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Almost Periodic Schrödinger Operators IV. The Maryland Model*

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The analysis of discrete Schrödinger operators of the form $(hu)(n) = u(n+1) + u(n-1) + \lambda \tan(\pi\alpha n + \theta)\, u(n)$ is discussed. Depending on Diophantine properties of α, the spectrum may be dense point, singular continuous or a mixture of the two. © 1985 Academic Press, Inc.

1. INTRODUCTION AND RESULTS

Exactly solvable models are useful laboratories that can teach one both positive and negative lessons: Certain phenomena that one might not expect or about which one might be unsure can be examined, while, on the other hand, one can find explicit counterexamples to "reasonable" conjectures. Of course, one must decide which aspects of the model are typical and which are artifacts of its special elements.

Thus the discovery of an exactly solvable almost periodic Schrödinger operator by Grempel, Fishman and Prange [19, 30] (henceforth GFP) is very significant. It is their model, which we dub the Maryland model, that we wish to study here. The basic model is the Jacobi matrix ($\equiv$ discrete Schrödinger operator) on $l^2(Z)$:

$$(hu)(n) = u(n+1) + u(n-1) + \lambda \tan(\pi\alpha n + \theta)\, u(n). \qquad (1)$$

In (1), α, λ, θ are parameters with $\theta \neq \pi/2$, $\pi/2 \pm \alpha$, $\pi/2 \pm 2\alpha$,... (so the potential is everywhere finite). We always take $\lambda \geqslant 0$ (if $\lambda \leqslant 0$, use $n \to -n$ symmetry). The potential $V(n) = \tan(\pi\alpha n + \theta)$ is technically not almost periodic since $V(\phi) = \tan(\phi)$ is not continuous or even in any L^p ($p \geqslant 1$) space, but it is almost periodic in some kind of extended sense. However, one should bear in mind that V is unbounded so even small λ isn't "weak coupling."

In an analysis, which should become a textbook example of how to conquer small divisors, GFP show that when α has suitable Diophantine Properties, h has an explicit set of exponentially localized states. Their analysis is essentially rigorous.

In this paper, we want to discuss a number of aspects of the model not treated in GFP. We found the bulk of these results in the fall of 1982 but did not publish them

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at the time several reasons. At roughly the same time, Pasteur and Figotin (PF) obtained a number of results about the model which they announced and sketched in [27]. There is considerable overlap of results (but not methods) between their work and ours, and we will point out the overlap where it occurs. We especially draw the reader's attention to the elegant formula PF obtain for the Green's function.

The themes we will treat here include:

(a) Completeness of the eigenfunctions found by GFP is not established in [19, 30]. We will prove it here. We note that for sufficiently *large* λ, results of Bellissard, Lima and Scoppola [3] also yield completeness. Also, Pastur and Prange [28] have informed me that one can deduce completeness from the PF Green's function formula.

(b) We want to discuss what happens when α does not have typical Diophantine properties (see below for the meaning of this). In later papers [29, 30] written subsequent to our own work on this case, GFP did discuss such α; as we will discuss in Section 4, while both our work and that in [29] provide some insight into such α, the most basic questions about the nature of the eigenfunctions remain open. PF [27] also discuss non-Diophantine α's and obtain results similar to ours in this case. Berry [6] also has results on this class of α.

(c) We want to note that the analysis works in arbitrary dimension. Independently, GFP [17] and PF [27] have recently also discussed this case.

(d) We want to discuss certain analyticity questions involving both analyticity in λ and analyticity of the Fourier transform of the eigenfunctions not addressed in [30].

(e) GFP discovered their model in their study of certain time dependent problems (see also [16]) and they obtain the basic equations by going back and forth between the time-dependent problem and equation (1). We want to present a more straightforward analysis also found by PF [27].

GFP correctly compute the density of states in [19, 30], although their calculation is not rigorous for two reasons: (a) Their calculation only uses the eigenfunctions they find and since they haven't proven completeness of these eigenfunctions, they do not know that they have properly counted states (b) The real density of state is normalized by counting states in n-space boxes; GFP label eigenfunctions in a natural way and they count states in label-space boxes. As for point (a), we will prove completeness in Section 2. As for (b), it is important that the eigenfunction labeled by n is in some weak sense localized near n (see Eq. (9) below)—under such conditions, we will show (also in Section 2) that label space normalization yields the correct density of states. Thus, in Section 2, we provide a rigorous justification of the GFP calculation of the density of states (at least if α has typical Diophantine properties; once one has the density of states for such α, one can deduce it for all *irrational* α since the density of states is continuous at such α—see [2, 15]).

While one can justify the GFP calculation of the density of states, we found a simple, direct calculation [33] which "explains" why the density of states in the model agrees with that in the Lloyd model, and which works directly for all irrational α. PF [27] also have a direct calculation of the density of states.

It is actually useful (although not essential) to know the density of states a priori, i.e., before trying to find eigenfunctions of (1). We therefore begin by quoting the result from [33]. We suppose that the reader is familiar with the definition of the integrated density of states (ids), k, and Lyaponov exponent, γ (see, e.g. [2]). It will also be useful to consider the ν-dimensional analog of (1); the operator on $l^2(Z^\nu)$ given by

$$(hu)(n) = \sum_{|j|=1} u(n+j) + \lambda \tan\left(\pi \sum_1^\nu \alpha_j n_j + \theta\right) u(n). \tag{2}$$

We recall that in ν-dimensions, the free ids (i.e., k when $V=0$), is given by

$$k_0(E) = \int_{S_E} \left[\prod_{i=1}^{\nu} \frac{d\phi_i}{2\pi}\right]; \qquad S_E = \left\{\phi \mid \sum 2\cos\phi_i \leqslant E\right\}.$$

The free Lyapunov obeys ($\nu = 1$)

$$\begin{aligned} \cosh\gamma_0(E) &= 1, & |E| &\leqslant 2, \\ &= \tfrac{1}{2}|E|, & |E| &\geqslant 2, \end{aligned}$$

or for short

$$\cosh\gamma_0(E) = \tfrac{1}{2}[|1+\tfrac{1}{2}E| + |1-\tfrac{1}{2}E|].$$

Then, it is proven in [33] that

THEOREM 1. *For any λ, ν, θ and any $\{\alpha_1, ..., \alpha_\nu\}$ so that $\{1, \alpha_1, ..., \alpha_\nu\}$ are independent over the rationals, one has that*

$$k_\lambda(E) = \frac{1}{\pi} \int \frac{\lambda}{(E-E')^2 + \lambda^2}\, k_0(E')\, dE'. \tag{3}$$

In one dimension, the Lyapunov exponent is given by

$$\gamma_\lambda(E) = \frac{1}{\pi} \int \frac{\lambda}{(E-E')^2 + \lambda^2}\, \gamma_0(E')\, dE'. \tag{4}$$

Remarks. 1. Equation (4) follows from (3) and the Thouless formula.

2. One can write γ_λ in "closed" form, viz:

$$\cosh\gamma_\lambda(E) = \tfrac{1}{4}[\sqrt{(2+E)^2+\lambda^2} + \sqrt{(2-E)^2+\lambda^2}]. \tag{5a}$$

In one dimension, one can also write k_λ in closed form

$$k_\lambda(E) = \tfrac{1}{2} + \pi^{-1} \operatorname{Arc\,sin}(E/2\cosh\gamma_\lambda(E)). \tag{5b}$$

Equation (5) can be proven in many ways; e.g., one can compute the invariant measure for the Lloyd model and obtain (5) from that ([20]). As we will see in Section 2, one can write (5) in an especially compact form; Let

$$z_\lambda = -\exp(-\gamma_\lambda + i\pi k_\lambda). \tag{5c}$$

Then $|z_\lambda| < 1$ and

$$z_\lambda + z_\lambda^{-1} = E + i\lambda. \tag{5d}$$

3. It is remarkable that k is independent fo α!

4. Since $\text{spec}(H) = \{E \mid k(E+\varepsilon) - k(E-\varepsilon) > 0 \text{ for all } \varepsilon > 0\}$, (3) shows that for $\lambda \neq 0$, $\text{spec}(H) = (-\infty, \infty)$. The spectrum is not a Cantor set so this "typical" aspect of a.p. Jacobi matrices (see [25, 11, 1, 4, 32]) is absent.

5. In one dimension, $\gamma > 0$ for all E. This implies [20, 26, 23, 34, 12] there is no a.c. spectrum ($\equiv$ "extended states") no matter what the value of α ($\nu = 1$).

6. Notice that $\lambda\, dE/\pi(E^2 + \lambda^2)$ is exactly the density for $\lambda \tan(\theta)$ if θ is uniformly distributed.

It is an important remark of Sarnak [31] that spectral properties of a.p. Schrödinger operators should depend on Diophantine properties of the frequencies. This has been realized already in the almost Mathieu equation (see [2]) and will be illustrated more completely in the Maryland model. The measure of irrationality we will use is

$$L(\alpha) \equiv \overline{\lim_{n\to\infty}}\ -n^{-1} \ln(|\sin(\pi\alpha n)|). \tag{6}$$

This measures the degree of rational approximation, since if $\alpha - p/q$ is small, then $|\sin(\pi\alpha q)|$ is approximately $\pi q\,|\alpha - p/q|$. $L(\alpha) > 0$ means there is a sequence of approximants p_n/q_n with

$$|\alpha - p_n/q_n| \sim e^{-Lq_n}.$$

What we have called "Liouville numbers" in [2] is just $\{\alpha \mid L(\alpha) = \infty\}$. It is a dense G_δ. Standard Diophantine analysis shows that $\{\alpha \mid L(\alpha) > 0\}$ has Lebesgue measure zero. The theory of continued fractions [22] can be used to show that for any fixed L_0, $\{\alpha \mid L(\alpha) = L_0\}$ is a dense, uncountable set.

The analysis of GFP is when $L(\alpha) = 0$. The basic result, proven in Section 2, is

THEOREM 2. *Let $\nu = 1$, $L(\alpha) = 0$. Then for any $\lambda \neq 0$ and any θ, h has a complete set of eigenfunctions $\varphi_{m,\lambda,\theta}(\cdot)(m = 0, \pm 1, \ldots)$ with eigenvalues $e_{m,\theta}(\lambda)$. These eigenfunctions and eigenvalues are real analytic in λ for $\lambda \neq 0$ and are uniquely determined (up to phase) by this analyticity and*

$$\lim_{|\lambda|\to\infty} \varphi_{m,\lambda,\theta}(n) = \delta_{mn}.$$

Moreover

$$\lim_{|n|\to\infty} |n|^{-1} \ln[|\varphi_{m,\lambda,\theta}(n)|^2 + |\varphi_{m,\lambda,\theta}(n+1)|^2] = -\gamma(e_{m,\theta}(\lambda)). \tag{7}$$

Remarks. 1. While our presentation will differ from that of GFP, this result and the overall strategy of proof is due to GFP except for one result: The completeness of the eigenfunctions.

2. Of course, the point of things is not merely the existence of φ and e, but an explicit formula for them. e is determined by

$$k_\lambda(e_{m,\theta}(\lambda)) = \left(\alpha m + \frac{1}{2} - \frac{\theta}{\pi}\right)_f \tag{8}$$

where $(x)_f$ = fractional part of the real number x. Of course, the eigenvalues are dense by (8) and the density of $\{(\alpha m + 1/2 - \theta/\pi)_f\}_{m=-\infty}^{\infty}$ (or alternatively by $\sigma(H) = (-\infty, \infty)$ and completeness). We will refer to (8) as the *quantization condition*. The formula for φ (see Section 2) is quite complicated.

3. φ_m is exponentially localized near m in the sense that

$$|\varphi_m(n)| \leqslant Ce^{-A|n-m|} \tag{9}$$

where $C < \infty$, $A > 0$ can be chosen uniformly for λ, α, θ fixed and all m with e_m in a fixed compact set.

There is one important lesson to learn from the above:

Lesson 1. Dense point spectra in almost periodic problems tend to move analytically in λ.

Since the point spectra are dense, this is far from obvious from a perturbation theoretic point of view (even assuming that one can justify perturbation theory). Second order perturbation theory $\sum_{m\neq n} (e_m - e_n)^{-1} |(\varphi_m, V\varphi_n)|^2$ is not trivially finite: Because of Eq. (9) one can see it is finite but it is not easy to imagine a proof of analyticity by direct control of the perturbation series. Craig [8] has informed us that in the regime where his KAM procedure [9] is applicable, one obtains eigenfunctions analytic in λ.

In Section 2, we will also note the ν-dimensional analog of Theorem 2:

THEOREM 3. *Let $\alpha_1, \ldots, \alpha_\nu$ be irrational numbers obeying*

$$\left|\sum_{i=0}^{\nu} n_i \alpha_i\right| \geqslant C(|n_0| + \cdots + |n_\nu|)^{-K}$$

for some fixed C, K and all $(n_0, n_1, \ldots, n_\nu) \neq (0, 0, \ldots, 0)$ where $\alpha_0 = 1$. Then for any $\lambda \neq 0$, and any θ, h has a complete set of eigenfunctions $\varphi_{m,\lambda,\theta}(\cdot)$ $(m \in Z^\nu)$ with *distinct eigenvalue* $e_{m,\theta}(\lambda)$. *They are analytic in* λ *and uniquely determined up to phase by*

$$\lim_{|\lambda| \to \infty} \varphi_{m,\lambda,\theta}(n) = \delta_{mn}.$$

Remarks. 1. Equation (9) (with m, n now in Z^ν) continues to hold.

2. Since V only depends on the sole direction $\alpha \cdot n$, one might be surprised by the exponential decay in directions perpendicular to $\alpha \cdot n$. The lattice is sufficiently effectve at coupling things in the orthogonal directions.

3. Spencer [36] has suggested that multidimensional random Hamiltonians have simple spectrum. The above supports his suggestion, at least in the localized state regime.

4. A moral one might draw from one dimension where random (iid) potentials always localize (see, e.g. [13]) and where suitable a.p. potentials have extended states (see, e.g. [14]) is that almost periodic potentials have less of a tendency to localize than random analogs. Given this and Theorem 3, it is tempting to believe that the Lloyd model [24] (the random analog of (2)) has only localized states in all dimension, contrary to the conventional wisdom [38]. Since we have been warned by Thouless [38] not to yield to this temptation, we will not make any conjectures, but only ask: Does the model have extended states (for $\nu \geqslant 3$), and if so, what is the essential difference between the Lloyd and Maryland models?

Returning to $\nu = 1$, (7) says that to leading order

$$\| \Phi(n) \| \equiv \sqrt{|\varphi(n)|^2 + |\varphi(n+1)|^2} \sim e^{-\gamma|n|}.$$

The standard method to go beyong leading order is to Fourier transform, note that $\varphi(n)$ will have an analytic transform, find the nearest singularities and deform a contour in an inverse Fourier transform. This will not work so easily; in Section 3, we will prove:

THEOREM 4. *Let* $\varphi(n)$ *be an eigenfunction given in Theorem* 1. *Then for a.e.* θ,

$$\hat{\varphi}(k) \equiv \sum e^{-ikn} \varphi(n) \tag{10}$$

written as a function of $e^{ik} = z$ *is analytic in the annulus*

$$e^{-\gamma} < |z| < e^{\gamma}$$

with natural boundaries on both circles bounding this annulus.

Remarks. 1. That natural boundaries might occur was suggested to us by Prange [28], who noted that small divisors yield Taylor series which have a structure similar to those in lacunary series [38, 21].

2. This phenomenon is a general feature of small divisor problems; in Section 3, we will first study the warm-up problem, f obeying

$$f(ze^{i\pi\alpha}) - f(z) = z/1 - z \tag{11}$$

which for α having typical Diophantine properties has a solution analytic in $|z| < 1$ with a natural boundary on $|z| = 1$.

3. The above theorem suggests that $e^{\gamma|n|}\varphi(n)$ has very irregular behavior.

In Section 4, we will discuss some results for the case where $L(\alpha) > 0$. By mimicking the arguments in [2], one can prove

THEOREM 5. *If $L(\alpha) = \infty$, then* (1) *has purely singular continuous spectrum.*

Lesson 2. The density of states cannot distinguish between pure point and singular continuous spectrum—indeed, there is a pair of potentials (namely, (1) with some α obeying $L(\alpha) = 0$ and (1) with some α obeying $L(\alpha) = \infty$) with equal k's but with one yielding dense point and the other purely singular continuous spectrum!

That the above is qualitatively true should not be surprising to the thoughtful reader—but we find it remarkable that there are these examples with strictly equal k's and different spectrum. The a.c. spectrum is determined by k since k determines γ by the Thouless formula and σ_{ac} is the essential closure of $\{E \mid \gamma(E) = 0\}$ ([23, 24]), but it is *global* features of k that determine σ_{ac} and not the local, as can be seen by noting that by Aubry duality (see, e.g., [2]) the k for the almost Mathieu equation in the coupling constant regime where it is expected to have dense point spectrum and the k in the expected a.c. regime differ only by scaling.

Interesting phenomena occur if $L(\alpha) \neq 0, \infty$. As a preliminary to the next theorem, it is useful to note that the minimum value of γ_λ obeying (4) occurs at $E = 0$ where $\cosh \gamma_\lambda(E) = \sqrt{1 + \frac{1}{4}\lambda^2}$ or equivalently $\sinh \gamma_\lambda(0) = \frac{1}{2}\lambda$. Thus, condition (12) below is equivalent to $\min_E \gamma_\lambda(E) \geq L(\alpha)$. In Section 4, we will prove

THEOREM 6. *Suppose that $L(\alpha) \neq 0, \infty$ and*

$$|\lambda| \geq 2 \sinh L(\alpha) \equiv \lambda_c. \tag{12}$$

Then the operator, h, of Eq. (1) *has a complete set of eigenfunctions $\varphi_{m,\lambda,\theta}(\cdot)$ ($m = 0, \pm 1, ...$) and corresponding eigenvalues, with the same analyticity and infinite λ properties as in Theorem* 1 *(analyticity only on $|\lambda| \geq \lambda_c$) but the limit in* (7) *is now*

$$-[\gamma(e_{m,\theta}(\lambda)) - L(\alpha)]$$

for a.e. θ.

Lesson 3. It can happen that a stochastic Jacobi matrix has dense point spectrum with exponentially decaying eigenfunction, but where the exponential decay is *not* given by the Lyaponov exponent.

Remarks. 1. The rate of decay is *never* faster than $-\gamma$ (see [10]).

2. It is often assumed in the physics literature that the "inverse of the localization length" (= negative of the limit in (7)) is identical to γ; what we see is that while this is probably "usually" true (and is proven for suitable one dimensional random systems [7, 10]), it can fail.

Thus, in the (e, λ) plane, the exterior of the ellipse with foci $(\pm 2, 0)$ and semimajor axis $2 \cosh L(\alpha)$, is a region of localized states. We believe that the interior of the ellipse consists of "exotic states," i.e., that $\{e \mid \gamma(e) < L(\alpha)\}$ is purely singular continuous spectrum, but we have been unable to prove it. Instead, we will show in Section 4 a weaker result. Define $\tilde{\gamma}_\lambda(e)$ by

$$\tilde{\gamma}_\lambda(e) = \int_0^\pi \frac{d\theta}{\pi} \ln \|A_{e,\lambda}(\theta)\| \tag{13}$$

where $A_{e,\lambda}(\theta)$ is the 2×2 matrix

$$A_{e,\lambda}(\theta) = \begin{pmatrix} e - \lambda \tan(\theta) & -1 \\ 1 & 0 \end{pmatrix}. \tag{14}$$

While $\ln \|A_{e,\lambda}(\theta)\|$ diverges at $\theta = \pi/2$, the divergence is only logarithmic, so $\tilde{\gamma}_\lambda(e) < \infty$. By the ergodic theorem, $\gamma_\lambda(e) \leqslant \tilde{\gamma}_\lambda(e)$ (and both diverge as $\ln |E|$ for $|e|$ large). In Section 4, we will prove:

THEOREM 7. *If $L(\alpha) \neq 0, \infty$, then for a.e. θ, h has dense point spectrum on $\{e \mid \gamma(e) > L(\alpha)\}$ and purely singular continuous spectrum on $\{e \mid \tilde{\gamma}(e) < \frac{1}{2} L(\alpha)\}$.*

Thus, we have for suitable α situations where we have a region of singular continuous spectrum and another region of point spectrum. In between, we believe there is also purely singular continuous spectrum, but all we know there is that there is no absolutely continuous spectrum. We will prove Theorem 7 by applying Gordon's method [18]. It is probably that one could replace $\tilde{\gamma}(e)$ by $\gamma(e)$ (but the factor of $\frac{1}{2}$ remains) by working harder in implementing Gordon's method; indeed, in Section 4 we will prove:

THEOREM 8. *Let $h_\theta = h_0 + f(2\pi\alpha n + \theta)$ for a Lipschitz continuous function, f, on R of period 2π. Suppose for some real e, $\gamma(e)$, the Lyaponov exponent for h_θ, obeys $\gamma(e) < \frac{1}{2} L(\alpha)$. Then e is not an eigenvalue of h_θ for any θ.*

The unfortunate factor of $\frac{1}{2}$ seems intrinsic to Gordon's method.

We note one not quite standard notation we use in this paper. If A is a finite set, $\#(A)$ denotes the number of points in it.

2. Frequencies with $L(\alpha)=0$

In this section, we will give the basic analysis of normalizable eigenfunctions and, in particular, prove Theorems 2 and 3. We emphasize again that the analysis has much in common with that of GFP [19, 30]. In the next proposition, the reader should think of the example

$$A=\lambda^{-1}(e-h_0), \qquad B=\text{mult. by } \tan(\pi\alpha n+\theta) \tag{2.1}$$

where e is a real number. Notice that A is bounded.

Proposition 2.1. *Let A be selfadjoint and bounded and B selfadjoint, both on a Hilbert space, $\mathscr{H}$. Let $u\in D(B)$ and $c\in\mathscr{H}$ be related by*

$$c=(1+iB)u. \tag{2.2}$$

Then

$$Au=Bu \tag{2.3}$$

if and only if

$$\frac{(1-iB)}{(1+iB)}\,c=\frac{(1+iA)}{(1+iA)}\,c. \tag{2.4}$$

Proof. Given (2.2) and the fact that A is bounded (and $\operatorname{Ker}(1+iA)=\{0\}$), we see that (2.4) is equivalent to

$$(1+iA)(1-iB)u=(1-iA)(1+iB)u.$$

Multiplying out and canceling, one sees this is equivalent to (2.3). ∎

Proposition 2.2. *If A,B are given by* (2.1) *and u in l^2 obeys* (2.3), *then*

$$\varlimsup_{n\to\infty}\frac{1}{n}\ln[|u(n)|^2+|u(n+1)|^2]^{1/2}=\varlimsup_{n\to\infty}\frac{1}{n}\ln[|c(n)^2+|c(n+1)|^2]^{1/2}. \tag{2.5}$$

Proof. By (2.2), $|u(n)|\leqslant|c(n)|$ while $c=(1+iA)u$ so

$$|c(n)|\leqslant D(|u(n)|+|u(n+1)|+|u(n+1)|).$$

This shows that (2.5) holds. ∎

To prove completeness of eigenfunctions, we will need to discuss non-l^2 solutions of

$$u(n+1)+u(n-1)+\lambda\tan(\pi\alpha n+\theta)\,u(n)=eu(n). \tag{2.6}$$

Notice that (2.6) is meaningful for *any* sequence u and so is $[(1-iB)/(1+iB)]u$. Moreover, $Q \equiv (1+iA)/(1-iA)$ has matrix elements Q_{nm} of the form $q(n-m)$ with q decaying exponentially so Qc can be defined for any $c(n)$ obeying

$$|c(n)| \leqslant A(1+|n|)^k \tag{2.7}$$

for some A, k (such a c is called *polynomially bounded*).

PROPOSITION 2.3. *Let u, c be sequences obeying* (2.2). *If u is polynomially bounded and obeys* (2.3), *then c is polynomially bounded and obeys* (2.4) *and conversely.*

Proof. One needs only follow the proofs of Propositions 2.1 and 2.2 noting that $(1+iA)$ and $(1+iA)^{-1}$ map polynomially bounded sequences to polynomially bounded sequences. ▮

What makes (2.4) so useful is that if $B =$ multiplication by $\tan(\pi\alpha n+\theta)$, then $(1-iB)/(1+iB)$ is multiplication by

$$e^{-2\pi i\alpha n-2i\theta}$$

and thus (2.4) becomes

$$\left(\frac{1-iA}{1+iA}\,c\right)(n) = e^{-2\pi i\alpha n-2i\theta}c(n). \tag{2.8}$$

Equation (2.8) is thus similar to the Schrödinger equation for the potential $V(n) = \lambda e^{2\pi i\alpha n}$ analyzed by Sarnak [31] and the analysis of GFP, given (2.6) is very similar to that of Sarnak several years before (GFP did not know of Sarnak's work).

Define the Fourier transform of a sequence $f(n)$ by

$$\hat{f}(k) = \sum_n e^{-ink}f(n). \tag{2.9}$$

Equation (2.8) becomes (q is a function of λ and e as well as k)

$$q(k)\,\hat{c}(k) = e^{-2i\theta}\hat{c}(k+2\pi\alpha) \tag{2.10}$$

where

$$q(k) = -\frac{2\cos k - e - i\lambda}{2\cos k - e + i\lambda}. \tag{2.11}$$

Now $q(k)$ is an analytic function of $z = e^{ik}$ in a neighborhood of $|z| = 1$ and $q(k) \neq -1$ for any k. Thus we can define $\zeta(k)$ (a function of λ, e also) by

$$q(k) = e^{-i\zeta(k)}, \qquad -\pi < \zeta(k) < \pi. \tag{2.12}$$

The next few results involve the study of $\zeta(k)$. We note first that in ν-dimensions, the analysis goes through with minimal change: αn must be interpreted as $\alpha \cdot n$ and $\cos k$ replaced by $\cos k_1 + \cdots + \cos k_\nu$.

PROPOSITION 2.4. *In ν-dimensions the function $\zeta(k; \lambda, e)$ obeys:*

(i) *ζ is analytic in $z_j = e^{ik_j}$ in a neighborhood of $\{z \mid |z_j| = 1\} \equiv C$,*

(ii) $(1/(2\pi)^\nu)(\prod_i \int_{-\pi}^{\pi} dk_i)\, \zeta(k; \lambda, e) = 2\pi(k_\lambda(e) - 1/2)$ (2.13)

where $k_\lambda(e)$ is the integrated density of states.

Proof. (i) q is analytic in a neighborhood of C and q is never -1, so the logarithm is analytic in a neighborhood of C.

(ii) Let $A(\lambda, e)$ denote the L.H.S. of (2.13). If ζ is given by (2.11), (2.12), then by direct calculation

$$\frac{\partial \zeta}{\partial e} = 2\lambda/(h_0 - e)^2 + \lambda^2.$$

Thus

$$\frac{\partial A}{\partial e} = \int \frac{2\lambda}{(h_0 - e)^2 + \lambda^2} \frac{d^\nu k}{(2\pi)^\nu}.$$

Using (3), we see that

$$\frac{\partial k_\lambda}{\partial e} = \int \frac{1}{\pi} \frac{\lambda}{(y - e)^2 + \lambda^2} \frac{\partial k_0}{\partial y}(y)\, dy.$$

Moreover,

$$\frac{\partial k_0}{\partial y} = \int \delta(h_0(k) - y) \frac{d^\nu k}{(2\pi)^\nu}$$

so we see that

$$\frac{\partial A}{\partial e} = 2\pi \frac{\partial k_\lambda}{\partial e}.$$

Equation (2.13) follows if we note that $k_\lambda(e) \to 0$ as $e \to -\infty$ while, for all k, $\zeta(k, \lambda, e) \to -\pi$ as $e \to -\infty$. ∎

We want to know much more about ζ in dimension $\nu = 1$. As a preliminary, we need:

LEMMA 2.. *Let $\lambda > 0$, e real. Let $z_0(e, \lambda)$ denote the solution of*

$$z_0 + z_0^{-1} = e + i\lambda \tag{2.14a}$$

obeying $|z_0| < 1$. *Then*

$$z_0 = -\exp(-\gamma + i\pi k). \tag{2.14b}$$

Proof. We will separately prove (2.14) for $\operatorname{Re} \ln z_0$ and $\operatorname{Im} \ln z_0$. However, this is actually redundant: By general principles [37, 2, 10], for λ fixed, $-\gamma + i\pi k$ is the boundary value of an analytic function in $\operatorname{Im} e > 0$ and $\ln z_0$ is analytic there, so checking either real *or* imaginary parts proves the formula up to an additive constant (which can be evaluated by taking $e \to -\infty$).

We will compute in the next proposition that $(1/2\pi) \int_{-\pi}^{\pi} \zeta(k; e, \lambda) = 2 \operatorname{Im}(\ln z_0) - \pi$, so $\operatorname{Im} \ln z_0 = \pi k$ follows from the last proposition. We will sketch the tedious calculation that shows $\operatorname{Re} \ln(-z_0) = \ln |z_0|$ is $-\gamma$ where γ is given by (5a).

Let $\omega_0 = e + i\lambda$. Let $y + i\sqrt{x}$ be the value of $\sqrt{\omega_0^2 - 4}$ with $\operatorname{Im}(\sqrt{\omega_0^2 - 4}) > 0$. Then

$$y^2 - x = e^2 - \lambda^2 - 4; \qquad y\sqrt{x} = \lambda e.$$

Thus, eliminating y, we see that x obeys

$$x^2 + (e^2 - \lambda^2 - 4)x - \lambda^2 e^2 = 0. \tag{2.15}$$

Moreover, $z_0 = \frac{1}{2}(\omega_0 - \sqrt{\omega_0^2 - 4})$ and $z_0^{-1}(\omega_0 + \sqrt{\omega_0^2 - 4})$

$$\frac{\operatorname{Re} z_0}{\operatorname{Im} z_0} = \frac{e - y}{\lambda - \sqrt{x}} = \frac{e(e - y)}{\sqrt{x}\,(y - e} = -\frac{e}{\sqrt{x}}$$

and thus, if $z_0 = |z_0|\, e^{i\eta}$, we see that

$$e^{i\eta} = \frac{e - i\sqrt{x}}{\sqrt{e^2 + x}}. \tag{2.16}$$

Using this,

$$\begin{aligned}
2\cosh(\ln |z_0|) &= |z_0| + |z_0|^{-1} \\
&= \operatorname{Re}\left(z_0 + \frac{1}{z_0}\right) \cos \eta - \operatorname{Im}\left(z_0 + \frac{1}{z_0}\right) \sin \eta \\
&= e^2 + x/\sqrt{e^2 + x} \\
&= \sqrt{e^2 + x}.
\end{aligned}$$

Using (2.15), one sees that

$$e^2 + \lambda = \tfrac{1}{4}[a + b + \sqrt{ab}]; \qquad a = (2 + e)^2 + \lambda^2;\ b = (2 - e)^2 + \lambda^2$$

so $\sqrt{e^2 + x} = \frac{1}{2}(\sqrt{a} + \sqrt{b})$ showing that $\ln |z_0|$ is $-\gamma$ if γ is given by (5a). ∎

The lemma lets us calculate the Fourier coefficients of $\zeta(k)$ explicitly:

PROPOSITION 2.6. $\zeta(k) = \sum_{n=-\infty} \check{\phi}_n e^{-ikn}$ *where*

$$\check{\phi}_0 = 2\pi(k_\lambda(e) - \tfrac{1}{2}) \tag{2.17}$$

$$\check{\phi}_n = (-1)^n \frac{2}{n} e^{-\gamma|n|} \sin(\pi nk); \qquad n \neq 0. \tag{2.18}$$

Proof. Let $z = e^{ik}$. Then the roots of $z + z^{-1} - e - i\lambda = 0$ are the quantities z_0, z_0^{-1} of (2.14) and the roots of $z + z^{-1} - e + i\lambda = 0$ are $\bar{z}_0, \bar{z}_0^{-1}$. Thus, by (2.11):

$$q(k) = -\frac{(z - z_0)(z - z_0^{-1})}{(z - \bar{z}_0)(z - \bar{z}_0^{-1})}$$
$$= -\frac{\bar{z}_0}{z_0} \frac{(1 - z_0/z)(1 - zz_0)}{(1 - \bar{z}_0/z)(1 - z\bar{z}_0)}.$$

For $|x| < 1$, $\ln(1 - x) = -\sum_1^\infty x^n$, so if $|z_0| < |z| < |z_0|^{-1}$, we see that $-i\zeta = \ln q$ obeys

$$-i\zeta = \ln\left(-\frac{\bar{z}_0}{z_0}\right) - \sum_{n=1}^{\infty} \frac{(z_0)^n - (\bar{z}_0)^n}{n} [z^n + z^{-n}].$$

Now, by the definition of z_0 as the root of (2.14a) with $|z_0| < 1$, it is easy to see that $\operatorname{Im} z_0 < 0$, so we can normalize $\operatorname{Im} \ln(-z_0)$ to lie in $0, \pi$. With this convention and the fact that ζ is normalized by $-\pi < \zeta < \pi$, we see that

$$\ln\left(-\frac{\bar{z}_0}{z_0}\right) = i\pi - 2 \operatorname{Im} \ln(-z_0).$$

From this one reads off

$$\check{\zeta}_0 = -\pi + 2 \operatorname{Im} \ln(-z_0) \tag{2.19}$$

$$\check{\phi}_n = \frac{2}{|n|} \operatorname{Im}(z_0^{|n|}). \tag{2.20}$$

Given our previous calculation of (2.17) (Proposition 2.4), (2.19) yields the promised proof that $\operatorname{Im} \ln(-z_0) = k$. Given (2.14), (2.20) implies (2.18). ∎

With these preliminaries about ζ out of the way we can return to the analysis of (2.10). For the time being, we will discuss possible *continuous* solutions of (2.10). Later, we will show that when $L(\alpha) \equiv 0$, any distributional solution of (2.10) is continuous. Since $|q(k)| = 1$, (2.10) implies that $|\hat{c}(k)| = |\hat{c}(k + 2\pi\alpha)|$. If α is irrational (or, in the multidimensional case, if $(1, \alpha_1, \ldots, \alpha_\nu)$ are rationally independent), this equality *plus continuity* implies that $|c(k)|$ is constant, so without loss we suppose $|\hat{c}(k)| = 1$. As k_i runs from 0 to 2π, $\hat{c}$ defines a map from the circle to itself (since $\hat{c}$ is periodic) which has a winding number m_i. Thus, for some $m = (m_1, \ldots, m_\nu)$, a ν-tuple of integers

$$\hat{c}(k) = \exp(-im \cdot k - i\psi(k)) \tag{2.21}$$

where ψ is periodic in k (and continuous). Taking logarithms in (2.10) we obtain the basic equation

$$\psi(k + 2\pi\alpha) - \psi(k) = \zeta(k) - 2\theta - 2\pi(m \cdot \alpha + m_0). \tag{2.22}$$

The integer m_0 enters because, when we take logs of continuous function, the two sides must agree up to an additive term $2\pi i m_0$.

The average of the left side of (2.22) is zero, since ψ is periodic. Thus, we find a *consistency condition* for (2.22) to have a solution; namely using (2.13)

$$k_\lambda(e) = \tfrac{1}{2} + m \cdot \alpha + m_0 - \frac{\theta}{\pi}.$$

Since $0 < k_\lambda(e) < 1$, m_0 must be chosen to be the integral part of $\alpha \cdot m + 1/2 - \theta/\pi$; thus

PROPOSITION 2.7. *A* necessary *condition for* (2.10) *to have a solution is that for some ν-tuple $m = (m_1, \ldots, m_\nu)$ of integers*

$$k_\lambda(e) = \left(\alpha \cdot m + \frac{1}{2} - \frac{\theta}{\pi}\right)_f. \tag{2.23}$$

Once we prove that if the hypotheses of Theorem 3 hold (or if $L(\alpha) = 0$ when $\nu = 1$), (2.23) is also sufficient, we have the quantization condition (8) for the eigenvalues. Before turning to the sufficiency of (2.23), we want to note:

PROPOSITION 2.8. (a) *If $\theta \neq (\pi(\alpha \cdot m + \frac{1}{2}))_f$ for any m, then* (2.23) *has a unique solution for each m. Let $e_{m,\theta}(\lambda)$ denote this solution.*

(b) *$e_{m,\theta}(\lambda)$, for fixed m, θ is a real analytic function of λ on* $(0, \infty)$.

(c) *For λ, θ fixed, $\{e_{m,\theta}(\lambda)\}_{m \in Z^\nu}$ is dense in* $(-\infty, \infty)$ *and*

$$\lim_{M \to \infty} (2M + 1)^{-\nu} \#\{m \mid e_{m,\theta}(\lambda) \leqslant e_0; |m_i| \leqslant M\} = k_\lambda(e_0). \tag{2.24}$$

Proof. (a) Since k is *strictly* monotone and runs from 0 to 1 as e goes from $-\infty$ to ∞, (a) is obvious

(b) $k_\lambda(e)$ is jointly analytic in λ, e for $(e, \lambda) \in (-\infty, \infty) \times (0, \infty)$ and $(\partial k_\lambda / \partial e)(e) > 0$ for all λ, e. Thus, the implicit function theorem yields analyticity.

(c) By a celebrated result of Weyl [40], the distribution of $\{(\alpha \cdot m + 1/2 - \theta/\pi)_f\}$ is uniform, i.e., for $0 \leqslant x_0 \leqslant 1$

$$\lim_{M\to\infty} (2M+1)^{-\nu} \# \left\{ x = \left(\alpha \cdot m + \frac{1}{2} - \frac{\theta}{\pi}\right)_f \,\middle|\, x \leqslant x_0 ; |m_i| \leqslant M \right\} = x_0.$$

Equation (2.24) follows from this and monotonicity of k. ∎

Since ψ is continuous and periodic, it has a Fourier series expansion

$$\psi(k) = \sum_n \check{\psi}_n e^{-ikn}.$$

Since adding constants to ψ does not affect solubility of (2.22) (and $\hat{c} = e^{-i\psi - imk}$ is multiplied only by a phase), we can suppose $\check{\psi}_0 = 0$. From (2.22) we see that

$$\check{\psi}_n = (e^{2\pi i \alpha \cdot n} - 1)^{-1} \check{\zeta}_n \qquad (n \neq 0). \tag{2.25}$$

Thus

PROPOSITION 2.9. *Equation* (2.22) *has a continuous solution*, ψ, *if and only if*

(i) (2.23) *holds*,

(ii) $\check{\psi}_n$ *obeying* (2.25) *is the Fourier series of a continuous function. Moreover, we have if* $\nu = 1$:

$$\overline{\lim_{|n|\to\infty}} \frac{1}{n} \ln |\check{\psi}_n| \leqslant -[\gamma(e) - L(\alpha)]. \tag{2.26}$$

The final statement follows if we note that $|e^{2\pi i \alpha \cdot n} - 1| = |\sin \pi\alpha \cdot n|$ and the definition of $L(\alpha)$ and if we note that $\overline{\lim}_{n\to\infty} 1/n \ln |\check{\zeta}_n| = \gamma(n)$ by the explicit form of $\check{\zeta}_n$ (Proposition 2.6).

In particular if $L(\alpha) = 0$, $\check{\psi}_n$ decays exponentially and therefore (2.22) has a solution so long as (2.23) holds. The final step needed before the proofs of Theorems 2 and 3 will ensure completeness:

PROPOSITION 2.10. *Suppose either* $\nu > 1$ *and the hypothesis of Theorem* 3 *holds or* $\nu = 1$ *and* $\gamma(e) > L(\alpha)$. *Let* c *be a polynomially bounded solution of* (2.4) (*with* A, B *given by* (2.1)). *Then* $\hat{c}$ *is a continuous function and* c *decays exponentially.*

Proof. $\hat{c}$ will be a distributional solution of (2.10). Pick θ_0, so, for the e of relevance

$$k_\lambda(e) = \left(1 - \frac{\theta_0}{\pi}\right)_f.$$

Then, by the last proposition and the discussion following it, we can find $d(n)$ decaying exponentially with $|\hat{d}(k)| = 1$ and

$$\hat{d}(k + 2\pi\alpha) = e^{-i\zeta(k) + 2i\theta_0} \hat{d}(k).$$

Since $\hat{d}$ is analytic, we can form the distribution $l = \hat{c}/\hat{d}$ and it obeys $l(k + 2\pi\alpha) = e^{2i(\theta - \theta_0)} l(k)$. Taking Fourier transforms, $e^{-2\pi i \alpha n} \check{l}_n = e^{2i(\theta - \theta_0)} \check{l}_n$, so $\check{l}_n \neq 0$ for exactly one n. Therefore, $\hat{c} = e^{-2\pi i k n} \hat{d}$ for some n and thus c is a translate of d and thus also exponentially decaying. ∎

Proof of Theorems 2 *and* 3. As discussed above for each $m \in Z^\nu$, we can solve (2.22) and ψ is analytic in k. Moreover, when $\nu = 1$, ψ is analytic in $\{k \mid e^{-\gamma} < |e^{ik}| < e^{\gamma}\}$. Thus $e^{i\psi}$ is analytic in k, so $c = (e^{i\psi})\check{}$ decays exponentially, and, at least at rate γ in case $\nu = 1$. By Proposition 2.2, the same is true of $u = (1 + iB)^{-1} c$. By general principles [10], no eigenfunction can decay faster than $e^{-\gamma|n|}$, so the eigenfunctions obey Eq. (7). It is easy to see that as $\lambda \to \infty$, $\varphi_m \to \delta_{mn}$. Analyticity of ψ jointly in λ, k follows from the analyticity of e and the explicit formulas for ψ. Thus, all we need is to prove completeness.

The eigenfunction expansion theory of Berezanskii, Browder, Gel'fand, Garding and Kac (see [5, 35] for discussion and history) guarantees one that the spectral measures are supported on $\{e \mid hu = eu$ has a polynomially bounded solution$\}$. But we showed in Proposition 2.10 that any such solution has $u \in l^2$, so the spectral measures are supported by the countable set of (point) eigenvalues, i.e., h has only point spectrum. Moreover, every eigenvalue has an eigenfunction decaying exponentially and so $\hat{c}$ is continuous. Thus the eigenvalues must obey (2.23) proving completeness of the φ_m. ∎

The final issue we want to discuss in this section is why the density of states obtained by "m labeling" agrees with the density obtained by n-space normalization; i.e., how one can verify that k, given by (2.17) is the density of states not a priori knowing it is. The key will be the proof of the estimate in Eq. (9).

PROPOSITION 2.11. *Let A be an infinite matrix indexed by $n \in Z^\nu$ and let E_Δ be its spectral projections. Let $\Lambda_R = \{n \mid |n_i| \leqslant R)$ and let $\mathscr{X}_R$ denote the function on Z^ν which is* 1 *if $n \in \Lambda_R$ and zero otherwise. Suppose that $k(e) \equiv \lim_{R \to \infty} (2R+1)^{-\nu}$ $\operatorname{Tr}(E_{(-\infty, e)} \mathscr{X}_R)$ exists. Suppose that A has a complete set of eigenfunctions φ_m indexed by $m \in Z^\nu$ with energies e_m so that for all E_0, there are $A > 0$ and C so that*

$$|\varphi_m(n)| \leqslant C e^{-A|n-m|} \tag{2.27}$$

so long as $|e_m| < E_0$. Then

$$k(e) = \lim_{R \to \infty} (2R+1)^{-\nu} \#\{m \mid e_m < e \text{ and } |m_i| \leqslant R\}.$$

Proof. It suffices to prove that

$$\lim_{R \to \infty} (2R+1)^{-\nu} \#\{m \mid a < e_m < b \text{ and } |m_i| \leqslant R\} = k(b) - k(a). \tag{2.28}$$

Define

$$N_m(R) = \sum_{|m_i| \leqslant R} |\varphi_m(n)|^2.$$

Then, by definition of k,

$$k(b) - k(a) = \lim_{R \to \infty} (2R+1)^{-\nu} \sum_{a < e_m < b} N_m(R)$$

while the left hand side of (2.28) is

$$(2R+1)^{-\nu} \sum_{\substack{a < e_m < b \\ |m_i| < R}} 1.$$

It is easy to see that (2.28) follows from

$$N_m(R) \leqslant 1 \qquad \text{all} \quad m, R$$

$$|N_m(R) - \mathscr{X}_R(m)| \leqslant C_1 \exp(-\tfrac{1}{2}A \operatorname{dist}(m, \partial\Lambda_R))$$

and that the latter inequality follows from (2.27). ∎

Thus the GFP calculation of k is justified by our proof of completeness and the proof of (2.27) ($\equiv$ Eq. (9)):

PROPOSITION 2.12. *Let $\varphi_m(n)$ denote the eigenfunction of Theorem 2 or 3 for λ, θ fixed. Then for each a, b there exist C, A so that (2.27) holds for all m with $a < e_m < b$.*

Proof. Let $\eta_m(n) = (1 + iB)\,\varphi_m(m+n)$. Then, for a normalization constant, ζ_m, discussed below

$$\hat{\eta}_m(k) = \zeta_m e^{-i\psi(k)}.$$

By the explicit formula for ψ, one has uniform bounds on ψ within a fixed annulus for all m with $a < e_m < b$. By the Payley–Weiner theorem (deforming the contour integral for $\eta(m)$ in the inverse transform), we obtain

$$|\eta_m(n)| \leqslant \zeta_m C e^{-A|n|}$$

which implies (2.27) for η_m (if we can obtain an upper bound on ζ_m), and thus for φ_m since $|\varphi_m(n)| \leqslant |\eta_m(n)|$.

To bound ζ_m, we note that $\zeta_m = \|\hat{\eta}_m\|^{L^2} = C\,\|\eta_m\|_{l^2}$. Moreover, since

$$\eta(n) = (1 + i\lambda^{-1}e)\,u(n) - i\lambda^{-1}(u(n+1) + u(n-1))$$

$\|\eta\|_{l^2} \leqslant \text{const}\,\|u\|_{l^2}$ where the constant is bounded as e goes through compacts. Thus ζ_m is bounded. ∎

Remark. Actually, one can justify the GFP calculation of the density of states and prove there is only point spectrum from (2.27) alone. For, only knowing the

GFP eigenfunctions are a subset of all eigenfunctions, without completeness, yields via our proof of Proposition 2.11

$$\lim(2R+1)^{-\nu}\#\{m \mid a<e_m<b; |m_i|\leqslant R\}\leqslant \lim_{R\to\infty}(2R+1)^{-\nu}\operatorname{Tr}(\mathscr{X}_R E^{\text{p.p}}_{(a,b)})$$

where $E^{\text{p.p.}}$ is the pure point part of E. From this, by taking $a\to-\infty$, $b\to\infty$, one finds that $(2R+1)^{-\nu}\operatorname{Tr}(\mathscr{X}_R E^{\text{p.p.}})\to 1$ and thus for a.e. θ, there is only point spectrum and k is given by (2.28). This argument does not prove completeness of the GFP eigenfunctions: Without additional argument, a set of eigenfunctions of zero density might fail to have $\hat{c}$ continuous.

3. Natural Boundaries

Our goal in this section is to prove Theorem 4. This involves a general aspect of small divisor problems. While we know of no explicit previous work, we would not be surprised if some existed. As a warm-up, we analyze Eq. (10):

THEOREM 3.1. *Suppose that $L(\alpha)=0$. Then there is a unique function f, analytic on $\{z \mid |z|<1\}$, obeying $f(0)=0$ and*

$$f(ze^{2\pi i\alpha})-f(z)=z/1-z. \tag{3.1}$$

f has a natural boundary on $|z|=1$.

Proof. Since $z/1-z=\sum_1^\infty z^n$, we can expand

$$f(z)=\sum_1^\infty a_n z^n; \qquad a_n=(e^{2\pi i\alpha n}-1)^{-1}.$$

$L(\alpha)=0$ implies that the series for f converges uniformly on each disc, $\{z \mid |z|<R\}$, with $R<1$. This proves existence and uniqueness.

Call an integer, m, *regular* if and only if $\lim_{r\uparrow 1}f(re^{2\pi i\alpha m})\equiv b_m$ exists and is finite. Equation (3.1) implies that any $m\neq 0$ is regular if and only if $m+1$ is also regular. Thus either all $m\geqslant 1$ are regular or none are regular. Similarly for all $m\leqslant 0$. Moreover, it cannot be that both 0 and 1 are regular since $\lim_{r\uparrow 1}[f(re^{2\pi i\alpha})-f(r)]=\infty$. Thus either no $m\geqslant 1$ or no $m\leqslant 0$ is regular (or both). It follows that arbitrarily near any $e^{i\theta}$, there are points where f has an infinite or no limit, so f cannot be analytic in any neighborhood of any $e^{i\theta}$. ∎

Remark. J. Avron has remarked that $f(z)=\lim_{\lambda\uparrow 1}f_\lambda(z)$ with $f_\lambda(z)=z\lambda\sum_{n=0}^\infty z^n(\mu^{n+1}-\lambda)^{-1}$ $(\mu\equiv e^{2\pi i\alpha})$ and since $(\gamma-w)^{-1}=\sum_{m=0}^\infty w^m\gamma^{-m-1}$ if $|\gamma|>|w|$, we have for $|z|<1$, $|\lambda|<1$:

$$f_\lambda(z)=z\lambda\sum_{n,m\geqslant 0}z^n\lambda^m\mu^{(n+1)(m+1)}=z\lambda\sum_{m=\theta}^\infty\lambda^m(\mu^{m+1}-z)^{-1}$$

i.e., for $|z| < 1$

$$f(z) = \lim_{\lambda \uparrow 1} z \sum_{m=1}^{\infty} \lambda^m (e^{2\pi i m \alpha} - z)^{-1}$$

which is very suggestive and links things to examples in Titchmarsh [39]. However, one should not take the occurrence of poles at $e^{2\pi i \alpha m}$ for $m \geqslant 1$ and not for $m \leqslant 0$ seriously. If we replace $\lambda(\mu^{n+1} - \lambda)^{-1}$ by $(\lambda\mu^{n+1} - 1)^{-1}$ we get a representation

$$f(z) = \lim_{\lambda \uparrow 1} z \sum_{n=0}^{\infty} \lambda^m (z - e^{-2\pi i m \alpha})^{-1}.$$

We can abstract the argument in the above proof to get:

THEOREM 3.2. *Let f be a function analytic in an annulus $\{z \mid a < |z| < b\}$ and suppose that for some θ we have*

(i) $\lim_{r \uparrow b} |f(re^{i\theta + 2\pi i \alpha}) - f(re^{i\theta})| = \infty$

(ii) $\lim_{r \uparrow b} f(re^{i\theta + 2\pi i \alpha(m+1)}) - f(re^{i\theta + 2\pi \alpha m})$ *exists and is finite for all* $m \neq 0$.

Then f has a natural boundary on the circle $\{|z| \mid |z| = b\}$.

Proof of Theorem 4. Look first at the phase ψ is given in (2.21). The right side of (2.22) has logarithmic singularities on the boundaries of the annulus at the points z_0, $\bar{z}_0$ (with $|z| = e^{-\gamma}$) and at z_0^{-1}, $\bar{z}_0^{-1}$ (with $| = e^{\gamma}$). Since $z_0 = e^{-\gamma + ik}$, Theorem 3.2 implies that $\{z \mid |z| = e^{-\gamma}\}$ and $z \mid |z| = e^{\gamma}\}$ are natural boundaries unless $2k$ is a multiple of α. Looking at (2.23), one sees that this can only happen if $\theta/\pi = n - 1/2 + l\alpha/2$ for some integers n, l. Eliminating this countable set of θ, we have natural boundaries for all m. Thus for a.e. θ, ψ_m has a natural boundary for each m.

If $\hat{c} = e^{-i\psi - imk}$ were analytic at any point, z_1, on the boundary, it would have either no zero or an isolated zero at z_1, so ψ_m would be analytic near z_1 and thus ψ_m would not have natural boundaries. We conclude that $\hat{c}$ has natural boundaries. Since

$$\hat{\varphi}(k) = \left\{1 + \frac{i\lambda^{-1}}{2} [2e - e^{ik} - e^{-ik}]\right\}^{-1} \hat{c}(k)$$

we see that $\hat{\varphi}$ has natural boundaries also. ∎

4. FREQUENCIES WITH $L(\alpha) \neq 0$

We begin with the analysis in the region where $\gamma_\lambda(e) > L(\alpha)$. The following combined with the analyticity arguments in Section 2 proves Theorem 6 and part of Theorem 7.

THEOREM 4.1. *Fix λ, α. In the region where $\gamma_\lambda(e) > L(\alpha)$ the operator, h, of*

Eq. (1) *has only point spectrum and for a.e.* θ, *the limit in Eq.* (7) *is* $-[\gamma(e_{n,\theta}(\lambda))-L(\alpha)]$.

Proof. Our analysis in Section 2, both of solutions with $\hat{c}$ continuous and of polynomially bounded solutions, works in the region $\gamma > L$ since the decay of coefficients of ζ overcomes the growth of $(e^{2\pi i\alpha m}-1)^{-1}$. Suppose that e is an eigenvalue with $L(k_\lambda(e))=0$. Since $L(\alpha)=0$ for a.e. α, k_λ is smooth and e is continuous in θ, $L(k_\lambda(e))=0$ for a.e. θ. $L(k)$ enters because of the occurrence of $\sin(\pi nk)$ in Eq. (2.18). By that equation, $\lim(1/n)\ln\check{\zeta}_n=-\gamma$ (limit and *not* $\overline{\lim}$) so that ψ has a singularity on the circles of radii $e^{\pm(\gamma-L)}$. We claim that $\hat{c}$ must also have singularities on that circle, for if $\hat{c}$ did not have singularities, ψ would just have logarithmic singularities and the density of n's for which $\check{\psi}_n \geqslant e^{-|n|(\gamma-L+\delta)}$ would be positive for some small δ. But, since $L>0$, this density is zero by the lemma below. It follows that $\hat{c}$ has singularities and thus so does $\hat{\varphi}_{e,\lambda,\theta}$. ∎

LEMMA 4.2. *Let* $a>0$ *and* α *irrational. Then*

$$N^{-1}\#\{q\leqslant N \,|\, |\sin(\pi\alpha q)|\leqslant e^{-aq}\}\to 0$$

as $N\to\infty$.

Proof. Fix c. By Weyl's theorem [40].

$$\lim_{N\to\infty} N^{-1}\#\{q\leqslant N \,|\, |\sin(\pi\alpha q)|\leqslant c\}=\frac{1}{2\pi}\,|\{\theta \,|\, |\sin(\theta)|\leqslant c\}|.$$

But

$$\overline{\lim}\, N^{-1}\#\{q\leqslant N \,|\, |\sin(\pi\alpha q)|\leqslant e^{-aq}\}\leqslant \overline{\lim}\, N^{-1}\#\{q\leqslant N \,|\, |\sin(\pi\alpha q)|\leqslant c\}$$

for any $c>0$, so the $\overline{\lim}$ is 0. ∎

The remainder of this section deals with situation $\gamma_\lambda(e)\leqslant L(\alpha)$. Since the set-up is intrinsically complicated and because our results are not sharp, the material is unfortunately denser than the preceding material.

Since $\gamma_\lambda(e)>0$ for all e, the Pastur–Ishii theorem [20, 26] implies there is no a.c. spectrum. On the other hand, since $dk_\lambda/de>0$ for all e, the spectrum is all of $(-\infty,\infty)$. Thus, Theorem 5 and the remaining part of Theorem 7 follow from

THEOREM 4.3. *There is a set, S, of* θ *of measure zero, so that if* $\tilde{\gamma}_\lambda(e)<\frac{1}{2}L(\alpha)$, e *is not an eigenvalue of* h_θ *for* any $\theta\notin S$.

As a warm-up for the proof of Theorem 4.3, we want to prove Theorem 8 which is of interest in its own right. To keep this discussion self-contained, we begin with the proof of a lemma of Gordon [18]:

LEMMA 4.4. *Let A be a* 2×2 *invertible matrix. Let x be a unit vector in* C^2. *Then* $\max(\|Ax\|, \|A^2x\|, \|A^{-1}x\|, \|A^{-2}x\|)\geqslant\frac{1}{2}$.

Proof. Let $a_2 A^2 + a_1 A + a_0 = 0$ be the characteristic equation for A normalized so $\max(|a_i|) = 1$ and so that some a_i with $|a_i| = 1$ has $a_i = 1$. Suppose $a_1 = 1$. Then applying the A-equation to $A^{-1}x$ we see that $x = -a_2 Ax - a_0 A^{-1}x$. Since $|a_0|$, $|a_2| \leqslant 1$, one of Ax and $A^{-1}x$ must have norm at least $\frac{1}{2}$. Similar arguments work if $a_0 = 1$ or $a_2 = 1$. ∎

In controlling transfer matrices, we will need:

LEMMA 4.5. *Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be $L \times L$ matrices with $\|B_j\| = \beta_j$ and $\|A_{l+j-1} \cdots A_j\| \leqslant Ce^{Dl}$ for some C, D. Then*

$$\|(A_n + B_n) \cdots (A_1 + B_1)\| \leqslant Ce^{Dn} \prod_{j=1}^{n} (1 + Ce^{-D}\beta_j) \tag{4.1}$$

and

$$\|(A_n + B_n) \cdots (A_1 + B_1) - A_n \cdots A_1\| \leqslant Ce^{Dn} \left\{ \left[\prod_{j=1}^{n} (1 + Ce^{-D}\beta_j) \right] - 1 \right\}. \tag{4.2}$$

Proof. Note that the norm of a product of n matrices, k of which are B's obeys $\|A_n \cdots B_{j_k} \cdots B_{j_1} \cdots A_1\| \leqslant C^{k+1} e^{D(n-k)} \prod_{l=1}^{k} \beta_{j_l}$. Summing up these estimates yields (4.1) and (4.2). ∎

We can now make concrete the argument of Gordon for operators h_θ defined by

$$h_\theta = h_0 + f(2\pi\alpha n + \theta) \tag{4.3}$$

where f obeys

$$f(x + 2\pi) = f(x); \qquad |f(x) - f(y)| \leqslant C_0 |x - y|. \tag{4.4}$$

Fix e and let $A(\theta)$ be the matrix

$$\begin{pmatrix} e - f(\theta) & -1 \\ 1 & 0 \end{pmatrix}$$

and suppose that

$$\|A(\theta + 2\pi\alpha(l-1)) \cdots A(\theta)\| \leqslant Ce^{Dl} \tag{4.5}$$

for some C, D and all θ. Then Gordon's argument and Lemma 4.5 yield

THEOREM 4.6. *Let f obey* (4.4) *and suppose* (4.5) *holds and that*

$$D < \tfrac{1}{2} L(\alpha). \tag{4.6}$$

Then e is not an eigenvalue of h_θ for any θ.

Proof. Pick ε with $2(D+\varepsilon) < L(\alpha)$ and choose rationals rationals $p_k/q_k = \alpha_k$ so

$$\left| \alpha - \frac{p_k}{q_k} \right| \leqslant e^{-2(D+\varepsilon)q_k}.$$

Suppose that u is any non-zero solution of $h_\theta u = eu$ and let u_k be the solution of $[h_0 + f(2\pi\alpha_k n + \theta)]\, u_k = eu_k$ with the same values as u at $n = 0, 1$. Let

$$\Phi_k(n) = \begin{pmatrix} u_k(n+1) \\ u_k(n) \end{pmatrix} \quad \Phi(n) = \begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix}.$$

Normalize u so $\|\Phi(0)\| = 1$. Then

$$\sup_{0 \leqslant n \leqslant 2q_k} \|\Phi_k(n) - \Phi(n)\|$$
$$\leqslant \sup_{0 \leqslant n \leqslant 2q_k} [\|A(\theta + 2\pi\alpha n) \cdots A(\theta + 2\pi\alpha) - A(\theta + 2\pi\alpha_k) \cdots A(\theta + 2\pi\alpha_k)\|$$

so using Lemma 4.5, (4.5) and (4.4) (which implies $\|A(\phi) - A(\phi')\| \leqslant C_0 |\phi - \phi'|$)

$$\sup_{0 \leqslant n \leqslant 2q_k} \|\Phi_k(n) - \Phi(n)\| \leqslant Ce^{2Dq_k}\{(1 + 2Ce_0 q_k e^{-2(D+\varepsilon)q_k})^{q_k} - 1\}. \tag{4.7}$$

Now $(1+x) \leqslant e^x$ and $|e^y - 1| \leqslant |y|\, e^y$ for $x, y \geqslant 0$ so

$$\text{RHS of (4.7)} \leqslant Ce^{2Dq_k}(2Ce_0)\, q_k^2 e^{-2(D+\varepsilon)q_k} \exp(W_k)$$

where $W_k = 2Ce_0 q_k^2 e^{-2(D+\varepsilon)q_k}$. Since $\varepsilon > 0$, we conclude that

$$\lim_{k \to \infty} [\sup_{0 \leqslant n \leqslant 2q_k} \|\Phi_k(n) - \Phi(n)\|] = 0. \tag{4.8}$$

Since $\det A(\theta) = 1$, (4.5) implies that

$$\|A(\theta - 2\pi\alpha(l-1))^{-1} \cdots A(\theta - 2\pi\alpha)^{-1} A(\theta)^{-1}\| \leqslant Ce^{Dl}$$

also, so $\sup_{0 \leqslant n \leqslant 2q_k}$ in (4.8) can be replaced by $\sup_{|n| \leqslant 2q_k}$. But by Lemma 4.4 (applied to the matrix $A(2\pi\alpha_k q_k + \theta) \cdots A(2\pi\alpha_k + \theta)$)

$$\min(\|\Phi_k(\pm q_k)\|\, \|\Phi_k(\pm 2q_k)\|) \geqslant \tfrac{1}{2}$$

and thus

$$\overline{\lim_{|n| \to \infty}} \|\Phi(n)\| \geqslant \frac{1}{2}$$

implying that u is not in l^2. ∎

Theorem 8 follows from Theorem 4.6 and

THEOREM 4.7. *Let h_θ be given by* (4.3) *with f bounded and continuous and $f(x+2\pi)=f(x)$. Let*

$$A_e(\theta)=\begin{pmatrix} e-f(\theta) & -1 \\ 1 & 0 \end{pmatrix}$$

and let $\gamma(e)$ be the Lyapunov exponent for h_θ. Then for all e:

$$\lim_{n\to\infty} n^{-1} \ln \sup_\theta \|A_e(\theta+2\pi\alpha n)\cdots A_e(\theta+4\pi\alpha)A(\theta+2\pi\alpha)\| = \gamma(e). \tag{4.9}$$

Remarks. 1. γ is defined, so that (4.9) holds for a.e. θ if "$\sup_\theta$" is dropped from the left side. That the $\overline{\lim}$ is at most $\gamma(e)$ even for the anomalous values of θ where the limit fails to exist or has an a typical value is a result of Craig–Simon [10]. We use their ideas to prove (4.9).

2. While we state this for an almost periodic f whose hull is the circle, the argument is valid for *any* almost periodic f.

Proof. Define $f_n(\theta, e)=\ln \|A_e(\theta+2\pi\alpha n)\cdots A_e(\theta+2\pi\alpha)\|$ and

$$f_n(e)=\sup_\theta f_n(\theta, e).$$

Each $f_n(e,\theta)$ is subharmonic in e, so since f_n is easily seen to be continuous in e, f_n is subharmonic. Moreover, $f_{n+m}(\theta, e)\leqslant f_m(\theta+2\pi\alpha n, e)+f_n(\theta, e)$ so

$$f_{n+m}(e)\leqslant f_n(e)+f_m(e).$$

Thus $\lim_{n\to\infty} n^{-1}f_n(e)=\inf_k 2^{-k}f_{2^k}(e)\equiv\tilde{\gamma}(e)$ exists and is subharmonic. Since γ is also subharmonic, we need only show that $\tilde{\gamma}(e)=\gamma(e)$ for e with $\operatorname{Im} e>0$. But, the Thouless argument that

$$n^{-1}f_n(\theta, e)\to\int \ln(e-e')\,dk'(e')$$

for $\operatorname{Im} e'\neq 0$ depends only on the rate of convergence of the finite volume density of states to the infinite volume density of states, i.e., if $dk_\theta^l(e)$ is the finite volume sum of delta functions, we are concerned with $\int \ln|e-e'|\,dk_\theta^l(e')$. But by the next lemma this is uniform in θ, so $n^{-1}f_n(\theta, e)$ converges to $\gamma(e)$ uniformly in θ if $\operatorname{Im} e\neq 0$. Thus $\sup_\theta n^{-1}f_n(\theta, e)\to\gamma(e)$. Therefore $\gamma(e)=\tilde{\gamma}(e)$ for all e with $\operatorname{Im} e\neq 0$ and so for all e. ∎

LEMMA 4.8. *Let $dk_\theta^l(e)$ be the density of states for h_θ on the interval $[0, l-1]$. Let $g(e)$ be a continuous function on R. Then*

$$\int g(e)\,dk_\theta^l(e)\to\int g(e)\,dk(e)$$

uniformly in θ.

Proof. By a standard limiting argument, we need only prove this for f of the form $g(e) = e^{-ite}$. But then, if h_θ^l is an $l \times l$ matrix:

$$\int e^{-ite}\, dk_\theta^l(e) = l^{-1}\,\mathrm{Tr}(e^{-ith_\theta^l}) = l^{-1} \sum_{j=0}^{l-1} (e^{-ith_\theta^l})(j,j).$$

Now, by expanding, $e^{ith_\theta^l}$ in a perturbation series

$$e^{-it(h_0^l + f_\theta)} = e^{-itf} - it \int_0^1 ds\, e^{-i\alpha tf} h_0^l e^{-i\alpha(1-t)f} + \cdots$$

we see that uniformly in θ

$$|e^{-ith_\theta^l}(j,j) - e^{-ith_\theta^\infty}(j,j)| \leqslant C \exp(-\min(|j|, |l-j|))$$

so

$$l^{-1} \sum_{j=0}^{l-1} |e^{-ith_\theta^l}(j,j) - e^{ith_\theta^\infty}(j,j)| \to 0$$

uniformly in θ.

Moreover (see, e.g., [2]), $e^{-ith_\theta^\infty}(j,j) = F(2\pi\alpha j + \theta)$ with F continuous on the circle. The convergence of the Riemann sums

$$l^{-1} \sum_{j=0}^{l-1} F(2\pi\alpha j + \theta)$$

to $\int F(\phi)\, d\phi/2\pi$ is uniform in θ. ∎

To prove Theorem 4.3, the first thing we must face is that $\tan(\theta)$ is not Lipschitz. So long as $\tan(\theta + \pi\alpha j)$ does not get too large, an estimate is possible. The goal of the next lemma is to show that by eliminating a set of measure zero of θ, we can be sure that $\tan(\pi\alpha j + \theta) - \tan(\pi\alpha_k j + \theta)$ is very small for $|j| \leqslant 2q_k$:

LEMMA 4.9. *Let $\alpha_k = p_k/q_k$ be the continued fraction approximants for a number α with $L(\alpha) > 0$. Define*

$$S_k(\varepsilon) = \left\{ \theta \;\middle|\; \left| \left(\frac{\theta}{\pi} + \alpha j \right)_f - \frac{1}{2} \right| \right\} \leqslant 2e^{-\varepsilon q_k/4} \qquad \text{some } j \text{ with } |j| \leqslant 2q_k.$$

Let

$$S = \bigcup_{\varepsilon>0} \bigcap_k \bigcup_{l=k}^{\infty} S_l(\varepsilon).$$

Then, S has measure zero and for any $L < L(\alpha)$, and $\theta \notin S$, there is a subsequence $\tilde{q}_k$ of the q_k and C_k so that

$$|\tan(\pi\alpha n + \theta) - \tan(\pi\tilde{\alpha}_k n + \theta)| \leqslant C_k e^{-L\tilde{q}_k} \tag{4.10}$$

for all n with $|n| \leqslant 2\tilde{q}_k$.

Proof. Clearly $|S_k(\varepsilon)| \leqslant (4\pi)(2q_k + 1) e^{-\varepsilon q_k/4}$, so $\sum_{l=k}^{\infty} |S_l(\varepsilon)| \to 0$ as $k \to \infty$, so each set $\bigcap_k \bigcup_{L=k}^{\infty} S_l(\varepsilon)$ has measure zero. Since the sets increase as ε decreases, $\bigcup_{\varepsilon>0} = \bigcup_{\varepsilon=2^{-n}}$ and so S has measure zero.

Given $L < L(\alpha)$, pick ε with $L + \varepsilon < L(\alpha)$ and then, given $\theta \notin S$, k_0 so $\theta \notin S_l(\varepsilon)$ if $l \geqslant k_0$. Pick a subsequence, $\tilde{q}_k$, of the $\{q_k\}_{k \geqslant k}$, so that

$$\left| \alpha - \frac{\tilde{p}_k}{\tilde{q}_k} \right| \leqslant (2\tilde{q}_k)^{-1} e^{-\tilde{q}_k(L+\varepsilon)}.$$

Then, for $|j| \leqslant 2q_k$, we have that

$$\left| \left(\frac{\theta}{\pi} - \beta_j\right)_f - \frac{1}{2} \right| \geqslant e^{-\varepsilon\tilde{q}_k/4} \qquad \text{all } \beta \text{ between } \alpha \text{ and } \alpha_k.$$

Thus, since $(d/dx)(\tan x) \leqslant C_1[(x/\pi)_f - 1/2]^{-2}$, we see that

$$|\tan(\pi\alpha j + \theta) - \tan(\pi\tilde{\alpha}_k j + \theta)| \leqslant C_1 e^{\varepsilon\tilde{q}_k/2} \pi e^{-\tilde{q}_k(L+\varepsilon)}$$

as desired. ∎

Proof of Theorem 4.3. By following the proof of Lemma 4.5, we see that if $\theta \notin S$ and if L, $\tilde{q}_k$ are picked obeying (4.10), then for $j \leqslant 2q_k$

$$\begin{aligned} &\|A(\theta + 2\pi\alpha j) \cdots A(\theta + 2\pi\alpha) - A(\theta + 2\pi\alpha_k j) \cdots A(\theta + 2\pi\alpha_k)\| \\ &\quad \leqslant \prod_{l=1}^{j} \|A(\theta + 2\pi\alpha l)\| \, \{(1 + C_k e^{-L\tilde{q}_k})^j - 1\}. \end{aligned} \tag{4.11}$$

Following the argument between (4.7) and (4.8) we see that for k large

$$\text{LHS of } (4.11) \leqslant C_k' q_k e^{-L\tilde{q}_k} \prod_{l=1}^{j} \|A(\theta + 2\pi\alpha l))\|.$$

By the ergodic theorem applied to $\ln \|A(\theta)\|$, we see that for a.e. θ

$$\lim_{j\to\infty} |j|^{-1} \ln \prod_{l=1}^{j} \|A(\theta + 2\pi\alpha l)\| = \tilde{\gamma}(e). \tag{4.12}$$

Parenthetically, we note that the set of θ which needs to be eliminated for (4.12) to hold can be chosen independently of e: For we can arrange that (4.12) hold for all rational e and then note that as $e_l \to e$, $\ln \|A_{e_l}(\theta)\|$ converges to $\ln \|A_e(\theta)\|$ uniformly in θ (at the point where $\ln \|A_e(\theta)\| = \infty$, one has for any fixed e, e' that $\ln[\|A_e(\theta)\|/\|A_{e'}(\theta)\|] = 0$).

Thus, if $L > 2\tilde{\gamma}(e)$, we know that for a.e. θ

$$\lim_{k\to\infty} [\text{LHS of (4.12) for } j = \pm q_k \pm 2q_k] = 0.$$

Thus, by mimicking the proof of Theorem 4.6, we see that h_θ has no l^2 eigenfunctions so long as we have $L(\alpha) > 2\tilde{\gamma}(e)$. ∎

We should close our discussion of Theorem 7 by explaining why we did not use the "obvious" argument that when $\gamma(e) < L(\alpha)$, there are no eigenfunctions. The same argument that leads to the proof of (2.26) shows that if $\gamma(e) < L(\alpha)$, then (2.22) can have no solution ψ in l^2. It is tempting to therefore conclude that (2.10) then has no solution. Certainly it has no continuous solution, but we need to show it has no L^2 solution. If $\hat{c}$ is an L^2 solution of (2.10), then it is not hard to show that $|\hat{c}(k)| = 1$ and thus $\hat{c}(k) = e^{i\beta(k)}$ where β obeys

$$\begin{aligned} \beta(k + 2\pi\alpha) - \beta(k) &= \varphi(k) - 2\theta - 2\pi m_0(k) \\ \beta(k + 2\pi) - \beta(k) &= 2\pi m_1(k) \end{aligned} \tag{4.13}$$

with $m_0(k)$ and $m_1(k)$ integrally valued. But the lack of continuity does not allow us to conclude that m_0 and m_1 are constant. Thus, we cannot conclude that a solution of (4.13) will yield a solution of (2.22). That is, we do not know how to rigorously eliminate the possibility of eigenfunctions decaying so slowly that $\hat{c}$ is discontinuous when e is in the set $\{e \mid \gamma(e) < L(\alpha);\ \tilde{\gamma}(e) > \frac{1}{2}L(\alpha)\}$.

Finally, we should complete our analysis of $L(\alpha) \neq 0$ by remarking that the most significant questions concerning the operators with $L(\alpha) \neq 0$ remain open. The BGK theory of eigenfunctions expansions ([35] and references therein) assure us that for each θ, λ, α there is a set S in R so that (i) for every $E \in S$, $h_\theta u = Eu$ has a polynomially bounded solution, (ii) S has measure one with respect to the spectral measure. When there is some singular continuous spectrum, S must be uncountable and of Lebesgue measure zero. Identifying what E lies in S we call finding the *quantization condition.* For $L(\alpha) = 0$, we have found S in Eq. (8). We will not regard the general α case as being "solved" until the quantization condition is solved in general. The structure of the eigenfunctions is also of considerable interest. Here the recent work of Prange *et al.* [29] may be of significance: They do not find the quantization condition but do find the structure of *one* particular eigenfunction—it remains to be seen if it is typical. We note that the open question discussed above (C36) in [35] may be solved negatively by an understanding of eigenfunctions.

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REFERENCES

1. J. AVRON AND B. SIMON, *Comm. Math. Phys.* **82** (1982), 101–120.
2. J. AVRON AND B. SIMON, *Duke Math. J.* **50** (1983), 369–391.
3. J. BELLISSARD, R. LIMA, AND E. SCOPPOLA, *Comm. Math. Phys.* **88** (1983), 465–477.
4. J. BELLISSARD AND B. SIMON, *J. Funct. Anal.* **48** (1982), 408–419.
5. J. BEREZANSKII, *Dokl. Akad. Nauk. SSSR* **108** (1956), 379–382.
6. M. BERRY, *Physica D*, in press.
7. R. CARMONA, *Duke Math. J.* **49** (1982), 191.
8. W. CRAIG, private communication.
9. W. CRAIG, *Comm. Math. Phys.* **88** (1983), 113–131.
10. W. CRAIG AND B. SIMON, *Duke Math. J.* **50** (1983), 551–560.
11. V. CHULAEVSKY, *Russian Math. Surveys* **36**, No. 5 (1981), 143.
12. P. DEIFT AND B. SIMON, *Comm. Math. Phys.* **90** (1983), 389–411.
13. F. DELYON, H. KUNZ, AND B. SOUILLARD, *J. Phys. A* **16** (1983), 25.
14. E. DINABURG AND YA. SINAI, *Funct. Anal. Appl.* **9** (1975), 279–289.
15. G. ELLIOTT, *C. R. Math. Rep. Acad. Sci. Canada* **4** (1982), 255.
16. S. FISHMAN, D. GREMPEL, AND R. PRANGE, *Phys. Rev. Lett.* **49** (1982), 509–512.
17. S. FISHMAN, D. GREMPEL, AND R. PRANGE, *Phys. Rev. A*, in press.
18. A. GORDON, *Usp. Math. Nauk.* **31** (1976), 257.
19. D. GREMPEL, S. FISHMAN, AND R. PRANGE, *Phys. Rev. Lett.* **49** (1982), 833.
20. K. ISHII, *Supp. Progr. Theoret. Phys.* **53** (1973), 77.
21. Y. KATZNELSON, "An Introduction to Harmonic Analysis," Wiley, New York, 1968.
22. A. KHINTCHINE, "Continued Fractions," Nordhoff, Amsterdam, 1963.
23. S. KOTANI, "Proceedings, Conference on Stochastic Process, Kyoto, 1982."
24. P. LLOYD, *J. Phys. C* **2** (1969), 1717–1725.
25. J. MOSER, *Comm. Math. Helv.* **56** (1981), 198.
26. L. PASTUR, *Comm. Math. Phys.* **75** (1980), 179.
27. L. PASTUR AND A. FIGOTIN, *JETP Lett.* **37** (1983), 686–688; An exactly soluble model of a multi-dimensional incommensurate structure, *Comm. Math. Phys.*, in press.
28. R. PRANGE AND J. AVRON; L. PASTUR AND A. FIGOTIN, private communications.
29. R. PRANGE, D. GREMPEL, AND S. FISHMAN, *Phys. Rev. B*, in press.
30. R. PRANGE, D. GREMPEL, AND S. FISHMAN, Maryland preprint.
31. P. SARNAK, *Comm. Math. Phys.* **84** (1982), 377–401.
32. B. SIMON, *Adv. Appl. Math.* **3** (1982), 463–490.
33. B. SIMON, *Phys. Rev. B* **27** (1983), 3859–3860.
34. B. SIMON, *Comm. Math. Phys.* **89** (1983), 227–234.
35. B. SIMON, *Bull. Math. Soc.* **7** (1982), 447–526.
36. T. SPENCER, private communication.
37. D. THOULESS, *J. Phys. C* **5** (1972), 77–81.
38. D. THOULESS, private communication.
39. E. TITCHMARSH, "The Theory of Functions," Oxford Univ. Press, Oxford, 1932.
40. H. WEYL, *Math. Ann.* **77** (1916), 313–352.