, one must appeal to the bound of Nelson–Glimm [12, 3] (see also [13, 1b]). For (b), either Theorem 3 or 4 may be used in either case and for (c), it is probably easiest to appeal to Theorem 7 (with C the number operator and $k = 1$ in the $(\phi^4)_2$ case), although Theorem 5 can be employed.

(b) *Perturbations of* $(p^2 + x^2)^k$. Let $H_0 = (a^\dagger a + \frac{1}{2})^k$, and let V be any polynomial in a, $a^\dagger$ which is of degree $4k$ or less and which defines a self-adjoint operator which is bounded below (e.g., $(p^2 + x^2)^k + \beta x^{4k}$). Then the conditions of Theorem 5 are obeyed with $C = H_0$ and, in particular, $V \leqslant C(H_0 + 1)^2$; so Theorems 3 and 1 may be applied.

6. A Critique

There is one major weakness in the results we have proven here. They are, in the end, restricted to a very special class of perturbations. In the case where $H_0 = C$ (in the terminology of Theorems 5, 7), the perturbation V is restricted to have the property that for some E_0, $(\psi, V\phi) = 0$ if $\psi = P_\Omega(H_0)\psi$; $\phi = P_{\Omega_1}(H_0)\phi$ with $\inf_{x \in \Omega_1; y \in \Omega_2} | x - y | > E_0$, where $P_\Omega(H)$ are the spectral projections for H_0. When $H_0 \neq C$, this is no longer strictly true [example: $(\phi^4)_2$ has V linking states of arbitrarily large free energy difference; it is the difference of the number of particles which is bounded], but the conditions of Theorems 5–7 are very restrictive indeed.

It would be interesting to know if the Borel summability of levels of $H_0 + \beta V$ is stable under changing V by a bounded operator. This would greatly increase the class of perturbations we can treat by this method, but unfortunately the question does not seem answerable by the methods of Section 4.

Of course, the class we consider, while restricted, seems to include several cases of direct physical interest. And, in general, our results suggest one can hope to determine levels of "singular" perturbation "directly" from the perturbation series.

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