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Ultracontractivity and the Heat Kernel for Schrödinger Operators and Dirichlet Laplacians

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Abstract connections between integral kernels of positivity preserving semigroups and suitable L^p contractivity properties are established. Then these questions are studied for the semigroups generated by $-\Delta + V$ and H_Ω, the Dirichlet Laplacian for an open, connected region Ω. As an application under a suitable hypothesis, Sobolev estimates are proved valid up to $\partial\Omega$, of the form $|\eta(x)| \leqslant c\varphi_0(x) \|H_\Omega^k \eta\|_2$, where φ_0 is the unique positive L^2 eigenfunction of H_Ω.

1. INTRODUCTION

Since the discovery of L^p properties of the Hermite semigroup by Nelson [23], and especially since the discovery of the connection with "logarithmic Sobolev inequalities" by Gross [18], the general area of hypercontractive semigroups has been extensively studied (see the bibliography of Gross [19]). There are several important themes which have not been explored which we feel are analytically significant. It is surprising these themes have not been studied, and we are even surprised that neither of the present authors, each of whom has a long interest in the subject, has previously examined them.

To describe these themes, we briefly recall several definitions which we will make formally later on. Consider a semigroup, e^{-tA}, on $L^2(X, dx)$ where dx is a Baire measure on a locally compact Hausdorff space, X. We suppose

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that e^{-tA} extends continuously as a map from $L^2 \cap L^p$ to L^2 to a contraction semigroup on each L^p $(1 \leqslant p < \infty)$. e^{-tA} is called hypercontractive [39] (resp. supercontractive [30]) if e^{-tA} maps L^2 to L^4 for some $t > 0$ (resp. all $t > 0$). (2 and 4 play no special role; once one has L^2 to L^4 information, one automatically obtains L^p to L^q information for any $1 < p < q < \infty$.)

Now suppose that e^{-tH} is a positivity preserving selfadjoint semigroup on $L^2(X, dx)$. Suppose that $H\varphi_0 = e\varphi_0$ for a positive function φ_0 in $L^2(X, dx)$. φ_0 is automatically the *ground state*, i.e., $e = \inf \operatorname{spec}(H)$. Define the probability measure $d\mu(x) = \varphi_0(x)^2\, dx$ and the unitary map $M_{\varphi_0}: L^2(X, d\mu) \to L^2(X, dx)$ by $M_{\varphi_0} f = f\varphi_0$.

Then $A = M_{\varphi_0}^{-1}(H - e) M_{\varphi_0}$ on $L^2(X, d\mu)$ generates a contractive semigroup, e^{-tA}, on each $L^p(X, d\mu)$ (see, e.g., [27, Theorem X.55]). One is often interested in hypercontractive properties of e^{-At} which we dub "intrinsic hypercontractive" properties of e^{-tH}. Thus Nelson's result [23] is that $H = -\Delta + x^2$ on $L^2(R^\nu, d^\nu x)$ generates an intrinsically hypercontractive semigroup.

Our main themes here are the following:

(a) *L^∞-properties.* We will examine when e^{-tA} maps L^2 to L^∞ for all $t > 0$, a property we call "ultracontractivity," following a suggestion of D. Robinson. We are especially interested in intrinsic ultracontractivity for Schrödinger semigroups, e^{-tH}, with $H = -\Delta + V$. Carmona [5] (following, in part, ideas of Herbst and Gross), showed that if such a semigroup is intrinsically hypercontractive, then $H \geqslant cx^2 - d$ (some $c > 0$) so in an average sense V must grow as fast as x^2. Rosen [30] proved intrinsic supercontractivity for a class of V including $V(x) = |x|^\alpha$ for any $\alpha > 2$. There is no literature on intrinsic ultracontractivity of Schrödinger semigroups on R^ν in part because there is a folk belief that this never occurs. We were quite suprised to realize that, despite this folk belief, which we shared, ultracontractivity is quite common. We will prove (see Section 6) that for $H = -\Delta + |x|^\alpha \ln(|x| + 2)^\beta$, one has no intrinsic hypercontractivity if $\alpha < 2$, intrinsic hypercontractivity but not intrinsic supercontractivity if $\alpha = 2$, $\beta = 0$, intrinsic supercontractivity but not intrinsic ultracontractivity if $\alpha = 2$, $0 < \beta \leqslant 2$ and intrinsic ultracontractivity if $\alpha = 2$, $\beta > 2$, or if $\alpha > 2$ (the $\alpha < 2$ result is due to Carmona [5], and the $\alpha = 2$, $\beta = 0$ result to Nelson [24]). Intrinsic ultracontractivity is especially interesting since it implies that $\varphi_n \varphi_0^{-1}$ is bounded for any L^2 eigenfunction φ_n and so these results are a contribution to the large literature on decay of eigenfunctions (see Sect. C3 of [37] for references). We note that intrinsic ultracontractivity for a very large class of one-dimensional Schrödinger operators on a *bounded* interval was proven by one of us [9] prior to this work.

(b) *Intrinsic contractivity of Dirichlet semigroups.* There appears to be no previous literature on the intrinsic contractivity properties of e^{-tH} for H, the Dirichlet Laplacian of a region Ω except for the one dimensional results in [9]. (Hooten [22] discusses contractivity properties for $L^p(\Omega, d^\nu x)$ rather than $L^p(\Omega, \varphi_0^2\, d^\nu x)$). In Section 9, we describe a bounded open region Ω in R^2 for which the associated Dirichlet semigroup is *not* intrinsically ultracontractive. We will also prove that under fairly weak geometric hypotheses on Ω (interior and exterior cone conditions), Dirichlet semigroups are intrinsically ultracontractive.

(c) *Sobolev estimates up to the boundary.* This involves another natural question, at first sight not related to L^p contractivity, which surprisingly has not been studied. Let H_Ω be the Dirichlet Laplacian of a bounded open region Ω in R^ν. Sobolev estimates imply that

$$\|\psi\|_\infty \leqslant c\, \|H_\Omega^k \psi\|_2 \tag{1.1}$$

so long as $k > \nu/4$, and under a very weak condition (see Appendix C), one knows that any $\psi \in D(H_\Omega^k)$ $(k > \nu/4)$ is a continuous function vanishing on $\partial\Omega$. The natural question to ask is how fast must such a ψ vanish? Even if $\nu = 1$, $\Omega = [0, 1]$, the rate will depend on k if k is very close to $\frac{1}{4}$, so it is natural to restrict oneself to the situation for k sufficiently large. It is not hard to see (see Sect. 9) and is well known that if $\partial\Omega$ is smooth, then any $\psi \in D(H_\Omega^k)$ $(k > \nu/4 + 1)$ vanishes at least linearly as one approaches $\partial\Omega$. For Ω with $\partial\Omega$ nonsmooth, the precise boundary behavior can be quite complicated, but it is natural to expect that no function in $D(H_\Omega^k)$, k large should go to zero more slowly that φ_0, the ground state of H_Ω. That is, we ask if there is sufficiently large k so that a Sobolev estimate up to the boundary holds:

$$|\psi(x)| \leqslant c\varphi_0(x)\, \|H_\Omega^k \psi\|_{L^2(\Omega, d^\nu x)}. \tag{1.2}$$

The point is that (1.2) is intimately related to intrinsic ultracontractivity! For letting $\eta = M_{\varphi_0}^{-1}\psi = \psi/\varphi_0$ and $A = M_{\varphi_0}^{-1}(H - e)\, M_{\varphi_0}$, Eq. (1.2) is equivalent to $|\eta(x)| \leqslant c\, \|(A+1)^k \eta\|_{L^2(\Omega, d\mu)}$ which, in turn, is equivalent to asking if $(A+1)^{-k}$ maps $L^2(\Omega, d\mu)$ to $L^\infty(\Omega, d\mu)$. Such an estimate will hold if and only if $\|e^{-tA}\eta\|_\infty \leqslant Ct^{-l}\, \|\eta\|_2$ for some l and all $0 < t < 1$. Thus (1.2) is equivalent to intrinsic ultracontractivity together with suitable information on the rate of divergence of $\|e^{-tA}\|_{\infty,2}$ ($\|B\|_{p,q}$ is the norm of B as a map from L^q to L^p). In Section 9, we prove estimates like (1.2) when Ω obeys an interior and an exterior cone condition; k depends on the geometry of the interior cone.

(d) *Behavior of the heat kernel.* Suppose that a positivity preserving semigroup e^{-tH} on $L^2(X, dx)$ has a continuous integral kernel $a_t(x, y)$ and H has a ground state $\varphi_0(x)$. Typically, a_t and φ_0 vanish as x (or x, y) go "to inifinity." Let $b_t(x) = \sqrt{a_t(x, x)}$. An optimist might hope that $a_t(x, y)$, $b_t(x)\, b_t(y)$, and $\varphi_0(x)\, \varphi_0(y)$ all vanish at the same rate in the sense that the ratio of any two is bounded. Under some additional hypotheses (normally true for Dirichlet and Schrödinger semigroups), we will show that the comparability of these three quantities is equivalent to intrinsic ultracontractivity of e^{-tH}. We will prove that partial comparability results often imply ultracontractivity and thus full comparability, for example, an upper bound $a_t(x, y) \leqslant c_t \varphi_0(x)\, \varphi_0(y)$ implies a lower bound $d_t \varphi_0(x)\, \varphi_0(y) \leqslant a_t(x, y)$!

This paper has become long because of this variety of themes. In addition, we felt it necessary to repeat some of the arguments of Gross [18], Eckmann [14], Rosen [30], and Carmona [5], in part because they often do not make constants explicit (and in going to L^∞, explicit constants are crucial), and in part because we wish to use recent methods [37] to discuss domain questions.

In Sections 2 and 3 we establish the connection between ultracontractivity and behaviour of the heat kernel in an abstract setting. In Sections 4 and 5 we extend ideas of Gross [18] and Rosen [30] to reduce the proof of intrinsic ultracontractivity to an estimate of the form

$$-\ln \varphi_0 \leqslant \delta H + g(\delta) \tag{1.3}$$

with some restrictions on how fast $g(\delta)$ can grow as $\delta \downarrow 0$ (a bound $g(\delta) \leqslant C\delta^{-l}$ for some l is certainly sufficient). In Section 6, we examine when (1.3) holds for Schrödinger operators. In Section 7, we obtain upper bounds on $-\ln \varphi_0$ for the Dirichlet Laplacian by geometric functions and in Section 8, we recall estimates of Davies [11] obtaining upper bounds on similar geometric quantities by H. By combining these in Section 9, we prove Sobolev estimates up to the boundary for a wide class of regions.

Appendices A–C contain various technical results that complement issues discussed in the text. While these appendices illuminate our main thread, they are not required in the main body of the paper.

A sketch of some of our arguments in pedagogical presentation can be found in [13].

2. An Abstract Framework

Although hypercontractivity was historically first studied for the harmonic oscillator in quantum field theory, and later for Schrödinger operators on $L^2(\mathbb{R}^N)$, the ideas are applicable to a variety of other situations. In order to describe all these applications in a unified manner, we treat the subject at an abstract level. For the same of the reader who wishes to proceed to Section 3 at once, we comment that all the hypotheses and results in this section apart from (2.1) are automatically satisfied for uniformly elliptic second-order operators on manifolds and for a large class of Schrödinger operators on $L^2(\mathbb{R}^N)$. See [15, 37] and references there.

Let X be a locally compact second countable Hausdorff space, and let dx be a regular Borel measure on X with support equal to X, i.e., the measure of every nonempty open set is strictly positive. Let H be a nonnegative selfadjoint operator on $L^2(X)$ such that for every $t > 0$, e^{-Ht} has a jointly continuous integral kernel $a_t(x, y)$. Thus

$$e^{-Ht}f(x) = \int_X a_t(x, y) f(y)\, dy$$

for all $f \in L^2(X)$ and $t > 0$. Let us also assume that $a_t(x, y) \geqslant 0$ for all $x, y \in X$ and $t > 0$, or equivalently that e^{-Ht} is positivity preserving. Finally, let us suppose that

$$\operatorname{Tr}[e^{-Ht}] < \infty \tag{2.1}$$

for all $t > 0$, or equivalently by Mercer's theorem (see [28, p. 65]),

$$\int_X a_t(x, x)\, dx < \infty$$

for all $t > 0$. Since out entire theory is based upon this last hypothesis, it is clear that we are not studying hypercontractivity in its most general setting. However, we investigate the status of this hypothesis in Appendix A and show that it is a consequence of hypercontractivity for all Schrödinger operators; see Theorem A.8. However, (2.1) rules out most quantum field theory situations.

Under these conditions H has purely discrete spectrum with eigenvalues $\{E_n\}_{n=0}^{\infty}$ which we write in increasing order and repeat according to multiplicity. Let $\{\varphi_n\}_{n=0}^{\infty}$ be the corresponding eigenfunctions, normalized by $\|\varphi_n\|_2 = 1$.

LEMMA 2.1. *The function φ_n is continuous for each $n \geqslant 0$, and we have*

$$a_t(x, y) = \sum_{n=0}^{\infty} e^{-E_n t}\varphi_n(x)\, \varphi_n(y) \tag{2.2}$$

for each $t > 0$, where the series is locally uniformly convergent on $X \times X$. If we define $b_t : X \to \mathbb{R}^+$ for each $t > 0$ by

$$b_t(x) = a_t(x, x)^{1/2} \tag{2.3}$$

then b_t is a continuous function in $L^2(X)$ and

$$|\varphi_n(x)| \leqslant e^{(1/2)E_n t} b_t(x) \tag{2.4}$$

for all n, x and all $t > 0$.

Proof. Because a_t is a kernel of positive type we have

$$0 \leqslant a_t(x, y) \leqslant b_t(x)\, b_t(y) \tag{2.5}$$

and

$$\|b_t\|_2^2 = \int_X a_t(x, x)\, dx = \mathrm{tr}[e^{-Ht}] < \infty.$$

Also

$$\begin{aligned} |\varphi_n(x)| &= \left| e^{E_n t} \int_X a_t(x, y)\, \varphi_n(y)\, dy \right| \\ &\leqslant e^{E_n t} \int_X b_t(x)\, b_t(y)\, |\varphi_n(y)|\, dy \\ &\leqslant e^{E_n t} b_t(x)\, \|b_t\|_2. \end{aligned} \tag{2.6}$$

The continuity of φ_n is obtained by applying the dominated convergence theorem of the formula

$$\varphi_n(x) = e^{E_n t} \int_X a_t(x, y)\, \varphi_n(y)\, dy.$$

Now the expansion (2.2) is certainly valid in the $L^2(X \times X)$ norm so the proof of (2.2) can be completed by showing that the rhs is locally uniformly convergent. If $K \subseteq X$ is compact and

$$c_t = \sup_{x \in K} b_{t/3}(x)\, \|b_{t/3}\|_2$$

then

$$|\varphi_n(x)| \leqslant c_t e^{E_n t/3}$$

for all $x \in K$ by (2.6), so

$$\sum_{n=0}^{\infty} |e^{-E_n t} \varphi_n(x)\, \varphi_n(y)| \leqslant \sum_{n=0}^{\infty} c_t^2 e^{-E_n t/3}$$

for all $x, y \in K$ and the uniform convergence of the series on $K \times K$ is a consequence of the Weierstrass M-test. Putting $x = y$ in (2.2) we see that

$$e^{-E_n t} |\varphi_n(x)|^2 \leqslant \sum_{m=0}^{\infty} e^{-E_m t} |\varphi_m(x)|^2$$
$$= a_t(x, x) = b_t(x)^2$$

which proves (2.4). ∎

In many situations $b_t(x) \to 0$ at "infinity" in X in which case our proof shows that the convergence in (2.2) is uniform and not just locally uniform.

COROLLARY 2.2. *For each $x \in X$, $b_t(x)$ is an analytic, logarithmically convex, monotonically decreasing function of t.*

Proof. These facts are all simple consequences of the formula

$$b_t(x)^2 = \sum_{n=0}^{\infty} e^{-E_n t} |\varphi_n(x)|^2. \quad ∎$$

We now make the assumption that the contraction semigroup e^{-Ht} is irreducible in the sense that if, for some Borel set $A \subset X$ $\{f \in L^2(X, d_\mu) \mid f = 0 \text{ on } X \backslash A\}$ is left invariant by e^{-Ht}, then up to null sets, A is ϕ or X. It is then a standard fact [29, p. 202, 8, p. 174] that E_0 has multiplicity one, and that $\varphi_0(x)$ is strictly positive except on a (closed) null set. By removing this set from X we can assume that

$$\varphi_0(x) > 0 \qquad \text{all} \quad x \in X. \tag{2.7}$$

Following [32] we can now deduce that e^{-Ht} is positivity improving in a rather strong sense.

LEMMA 2.3. *If $t > 0$, then*

$$a_t(x, y) > 0 \qquad \text{all} \quad x, y \in X.$$

Proof. The core of our proof is the assertion that if $a_t(x, y) > 0$, then $a_{t+s}(x, y) > 0$ for all $s > 0$. To prove this, note that

$$a_s(y, y) \geqslant e^{-E_0 s} \varphi_0(y)^2 > 0,$$

so if $a_t(x, y) > 0$ and z lies in a small enough neighborhood N of y, then $a_t(x, z) > 0$ and $a_s(z, y) > 0$ by continuity. Therefore

$$\begin{aligned} a_{t+s}(x, y) &= \int_X a_t(x, z)\, a_s(z, y)\, dz \\ &\geqslant \int_N a_t(x, z)\, a_s(z, y)\, dz \\ &> 0. \end{aligned}$$

We combine the above observation with the fact that $a_t(x, y)$ is an analytic function of t to conclude that either $a_t(x, y) > 0$ for all $0 < t < \infty$ or $a_t(x, y) = 0$ for all $0 < t < \infty$. It remains to eliminate the second possibility.

If we put

$$f_t(z) = a_t(z, y)$$

then $f_t \in L^2(X)$ by (2.5). Moreover

$$\begin{aligned} f_{t+s}(z) &= \int a_s(z, w)\, a_t(w, y)\, dw \\ &= e^{-Hs} f_t(z) \end{aligned}$$

and

$$\begin{aligned} \langle f_t, \varphi_0 \rangle &= \int_X a_t(z, y)\, \varphi_0(z)\, dz \\ &\geqslant \int_N a_t(z, y)\, \varphi_0(z)\, dz \\ &> 0, \end{aligned}$$

where N is chosen to be a small neighborhood of y within which $a_t(z, y)$ is nonzero. Similar considerations apply to the function

$$g_t(z) = a_t(x, z).$$

Now

$$\begin{aligned} \lim_{s \to \infty} a_{2(t+s)}(x, y)\, e^{2E_0 s} &= \lim_{s \to \infty} \int_X a_{t+s}(x, z)\, a_{t+s}(z, y)\, e^{2E_0 s}\, dz \\ &= \lim_{s \to \infty} \langle e^{-Hs} g_t, e^{-Hs} f_t \rangle\, e^{2E_0 s} \\ &= \langle g_t, \varphi_0 \rangle \langle \varphi_0, f_t \rangle \end{aligned}$$

by the spectral theorem. We showed that this last quantity is nonzero, so $a_t(x, y)$ cannot be identically zero as a function of t. ∎

3. Ultracontractivity, Hypercontractivity, and the Heat Kernel

We now investigate hypercontractivity of the semigroup e^{-Ht} under all the assumptions of the last section. We emphasize that since we assume (2.1) we are not treating the fully general cases. However, in all the applications to Schrödinger operators and elliptic operators on manifolds, (2.1) is rather easy to verify, and for *such* operators (2.1) is actually a consequence of hypercontractivity by Theorem A.8.

For historical reasons, we assume that e^{-Ht} is a contraction semigroup on $L^p(X)$ for all $1 \leqslant p \leqslant \infty$, but will not use this condition in the next theorem. We then recall that e^{-Ht} is said to be *hypercontractive* if, for all $2 < p < \infty$ there exists $T_p < \infty$ such that $t > T_p$ implies e^{-Ht} is bounded from $L^2(X)$ to $L^p(X)$, and that e^{-Ht} is said to be *supercontractive* if $T_p = 0$ for all $2 < p < \infty$. We now introduce the even more restrictive condition of *ultracontractivity* as corresponding to the requirement that e^{-Ht} is bounded from $L^2(X)$ to $L^\infty(X)$ for *all* $t > 0$. Apart from [9, 10], it appears that this concept has not been investigated before, and we shall see that for Schrödinger operators it is usually true whenever hypercontractivity is true, with the exception of certain borderline cases such as the harmonic oscillator. Our first theorem introduces a variety of conditions equivalent to supercontractivity.

Theorem 3.1 *Suppose $2 < p < \infty$. Then the following are equivalent.*

(i) *For all $t > 0$ there exists $c_t < \infty$ such that*

$$\|\varphi_n\|_p \leqslant c_t e^{E_n t} \qquad \text{all } n.$$

(ii) $\|b_t\|_p < \infty$ *for all* $t > 0$.

(iii) $\|a_t\|_{p,2} \equiv \{\int\{\int |a_t(x, y)|^2\, dy\}^{p/2}\, dx\}^{1/p}$ *is finite for all* $t > 0$.

(iv) e^{-Ht} *is bounded from* $L^2(X)$ *to* $L^p(X)$ *for all* $t > 0$.

Proof. (i) ⇒ (ii)

$$\begin{aligned}\|b_t\|_p^p &= \|b_t^2\|_{p/2}^{p/2} \\ &\leqslant \left\{ \sum_{n=0}^{\infty} e^{-E_n t} \||\varphi_n|^2\|_{p/2} \right\}^{p/2}\end{aligned}$$

$$= \left\{ \sum_{n=0}^{\infty} e^{-E_n t} \|\varphi_n\|_p^2 \right\}^{p/2}$$

$$\leqslant \left\{ \sum_{n=0}^{\infty} e^{-E_n t/3} c_{t/3}^2 \right\}^{p/2}$$

$$= c_{t/3}^p \{\mathrm{tr}[e^{-Ht/3}]\}^{p/2}$$

$$< \infty.$$

(ii) $\Rightarrow$ (iii) $0 \leqslant a_t(x, y) \leqslant b_t(x)\, b_t(y)$ so

$$\|a_t\|_{p,2} \leqslant \|b_t\|_p \|b_t\|_2$$

$$= \|b_t\|_p \{\mathrm{tr}[e^{-Ht}]\}^{1/2}$$

$$< \infty.$$

(iii) $\Rightarrow$ (iv) if $g_t(x) = \{\int |a_t(x, y)|^2\, dy\}^{1/2}$ then

$$|e^{-Ht} f(x)| = \left| \int_X a_t(x, y)\, f(y)\, dy \right|$$

$$\leqslant g_t(x) \|f\|_2$$

for all $f \in L^2(X)$. Therefore

$$\|e^{-Ht} f\|_p \leqslant \|g_t\|_p \|f\|_2$$

$$= \|a_t\|_{p,2} \|f\|_2.$$

(iv) $\Rightarrow$ (i) since $\varphi_n = e^{E_n t} e^{-Ht} \varphi_n$ we see that

$$\|\varphi_n\|_p = e^{E_n t} \|e^{-Ht} \varphi_n\|_p \leqslant e^{E_n t} \|e^{-Ht}\|_{p,2}. \quad \blacksquare$$

Notes. (1) It is clear from the proofs that one can obtain precise quantitative connections between the constants involved in the different parts of the above theorem, provided one has effective bounds on $\mathrm{tr}[e^{-Ht}]$; these can often be obtained by a use of the Golden–Thompson inequality.

(2) Although we have chosen, for simplicity, to write down the conditions for all $t > 0$, each step involves only one value of t, and the complete circle of implications only need involve losing a factor of $(2 + \varepsilon)$ on t.

We now introduce intrinsic versions of the notions of hyper-, super-, and ultracontractivity. If we define the probability measure μ on X by

$$d\mu(x) = \varphi_0(x)^2\, dx$$

then there is a unitary map

$$U: L^2(X, d\mu) \to L^2(X, dx)$$

defined by

$$Uf(x) = \varphi_0(x) f(x)$$

and this enables us to lift all the ideas to $L^2(X, d\mu)$. In an obvious notation we then get

$$\tilde{H} = U^{-1}(H - E_0) U,$$

$$\tilde{\varphi}_n(x) = \varphi_n(x)/\varphi_0(x),$$

$$\tilde{a}_t(x, y) = \frac{a_t(x, y)}{\varphi_0(x)\,\varphi_0(y)},$$

$$\tilde{b}_t(x) = b_t(x)/\varphi_0(x).$$

It then follows [27, p. 255] that $e^{-t\tilde{H}}$ is a contraction on all L^p spaces. The point of this change of measure is that it leads to the normalization $\tilde{\varphi}_0 = 1$. We then say that H is intrinsically hyper-, super-, or ultracontractive if $\tilde{H}$ is, respectively, hyper-, super-, or ultracontractive in the sense previously discussed. The following theorem provides various equivalent criteria for intrinsic ultracontractivity.

THEOREM 3.2. *We have the implications*

$$\text{(i)} \Rightarrow \text{(ii)} \Leftrightarrow \text{(iii)} \Leftrightarrow \text{(iv)} \Leftrightarrow \text{(v)} \Rightarrow \text{(vi)}$$

between the following conditions (the constants c_1–c_5 are t-dependent).

(i) *There exists M such that $f \in \mathrm{Dom}(H^M)$ implies*

$$|f(x)| \leqslant c\varphi_0(x)\, \|(H+1)^M f\|_2 \qquad all \quad x \in X.$$

(ii) *For all $t > 0$ there exists c_1 such that*

$$|e^{-Ht} f(x)| \leqslant c_1 \|f\|_2\, \varphi_0(x) \qquad all \quad x \in X$$

for all $f \in L^2(X, dx)$.

(iii) *For all $t > 0$ there exists c_2 such that*

$$b_t(x) \leqslant c_2 \varphi_0(x) \qquad all \quad x \in X.$$

(iv) *For all $t > 0$ there exists c_3 such that*

$$a_t(x, y) \leqslant c_3 \varphi_0(x)\, \varphi_0(y) \qquad all \quad x, y \in X.$$

(v) *For all* $t > 0$ *there exists* c_4 *such that*

$$a_t(x, y) \geqslant c_4 b_t(x)\, b_t(y) \qquad \textit{all} \quad x, y \in X.$$

(vi) *For all* $t > 0$ *there exists* c_5 *such that*

$$a_t(x, y) \geqslant c_5 \varphi_0(x)\, \varphi_0(y) \qquad \textit{all} \quad x, y \in X.$$

Proof. (i) $\Rightarrow$ (ii) If $g = e^{-Ht}f$, then

$$\|(H+1)^M g\|_2 \leqslant \|(H+1)^M e^{-Ht}\|\, \|f\|_2$$

so

$$|g(x)| \leqslant c\varphi_0(x)\, \|(H+1)^M e^{-Ht}\|\, \|f\|_2.$$

(ii) $\Leftrightarrow$ (iii) We note that (ii) requires $e^{-\tilde{H}t}$ to be bounded from $L^2(X, d\mu)$ to $L^\infty(X, d\mu)$ for all $t > 0$, while (iii) requires $\tilde{b}_t$ to be bounded. We may therefore apply Theorem 3.1 to $\tilde{H}$ with $p = \infty$.

(iii) $\Leftrightarrow$ (iv) This follows from (2.5).

(iv) $\Rightarrow$ (v) Let the compact subset K of X be large enough so that

$$\int_K \varphi_0(x)^2\, dx \geqslant 1 - \frac{e^{-E_0 t}}{2c_3}.$$

Then

$$\begin{aligned} e^{-E_0 t}\varphi_0(x) &= \int_X a_t(x, y)\, \varphi_0(y)\, dy \\ &\leqslant \int_{X \setminus K} c_3 \varphi_0(x)\, \varphi_0(y)^2\, dy + \int_K a_t(x, y)\, \varphi_0(y)\, dy \\ &\leqslant \tfrac{1}{2} e^{-E_0 t}\varphi_0(x) + \int_K a_t(x, y)\, \varphi_0(y)\, dy \end{aligned}$$

so

$$\int_K a_t(x, y)\, \varphi_0(y)\, dy \geqslant \tfrac{1}{2} e^{-E_0 t}\varphi_0(x) \tag{3.1}$$

for all $x \in X$. Now

$$\gamma \equiv \min \left\{ \frac{a_t(z, w)}{\varphi_0(z)\, \varphi_0(w)} : z, w \in K \right\}$$

is strictly positive by Lemma 2.3, and for any $x, y \in X$ we have

$$\begin{aligned} a_{3t}(x, y) &\geqslant \int_K \int_K a_t(x, z)\, a_t(z, w)\, a_t(w, y)\, dz\, dw \\ &\geqslant \gamma \int_K \int_K a_t(x, z)\, \varphi_0(z)\, \varphi_0(w)\, a_t(w, y)\, dz\, dw \\ &\geqslant \tfrac{1}{4} \gamma e^{-2E_0 t} \varphi_0(x)\, \varphi_0(y) \qquad \text{by (3.1).} \end{aligned}$$

The argument is completed by using (iii).

(v) ⇒ (iii)

$$\begin{aligned} \varphi_0(x) &= e^{E_0 t} \int_X a_t(x, y)\, \varphi_0(y)\, dy \\ &\geqslant e^{E_0 t} \int_X c_4 b_t(x)\, b_t(y)\, \varphi_0(y)\, dy \\ &\geqslant c_4 e^{E_0 t/2} b_t(x) \int_X \varphi_0(y)^2\, dy \\ &= c_4 e^{E_0 t/2} b_t(x) \qquad \text{by (2.4).} \end{aligned}$$

(v) ⇒ (vi) This follows easily from (2.4). ∎

Notes. (1) Every step of this proof is under explicit quantitative control except (iv) ⇒ (v). For the strict positivity of γ is proved by compactness arguments combined with the indirect methods of Lemma 2.3.

(2) Condition (iv) is equivalent to the boundedness of the integral kernel $\tilde{a}_t(x, y)$ which is in turn equivalent to the boundedness of $e^{-\tilde{H}t}$ as an operator from $L^1(X, d\mu)$ to $L^\infty(X, d\mu)$. It is in fact easy to see that $e^{-\tilde{H}t}$ is bounded from $L^p(X, d\mu)$ to $L^q(X, d\mu)$ for all p, q under the above conditions.

The following theorem gives some support to our claim that hypercontractive semigroups are usually also ultracontractive, except in critical cases. This theorem does not utilize the trace hypothesis (2.1).

THEOREM 3.3. *Suppose that e^{-Ht} is a contradiction on $L^p(X)$ for all $1 \leqslant p \leqslant \infty$ and $0 \leqslant t < \infty$. If $e^{-H^\alpha c}$ is bounded from L^2 to L^4 for some $c > 0$ and some $0 < \alpha < 1$, then e^{-Ht} is ultracontractive.*

Proof. We have

$$\|e^{-Ht}\|_{4,2} \leqslant e^{g(t)} \|e^{-H^\alpha c}\|_{4,2},$$

where

$$\begin{aligned} g(t) &= \sup_{0 \leqslant s \leqslant \infty} (cs^\alpha - st) \\ &= c_1 t^{-\alpha/(1-\alpha)}. \end{aligned}$$

It follows by interpolation that

$$\|e^{-Ht}\|_{2n+2,2n+1} \leqslant e^{g(t)/2^n}$$

for all integers $n \geqslant 0$. Putting

$$t_n = 6t\pi^{-2}(n+1)^{-2}$$

we see that $\sum_{n=0}^{\infty} t_n = t$ so that

$$\begin{aligned}
\|e^{-Ht}\|_{\infty,2} &\leqslant \prod_{n=0}^{\infty} \|e^{-Ht_n}\|_{2n+2,2n+1} \\
&\leqslant \exp\left(\sum_{n=0}^{\infty} g(t_n)\, 2^{-n}\right) \\
&= \exp\left(\sum_{n=0}^{\infty} \frac{c_1}{2^n}\left(\frac{1}{6}\pi^2(n+1)^2\right)^{\frac{\alpha}{1-\alpha}}\right) \\
&< \infty \quad \blacksquare
\end{aligned}$$

Our next theorem proves the invariance of *intrinsic* ultracontractivity under bounded multiplicative perturbations. Simple perturbations of the harmonic oscillator Hamiltonian show that the second conclusion of the theorem is not valid under the condition of intrinsic hypercontractivity. It is not clear whether intrinsic hypercontractivity is preserved under bounded perturbations.

THEOREM 3.4. *If H is intrinsically ultracontractive and V is a bounded potential, then $H' = H + V$ is also intrinsically ultracontractive. Moreover,*

$$c^{-1}\varphi_0 \leqslant \varphi_0' \leqslant c\varphi_0, \tag{3.2}$$

where $1 < c < \infty$.

Proof. If $\|V\|_\infty = \gamma$, then it follows from the Trotter product formula that

$$e^{-\gamma t} a_t(x, y) \leqslant a_t'(x, y) \leqslant e^{\gamma t} a_t(x, y).$$

Also

$$\frac{a_t'(x, y)}{\varphi_0'(x)\,\varphi_0'(y)} \leqslant e^{\gamma t} \frac{a_t(x, y)}{\varphi_0(x)\,\varphi_0(y)} \frac{\varphi_0(x)}{\varphi_0'(x)} \frac{\varphi_0(y)}{\varphi_0'(y)},$$

so the intrinsic ultracontractivity of H' will be established if

$$\varphi_0(x) \leqslant c\varphi_0'(x) \qquad \text{all} \quad x \in X.$$

By Theorem 3.2(vi) we have

$$\begin{aligned}\varphi_0'(x) &= e^{E_0' t} \int_X a_t'(x, y)\, \varphi_0'(y)\, dy \\ &\geqslant e^{E_0' t - \gamma t} \int_X a_t(x, y)\, \varphi_0'(y)\, dy \\ &\geqslant e^{E_0' t - \gamma t} c_5 \int_X \varphi_0(x)\, \varphi_0(y)\, \varphi_0'(y)\, dy \\ &= c\varphi_0(x),\end{aligned}$$

where the strict positivity of both φ_0 and φ_0' imply that $c > 0$. The reverse bound in (3.2) then follows by symmetry. ∎

4. Logarithmic Sobolev Inequalities and Ultracontractivity

Gross [18] discovered an effective technique for checking when e^{-tA} maps L^p into L^q. We want to describe his method in this section. We even provide proofs for several reasons: (a) we want to discuss domain questions for Schrödinger operators using some recent technology [37]; (b) for our purposes it is more useful to integrate in p than t as Gross does (although the results are equivalent); (c) we want to describe how $q = \infty$ is allowed; (d) since Carmona [5] and Eckmann [14] misquote Gross' result (see below), we feel it is important to carefully state the estimates.

Definition. Let A be a nonnegative selfadjoint operator on $L^2(X, d\mu)$ with μ a probability measure. We say that e^{-tA} is a positivity preserving L^p contractive semigroup (PLC) if and only if (i) $e^{-tA} f \geqslant 0$ if $f \geqslant 0$; (ii) $\|e^{-tA} f\|_p \leqslant \|f\|_p$ for all $f \in L^2 \cap L^p$ and all $p \in [1, \infty]$.

Automatically, e^{-tA} is strongly continuous on each L^p, $p \neq \infty$ and holomorphic in the sector $|\text{Arg } t| < (\pi/2)(1 - |2/p - 1|)$ (by an application of the Stein interpolation theorem); see [27, p. 255]. (The condition $e^{-tA} 1 = 1$ is not used in the proof of Theorem X.55(c).) Neither hypothesis above is absolutely essential to the result below, but the two conditions make the domain considerations especially simple and they hold in our examples of interest.

Lemma 4.1. *Let* e^{-tA} *be a* PLC, *and define* $\mathscr{D} = \bigcup_{w>0} e^{-wA}[L^\infty]$ *and* $\mathscr{D}_+ = \bigcup_{w>0} e^{-wA}[L^\infty]_+$. *Then*

(a) *e^{-sA} maps $\mathscr{D}$ to $\mathscr{D}$ and $\mathscr{D}_+$ to $\mathscr{D}_+$,*

(b) *$\mathscr{D}$ is contained in $D_p(A)$, the L^p-domain of the generator of e^{-tA} for all $p \in (1, \infty)$,*

(c) *$\mathscr{D}$ is dense in each L^p, $p \neq \infty$ and is a core for A on L^p, $p \neq \infty$,*

(d) *$\mathscr{D}_+$ is dense in L^p_+.*

Proof. (a) is immediate since e^{-sA} commutes with e^{-wA} and maps L^∞ (resp. L^∞_+) to itself.

(b) $e^{-sA}[e^{-wA}\varphi]$ is L^p holomorphic for $|s|$ small by the remark above, so $e^{-wA}\varphi \in D(A_p)$.

(c) Density is proven in Theorem X.55(b) of [27], and the core statement then follows from (a), (b), and Theorem X.49 of [27].

(d) L^∞_+ is dense in L^p_+ and if f is in L^∞_+, $e^{-wA}f \to f$ in L^p as $w \downarrow 0$. ∎

For $f \geqslant 0$, we define $f_p = f^{p+1}$. Gross' basic theorem can be restated as

THEOREM 4.2. *Let e^{-tA} be a PLC. Let $r \in (2, \infty]$ and suppose there are continuous functions $c(p)$ and $\Gamma(p)$ on $(2, r)$ so that for $f \in \mathscr{D}_+$,*

$$\int f^p \ln f \leqslant c(p)\langle Af, f_p\rangle + \Gamma(p)\,\|f\|_p^p + \|f\|_p^p \ln \|f\|_p. \tag{4.1}$$

Suppose that

$$t = \int_2^r \frac{c(p)}{p}\,dp, \qquad M = \int_2^r \Gamma(p)\frac{dp}{p} \tag{4.2}$$

are both finite. Then e^{-tA} maps L^2 to L^r and

$$\|e^{-tA}f\|_r \leqslant e^M \|f\|_2. \tag{4.3}$$

Remarks. (1) $e^{-sA}f$ is smooth in s on any L^p, so Af can be interpreted as the L^p derivative which lies in all $L^p (p < \infty)$ and in particular in L^2; $f_p \in L^\infty \subset L^2$ so $\langle Af, f_p\rangle$ is intended as the L^2 inner product. Af can be interpreted as the L^p derivative or as the L^2 derivative.

(2) Gross [18] writes $c(p)(\langle Af, f_p\rangle + \gamma(p)\,\|f\|_p^p)$ and $M = \int_2^r \gamma(p(s))\,ds$, where $dp(s)/ds = p/c(p)$ which is equivalent to our (4.3). When Eckmann [14] and Carmona [5] quote Gross' results, they incorrectly use $M = \int_2^r \gamma(p(s))\,ds$ even though *their* γ is equal to our Γ rather than Gross' γ.

Proof. Suppose first that $f \in \mathscr{D}_+$. Then $e^{-sA}f \in L^\infty$ and L^p holomorphic in s. Define $p(s)$ by

$$\frac{dp(s)}{ds} = \frac{p(s)}{c(p(s))}; \qquad p(0) = 2, \tag{4.4a}$$

for $s \in [0, t)$, so $\lim_{s \to t} p(s) = r$ by (4.2) and each such $p(s) < \infty$. Let

$$M(s) = \int_2^{p(s)} \Gamma(p) \frac{dp}{p} \tag{4.4b}$$

and

$$G(s) = e^{-M(s)} \|e^{-sA} f\|_{p(s)} \equiv e^{-M(s)} \|f(s)\|_{p(s)}.$$

Then G is C^1 since $p(s)$, $M(s)$ are C^1 and $e^{-sA}f$ is L^q holomorphic in s for all $q \in [2, \infty)$. Since $G(0) = \|f\|_2$, (4.3) will hold for all $f \in \mathscr{D}_+$ if we show that $dG(s)/ds \leqslant 0$. This is because $f \in L^r$ if and only if $\sup_{q<r} \|f\|_q < \infty$ and this sup is the L^r-norm. But a straightforward calculuation (domain considerations are no problem by the holomorphy) shows that

$$\begin{aligned}\frac{d}{ds} \ln G(s) = {} & [c(p) \|f(s)\|_{p(s)}^{p(s)}]^{-1} \{-\Gamma(p) \|f(s)\|_{p(s)}^{p(s)} \\ & + \int |f(s)|^{p(s)} \ln f(s)\, d\mu(x) \\ & - c(p(s)) \langle Af(s)_{p(s)}, f(s)_{p(s)} \rangle - \|f(s)\|_{p(s)}^{p(s)} \ln \|f(s)\|_{p(s)}\}\end{aligned}$$

is nonpositive by (4.1) and $f(s) \in \mathscr{D}_+$.

We have thus proven (4.3) for $f \in \mathscr{D}_+$. By Lemma 4.1(d), the result holds for $f \in L^2_+$ and so for all f since $|e^{-sA} f| \leqslant e^{-sA} |f|$ pointwise. ∎

Henceforth, we restrict ourselves to a special class of A's. Let H be either (i) $-\Delta + V$ on $L^2(R^\nu)$ for some V with $V_- \in K_\nu$, $V_+ \in K_\nu^{\text{loc}}$ [37] so that $H\varphi = E\varphi$ has an L^2 solution ψ_0 with $E = \inf \operatorname{spec}(H)$; or (ii) a Dirichlet Laplacian on an open, connected region, X, with $\inf \operatorname{spec}(H)$ an eigenvalue with eigenfunction ψ_0 (e.g., X could be bounded but that is not necessary [36]). Let $A = M_{\psi_0}^{-1}(H - E) M_{\psi_0}$, i.e., the $\tilde{H}$ on $L^2(R^\nu, \psi_0^2\, dx)$ or $L^2(X, \psi_0^2\, dx)$ of Section 3. We will call A a *regular Dirichlet form.* Since $e^{-tA} 1 = 1$, it is easy to prove [27, p. 255] that e^{-tA} is a PLC. Let $X = R^\nu$ in the Schrödinger case and $d\mu = \psi_0^2\, dx$.

We are heading towards the following regularity result: $Q(A)$ denotes the L^2 quadratic form domain for A.

PROPOSITION 4.3. *Let A be a regular Dirichlet form. Let $\varphi \in Q(A) \cap L^\infty$ with $\varphi \geqslant 0$. Suppose that $a > 1$. Then $\varphi^a \in Q(A)$ and for any $\psi \in Q(A)$:*

$$(\varphi^a, A\psi) = a \int [(\varphi^{a-1} \nabla\varphi) \cdot \nabla\psi]\, d\mu.$$

For technical reasons, the natural proofs differ in the Schrödinger and Dirichlet cases. We begin with

PROPOSITION 4.4S. *Let A be a regular Dirichlet form associated to a Schrödinger operator. Then $Q(A)$ is precisely the set of $\varphi \in L^2(X, d\mu)$ with $\nabla\varphi$ (distributional gradient) in $L^2(X, d\mu)$ and*

$$(\varphi, A\varphi) = \int |\nabla\varphi|^2 \, d\mu(x); \tag{4.5}$$

$C_0^\infty(X)$ is a form core for A.

Proof. By Harnack's inequality [37], ψ_0 is bounded and locally bounded away from zero. Suppose first that ψ_0 is C^∞. Let $\eta \in C_0^\infty(R^\nu)$. Then an easy calculation using the fact that ψ_0 is bounded away from zero on supp η shows that

$$(\eta, (H-E)\eta)_{L^2(X, d^\nu x)} = \int |\nabla\psi_0^{-1}\eta|^2 \, \psi_0^2 \, dx. \tag{4.6}$$

We claim that (4.6) still holds whenever $V_- \in K_\nu$, $V_+ \in K_\nu^{\text{loc}}$, for in that case $\nabla\psi_0 \in L^2_{\text{loc}}$ (see [37, p. 467]). Let ψ_ε be a sequence of strictly positive C^∞ functions so that $\psi_\varepsilon \to \psi$ uniformly locally, $\nabla\psi_\varepsilon \to \nabla\psi$ in L^2_{loc} and $\Delta\psi_\varepsilon \to \Delta\psi$ in L^1_{loc}. Equation (4.6) holds for $V_\varepsilon \equiv E + \psi_\varepsilon^{-1}\Delta\psi_\varepsilon$ and ψ_ε since ψ_ε is C^∞. Since ψ_ε and ψ_0 are bounded away from zero on supp η, one can take limit in (4.6) for $\varepsilon \neq 0$ and obtain (4.6) in general.

Thus we have established (4.5) if $\varphi \in \psi_0^{-1}[C_0^\infty]$ but C_0^∞ is known to be a form core for H [37, p. 460], so $\psi_0^{-1}[C_0^\infty]$ is a form core for A and thus (4.5) holds for all φ in $Q(A)$. Moreover, if $\varphi \in C_0^\infty$, then $\eta = \psi_0\varphi$ obeys $\eta \in L^\infty$, $\nabla\eta \in L^2$, and supp η finite, so $\eta \in Q(-\Delta) \cap Q(V)$ and thus $\eta \in D(H)$ so $\varphi \in Q(A)$.

We note that all the above steps hold in the Dirichlet case without extra assumptions; we will use this later.

Returning to the Schrödinger case, all that remains to be proven is that given $\varphi \in L^2(R^\nu, d\mu)$ with $\nabla\varphi \in L^2(R^\nu, d\mu)$, we can find $\varphi_n \in C_0^\infty$ so $\varphi_n - \varphi$ and $\nabla(\varphi_n - \varphi)$ go to zero in $L^2(R^\nu, d\mu)$ (this verifies both what $Q(A)$ is and that C_0^∞ is a core for A). Pick $f \in C_0^\infty$, identically 1 near $x = 0$, let $f_n(x) = f(x/n)$ and let j_δ be a standard mollifier. Then $j_\delta * (f_n\psi) \in C_0^\infty$. As first $\delta \to 0$ and then $n \to \infty$, it and its derivative converge to ψ and $\nabla\psi$ in $L^2(R^n, d\mu)$. ∎

Proof of Proposition 4.3 (*Schrödinger case*). If $\varphi \in L^\infty \cap Q(A)$, then the distributional gradient of φ^a is easily seen to be $a\varphi^{a-1}\nabla\varphi$ and since this is in $L^2(R^\nu, d\mu)$, $\varphi^a \in Q(A)$ and the formula for $(\varphi^a, A\psi)$ follows from (4.5). ∎

PROPOSITION 4.4D. *Let A be a regular Dirichlet form associated to a Dirichlet Laplacian. Then $Q(A)$ is contained in the set of $\varphi \in L^2(X, d\mu)$ with $\nabla\varphi$ (distributional gradient) in $L^2(X, d\psi)$ and* (4.5) *holds. $C_0^\infty(X)$ is a form core for A.*

Remark. This differs from Proposition 4.4S in that we do *not* assert that any φ with $\nabla\varphi$ in L^2 lies in $Q(A)$. Under an additional regularity condition, we prove this fact in Appendix C.

Proof. As we remarked in the proof of Proposition 4.4S, the arguments there establish all facts except for the fact that C_0^∞ is a form core. But since ψ_0 is C^∞ and $\psi_0^{-1}[C_0^\infty]$ is a form core for A, we see that $C_0^\infty = \psi_0^{-1}[C_0^\infty]$ is a form core for A. ∎

LEMMA 4.5. *Let A be a regular Dirichlet form. Let $\varphi \in Q(A)$ with $\varphi \in L^\infty$. Then there exist $\varphi_n \in C_0^\infty(X)$ so that* (i) $\varphi_n \to \varphi$, $\nabla\varphi_n \to \nabla\varphi$ *in* $L^2(X, d\mu)$; (ii) $\varphi_n(x) \to \varphi(x)$ *for a.e. x*; (iii) $\sup_n \|\varphi_n\|_\infty < \infty$.

Proof. Without loss, take φ real valued. By Proposition 4.4, we can find $\tilde{\varphi}_n$ obeying (i) and then, by passing to a subsequence (using the Reisz–Fisher theorem), we can suppose that (ii) holds. Let $\|\varphi\|_\infty = a$ and pick $g \in C_0^\infty$ with $0 \leqslant g \leqslant 1$ and $g(s) = 1$ for $|s \leqslant a$. Let $F(x) = \int_0^x g(y)\, dy$ and $\varphi_n(x) = F(\tilde{\varphi}_n(x))$. Then $\|\varphi_n\|_\infty \leqslant \|F\|_\infty < \infty$ and $|\varphi_n - \varphi| \leqslant |\tilde{\varphi}_n - \varphi| \to 0$ in L^2. Finally $\nabla\varphi_n - \nabla\varphi = g(\varphi_n(x))\, \nabla\tilde{\varphi}_n - \nabla\varphi = [g(\varphi_n(x)) - 1]\, \nabla\varphi + g(\varphi_n(x))[\nabla\tilde{\varphi}_n - \nabla\varphi]$ is seen to go to zero in L^2 by eploying the dominated convergence theorem. ∎

Proof of Proposition 4.3 (*Dirichlet case*). Let φ_n be the sequence given by Lemma 4.5. Define $h(x) = x\,|x|^{a-1}$ which is C^1 and $\gamma_n = h(\varphi_n)$. Then $\gamma_n \to \gamma_n^a$ by dominated convergence and $\nabla\gamma_n = h'(\varphi_n)\, \nabla\varphi_n \to a\varphi^{a-1}\nabla\varphi$ by an argument identical to the one above proving that $\nabla\varphi_n \to \nabla\varphi$. ∎

The basic idea of the following is due to Gross [18]:

THEOREM 4.6. *Let A be a regular Dirichlet form. Suppose that*

$$\int |f|^2 \ln |f| \leqslant \varepsilon(f, Af) + b\,\|f\|_2^2 + \|f\|_2^2 \ln \|f\|_2 \tag{4.7}$$

for all $f \in C_0^\infty(X)$. Let $c(p) = \varepsilon$; $\Gamma(p) = 2b/p$. Then (4.1) *holds for all $f \in \mathscr{D}_+$ and all $p \in [2, \infty)$.*

Proof. Since $C_0^\infty(X)$ is a form core for A, Fatou's lemma implies that (4.7) holds for all f in $Q(A)$. Now let $f \in \mathscr{D}_+$. By Proposition 4.5, $f^{p/2}$ and

f^{p-1} lie in $Q(A)$. Moreover, since $\nabla f^{p/2}=(p/2)f^{p/2-1}\nabla f$ and $\nabla f^{p-1}=(p-1)f^{p-2}\nabla f$, we see that

$$(f^{p/2}, Af^{p/2}) = (f_p, Af)(p/2)^2(1/(p-1)). \tag{4.8}$$

Since $f^{p/2}\in Q(A)$, we can plug it into (4.7) and find

$$(p/2)\int f^p \ln f \leqslant \varepsilon(p/2)^2(p-1)^{-1}(f_p, Af) + b\,\|f\|_p^p + (p/2)\,\|f\|_p^p \ln \|f\|_p$$

which is precisely (4.1) for $c(p)=(p/2)(p-1)^{-1}\varepsilon$, $\Gamma(p)=2b/p$.

Equation (4.8) shows that $(f_p, Af)>0$. Since $\frac{1}{2}\leqslant\frac{1}{2}(p-1)^{-1}p\leqslant 1$, we can replace $c(p)=\frac{1}{2}(p-1)^{-1}p\varepsilon$ by ε (since $\frac{1}{2}(p-1)^{-1}p\geqslant\frac{1}{2}$ we don't lose very much by doing this and the arithmetic is simpler). ∎

From Theorems 4.2 and 4.6, we see that

THEOREM 4.7. *Let A be a regular Dirichlet form obeying* (4.7) *for all ε, some $b(\varepsilon)$, and all f in C_0^∞. Given t, suppose we can find $c(p)$ so that*

$$t=\int_2^\infty \frac{c(p)}{p}\,dp. \tag{4.9}$$

Then

$$\|e^{-tA}f\|_\infty \leqslant e^M\,\|f\|_2,$$

where

$$M=\int_2^\infty 2b(c(p))\,p^{-2}\,dp. \tag{4.10}$$

EXAMPLES. (1) Let $b(c)=A_0+A_1\ln(c^{-1})$. Take $c(p)=t(\ln 2)/(\ln p)^2$ so (4.9) holds. Then $M=A_2+A_1\ln(t^{-1})$, where

$$A_2=A_0-A_1\ln(\ln 2)+4A_1\int_2^\infty \ln(\ln p)\,p^{-2}\,dp.$$

and thus

$$b(c)=A_0+A_1\ln(c^{-1}) \Rightarrow \|e^{-itA}f\|_\infty \leqslant Bt^{-A_1}. \tag{4.11}$$

(2) Let $b(c)=Ac^{-l}$ for c small. Then, choosing $c(p)$ as above, $M=d_lAt^{-l}$. Now $e^{-x^\alpha}=\int_0^\infty g_\alpha(y)\,e^{-yx}\,dy$ where, for small y, the leading

behavior of $g_\alpha(y)$ is $\exp(-ct^{-b})$ with $b = a/(1-\alpha)$ (this is connected with the calculation in the proof of Theorem 3.3). Thus, if $b > l$, we can prove that e^{-tA^α} is ultracontractive. Solving for α we see that

$$b(c) = Ac^{-l} \text{ for } c \text{ small} \Rightarrow \|e^{-tA^\alpha} f\|_\infty \leqslant C_\alpha(t)\, \|f\|_2 \text{ if } \alpha < l/l+1.$$

This is of course connected with Theorem 3.3.

(3) $b(c) = \exp(c^{-a}) c$ small. Pick $c(p) = td(\alpha)/(ln\ p)^\alpha$. with $\alpha > 1$. Then $M < \infty$ if $a\alpha < 1$. As a result, we see that $a < 1 \Rightarrow e^{-tA}$ is ultracontractive. It is interesting to see that this borderline is the same as for Trudinger-type estimates; see [2]. In Section 6, we will find examples where $b(c) = \exp(c^{-1})$ and e^{-tA} is *not* ultracontractive.

5. Rosen's Lemma

In the last section we reduced ultracontractive estimates for regular Dirichlet forms to the L^2 estimate (4.7). Rosen [30] proved such estimates by exploiting ordinary Sobolev estimates and certain *lower* bounds on ψ_0 (earlier, by a more involved argument, Eckmann [14] has exploited Gross' logarithmic Sobolev estimate for $V = x^2$ and ψ_0 estimates). Hence, for the reader's convenience and because neither Rosen [30] nor Carmona [5] makes constants explicit, we prove the result using the approach of Carmona [5]:

THEOREM 5.1 (Rosen's lemma). *Let A be a regular Dirichlet form. Suppose that for all $\delta > 0$,*

$$-\log \psi_0 \leqslant \delta H + g(\delta) \tag{5.1}$$

as operators on $L^2(R^\nu, d^\nu x)$. Then (4.7) holds with

$$b(\varepsilon) = c + a_{1,\nu} - \frac{\nu}{4} \log \varepsilon + g\left(\frac{\varepsilon}{2}\right) \tag{5.2}$$

for all $\varepsilon < 1$. Here $a_{1,\nu}$ is a universal ν-dependent constant. In the Dirichlet case or when $V \geqslant 0$, $c = 0$ and in general it depends only on the K_ν-norm on V_-.

Proof. Since both $L^p(X, d^\nu x)$ and $L^p(X, d\mu)$ norms appear, we use the symbol $\|\cdot\|_p$ for the former and $\|\cdot\|_{p;\psi}$ for the latter. By replacing f by $f/\|f\|_{2;\psi}$ we need only show (4.7) if $\|f\|_{2;\psi} = 1$. Moreover, (4.7) holds if we show that $\varepsilon A + b - \log |f| \geqslant 0$ since we can take expectations in the vector f. But this operator on $L^2(X, d\mu)$ is unitarily equivalent to $\varepsilon H + b - \log |f|$ on $L^2(X, d^\nu x)$, so we are reduced to proving

$$\|f\|_{2;\psi} = 1 \Rightarrow \log |f| \leqslant \varepsilon H + b(\varepsilon). \tag{5.3}$$

Sobolev's inequality [27, p. 31] says that

$$\int \frac{|f(x)|\,|f(y)|}{|x-y|^\alpha}\, d^\nu x\, d^\nu y \leqslant C_\alpha \|f\|_p^2$$

if $0 < \alpha < \nu$ and $2p^{-1} + \nu^{-1}\alpha = 2$. If we take $\alpha = \nu - \frac{1}{2}$ and note that $(-\Delta)^{-1/4}$ is convolution with $d_\nu |x|^{-\alpha}$, then the above reads

$$(f, (-\Delta)^{-1/4} f) \leqslant C_\nu^{(1)} \|f\|_p^2,$$

where $p^{-1} = \frac{1}{2} + (4\nu)^{-1}$ (we take $\frac{1}{4}$ rather than $\frac{1}{2}$ or 1 so we can include $\nu = 1, 2$). Writing $f = g^{1/2}\psi$ with $g \in L^{2\nu}$ and $\psi \in L^2$, we see that

$$\| g^{1/2}(-\Delta)^{-1/4} g^{1/2} \|_{2,2} \leqslant C_\nu^{(1)} \| g \|_{2\nu}$$

or for suitable k_ν (which can be computed from best constants in Sobolev inequality)

$$g \leqslant k_\nu^{1/4} \| g \|_{2\nu} (-\Delta)^{1/4}.$$

Since $x \leqslant x^4 + 1$,

$$g \leqslant 1 + k_\nu \| g \|_{2\nu}^4 (-\Delta). \tag{5.4}$$

Given f with $\|f\|_{2;\psi} = 1$, let $A_\beta = \{x \mid |f|\beta\psi_0 \geqslant 1\}$ and let $\mathscr{X}_\beta$ be its characteristic function. Then, by (5.4)

$$\begin{aligned} \log(\beta\,|f|\,\psi_0) &\leqslant \mathscr{X}_\beta \log(\beta\,|f|\,\psi_0) \\ &\leqslant 1 + k_\nu \|\mathscr{X}_\beta \log(\beta\,|f|\,\psi_0)\|_{2\nu}^4 (-\Delta). \end{aligned} \tag{5.5}$$

For all $y \geqslant 1$, there exists a constant d_ν so that

$$(\log y)^{2\nu} \leqslant d_\nu y^2.$$

Thus

$$\|\mathscr{X}_\beta \log(\beta\,|f|\,\psi_0)\|_{2\nu}^{2\nu} \leqslant d_\nu \|\beta\,|f|\,\psi_0\|_2^2 = d_\nu \beta^2 \|f\|_{2;\psi}^2 = d_\nu \beta^2. \tag{5.6}$$

Moreover, we have

$$-\Delta \leqslant 2H + \tilde{c}, \tag{5.7}$$

where $\tilde{c}=0$ if $V \geqslant 0$ or in the Dirichlet case and, in general, where $\tilde{c}$ only depends on the K_ν norm of V_-. Equations (5.5)–(5.7) imply that

$$\log|f| \leqslant 2k_\nu d_\nu^{2/\nu}\beta^{4/\nu}(H+\tfrac{1}{2}\tilde{c}) - \log\beta - \log\psi_0 + 1.$$

Now, use (5.1) with $\delta=\varepsilon/2$ and choose β so $2k_\nu d_\nu^{2/\nu}\beta^{4/\nu}=\frac{1}{2}\varepsilon$; Eq. (5.3) results with $b(\varepsilon)$ given by (5.2). ∎

If we combine Rosen's lemma with the three examples at the end of Section 4, we see (a)–(c) below; (d), which is Rosen's result [30], has a similar proof:

THEOREM 5.2. *Let A be a regular Dirichlet form so that for all $\delta>0$*

$$-\log|\psi_0| \leqslant \delta H + g(\delta). \tag{5.8}$$

(a) *If $g(\delta)\leqslant O(\exp(\delta^{-a}))$ for some $a<1$ and δ small, then e^{-tA} is ultracontractive.*

(b) *If $g(\delta)\leqslant O(\delta^{-l})$ for δ small, then A^α generates an ultracontractive semigroup for any $\alpha>1-(l+1)^{-1}$.*

(c) *If $g(\delta)\leqslant B_0+B_1\log(\delta^{-1})$*, then for t small

$$\|e^{-tA}g\|_\infty \leqslant Ct^{-(B_1+(1/4)\nu)}\|g\|_2 \tag{5.9}$$

and

$$\|e^{-tA}g\|_\infty \leqslant C't^{-((1/2)\nu+2B_1)}\|g\|_1. \tag{5.10}$$

(d) *Without any restriction on g, e^{-tA} is supercontractive.*

Remarks. (1) (5.10) follows from (5.9) if we duality to note that

$$\|e^{-tA}\|_{2,1}=\|e^{-tA}\|_{\infty,2} \qquad \text{and} \qquad \|e^{-tA}\|_{\infty,1}\leqslant\|e^{-\frac{1}{2}tA}\|_{\infty,2}\|e^{-\frac{1}{2}tA}\|_{2,1}.$$

(5.10) is of particular interest since it tells us about the sup norm of the integral kernel. The $\frac{1}{2}\nu$ in the exponent is in a real sense a reflection of the $t^{-(1/2)\nu}$ divergence of this sup norm for $e^{t\Delta}$. Its appearance suggests that our arguments have not lost anything as far as leading order is concerned.

(2) In the next section, we will find an example with $g(\delta)=O(\exp(\delta^{-1}))$ for which e^{-tA} is *not* ultracontractive.

6. ULTRACONTRACTIVITY FOR SCHRÖDINGER OPERATORS

Armed with Theorem 5.2, we can examine intrinsic ultracontractive properties of Schrödinger semigroups. For virtually all these applications, we will prove

$$-\log \psi_0 \leqslant \delta V_+ + g(\delta) \tag{6.1}$$

which implies (5.1) with an inessential change in $g(\delta)$. We begin with the one dimensional case.

THEOREM 6.1. (a) *If* $V(x) = |x|^a$ *and* $\nu = 1$, *then* e^{-tH} *is intrinsically ultracontractive if* $a > 2$, *intrinsically hypercontractive* (*but not supercontractive*) *if* $a = 2$, *and not even intrinsically hypercontractive if* $a < 2$.

(b) *Let* $V(x) = x^2[\ln|x| + 2)]^b$. *Then* e^{-tH} *is intrinsically ultracontractive if* $b > 2$ *and intrinsically supercontractive* (*but not ultracontractive*) *if* $0 < b \leqslant 2$.

Proof. The result for $a < 2$ (due to Carmona [7]) follows from Theorem A.8 and the result for $a = 2$ follows from Nelson's result [24] which specifies precisely that in case $A = \tilde{H}$ with $H = -d^2/dx^2 + x^2$, then $e^{-tA}: L^p \to L^q$ if and only if $e^{-4t} \leqslant (p-1)/(q-1)$.

In all other cases ($a > 2$ or $b > 0$), one can prove that if ψ is in L^2 near $\pm\infty$ and obeys

$$-\psi'' + V\psi = E\psi \tag{6.2}$$

then near $\pm\infty$,

$$|\psi(x)| \sim C_\pm (V(x) - E)^{-1/4} \exp(-W) \tag{6.3a}$$

$$W(x) = \int_0^x \{\sqrt{V(y)} - \tfrac{1}{2}E[V(y)]^{-1/2}\}\, dy \tag{6.3b}$$

in the sense of the ratio going to 1. (6.3) can be proven either by variation of parameters or by subharmonic comparison arguments (see Appendix B).

Case 1: $a > 2$. Then $W(x) \sim x^{1+(1/2)a} + C$ and so $-\log \psi_0 \sim x^{1+(1/2)a} \leqslant \delta x^a + C_a \delta^{-g(a)}$ with $g(a) = (a+2)/(a-2)$. Thus A is intrinsically ultracontractive. Indeed A^α is intrinsically ultracontractive if $\alpha > a^{-1} + \frac{1}{2}$ (by Theorem 5.2(b)).

Case 2: $b > 0$. Then $-\log \psi_0 \sim x^2(\ln x)^{(1/2)b}$. By elementary calculus, for δ small the maximum of $-x^2(\ln x)^{(1/2)b} + \delta x^2(\ln x)^b$ occurs at $x \sim \exp(c\delta^{-2/b})$ and so $g(\delta)$ can be chosen $C_b \exp(D_b \delta^{-2/b})$. If $b > 2$, we obtain intrinsic ultracontractivity and for $0 < b \leqslant 2$ we certainly have intrinsic supercontractivity.

Case 3: $0 < b \leqslant 2$. Let ψ_1 be the second eigenfunction for H. Then $\psi_1/\psi_0 \sim \tilde{C}_\pm \exp(\frac{1}{2}(E_1 - E_0) \int_0^x V(y)^{-1/2}\, dy)$. Since $\int_0^\infty V(y)^{-1/2}\, dy = \infty$ (precisely if $b \geqslant 2$), ψ_1/ψ is not bounded and so e^{-tA} cannot be intrinsically ultracontractive. ∎

Remarks. (1) $V(x) = x^2(\ln x)^2$ is the promised example with $g(\delta) = C\exp(\delta^{-1})$ and no ultracontractivity.

(2) One can show by the same methods that if $V(x) \sim x^2(\ln x)^2(\ln\ln x)^c$, then there is not ultracontractivity if $c \leqslant 2$ and there is of $c > 2$.

To obtain more general ultracontractivity results, one must obtain lower bounds on ψ_0. In this regard, the following comparison theorem is useful:

LEMMA 6.2 (see e.g., [33]). *If $W(x)$, $V(x) - W(x)$ both go to infinity at ∞ and if ψ_0^W, ψ_0^V are the ψ_0's corresponding to $-\Delta + W$ and $-\Delta + V$, then*

$$\psi_0^V(x) \leqslant c\psi_0^W(x)$$

for some constant c.

THEOREM 6.3. *Suppose that for some $C_1, C_3 > 0$, C_2, C_4, we have that*

$$C_3|x|^b + C_4 \leqslant V_1(\mathbf{x}) \leqslant C_1|x|^a + C_2,$$

where $\frac{1}{2}a + 1 < b$. Then $-\Delta + V$ is intrinsically ultracontractive.

Proof. $-\Delta + C_1|x|^a + C_2$ has a ψ_0 which looks like $x^{-a/4}\exp(-\sqrt{C_1}(a+1)^{-1}|x|^{(1/2)a+1})$ at ∞ (since $a > 2$ is implied by $\frac{1}{2}a + 1 < b \leqslant a$). By the lemma, the ψ_0 for $-\Delta + V$ obeys

$$-\ln\psi_0 \leqslant C|x|^{(1/2)a+1}. \tag{6.4}$$

As in Case 1 in the proof of Theorem 6.1, we obtain

$$g(\delta) \leqslant C\delta^{-\alpha}, \qquad \alpha = (a+2)/[2b-a-2]. \quad \blacksquare$$

Remark. (6.4) can also be obtained by a simple path space estimate. One need only write $\psi_0(x) = E_x(\exp(-\int_0^t (V(b(s)) - E)\,ds)\,\psi_0(b(t)))$. Pick $t = x^{1-(1/2)a}$ and estimate the contribution of paths with $|b(s) - (1 - s/t)x| < 1$.

Sokal [42] has noted a rather striking implication of Theorem 6.3 for $P(\varphi)_1$-stochastic processes (see [43, p. 57] for definition). Consider the conditional distribution for $q(t)$ given that $q(0) = a$. This is just

$$\begin{aligned} d\mu_a(x) &\equiv \psi_0(a)^{-1}\exp(-|t|\,(H - E_0))(a, x)\,\psi_0(x)\,dx \\ &\equiv F_a(x)\,dx \end{aligned}$$

with $H = -d^2/dx^2 + P(x)$ on $L^2(-\infty, \infty)$; P a polynomial with

$\lim_{|x|\to\infty} P(x) = \infty$. Ultracontractivity (i.e., $\deg P > 2$) implies a bound, uniformly in a (but not in t)

$$d_t \psi_0(x)^2 \leqslant F_a(x) \leqslant c_t \psi_0(x)^2$$

(and it is easy to show $\lim_{a\to+\infty} F_a(x)$ exists: One proof uses the eigenfunction expansion, the fact that $\psi_n \psi_0^{-1}$ has a limit any eigenfunction n and ultracontractivity to control the convergence of the series; another proof [42] uses the FKG inequalities to note that $\int g(x) F_a(x)\, d\mu(x)$ is increasing in a if g is increasing together with the above uniform bounds on F_a). Thus, taking $q(0) = a \to \infty$ does *not* drag $q(t)$ off to infinity; rather, the process "recovers" to its typical values very quickly. This is connected to the discussion in Rosen & Simon [41]. This behavior is in sharp contrast to the behavior of $P(\varphi)_1$ *lattice* theories which are definitely *not* ultracontractive [42].

Two dimensional operators of the form $-\Delta + x^k y^k$ have recently produced some interest, they have compact resolvent although $x^k y^k \not\to \infty$ at ∞ [36].

THEOREM 6.4. *None of the operators $H = -\Delta + x^k y^k$ is intrinsically hypercontractive.*

Proof. Let $f, g \in C_0^\infty(R)$ with $\int f^2\, dx = \int g^2\, dx = 1$ and let

$$\varphi_{x_0}(x, y) = f(x - x_0)\, g(y x_0^{(1/2)\beta})\, x_0^{(1/4)\beta},$$

where $\beta = 2k/k + 2$. Then for x_0 large,

$$(\varphi_{x_0}, H\varphi_{x_0}) = O(x_0^\beta)$$

while

$$(\varphi_{x_0}, (x^2 + y^2)\, \varphi_{x_0} = O(x_0^2).$$

Since $\beta < 2$, it is false that $H \geqslant cx^2 - d$ for any c, d so by Theorem A.8, H is not intrinsically hypercontractive. ∎

The above suggests that the $k \to \infty$ limit, i.e., the Dirichlet Laplacian for the region $\{(x, y) \mid |xy| \leqslant 1\}$ is hypercontractive. Indeed, this is true; see Section 9.

As a final subject, we want to discuss uniform hypercontractive estimates in the semiclassical limit. The usefulness of such estimates in the theory of multiple wells was pointed out by one of us [9] and this question was one of our motivations in initiating this work. Let $V(x)$ be a C^∞ function of R^ν obeying

(a) $V(x) \geqslant 0$; for some R and $\varepsilon > 0$, $V(x) \geqslant \varepsilon$ if $|x| \geqslant R$;

(b) $V(x)=0$ for some $x^{(1)},\dots,x^{(k)}$ only and $(\partial^2 B/\partial x_i\,\partial x_j)(x^{(a)})$ is nonsingular for $a=1,\dots,k$. Define

$$H(g)=-\tfrac{1}{2}\Delta+gV(g^{-1/2}x). \tag{6.5}$$

Define ψ_g to be the ψ_0 for $H(g)$. We will suppose

$$\lim_{g\to\infty}\int_{|x-g^{1/2}x^{(a)}|\leqslant 1}|\psi_g(x)|^2\,d^\nu x>0 \tag{6.6}$$

for $a=1,2,\dots,k$ so the ground state lies in all the semiclassical wells (see [20, 10, 35, 21] for a discussion of such tunneling problems).

Introduce the symbol $\|\cdot\|_{p;g}$ by

$$\|f\|_{p;g}^p=\int|f(x)|^p\,|\psi_g(x)|^2\,d^\nu x$$

and as usual

$$\tilde{H}(g)=\psi_g^{-1}(H(g)-E_0(g))\,\psi_g.$$

We will also need to assume that for $|x|\geqslant R$,

$$C_1\,|x|^b\leqslant V(x)\leqslant C_2\,|x|^a,\qquad \tfrac{1}{2}a+1\leqslant b\leqslant a. \tag{6.7}$$

THEOREM 6.5. *Let* (6.6) *and* (6.7) *hold. For a suitable C, T we have for all $g\geqslant 1$,*

$$\|e^{-T\tilde{H}(g)}f\|_{4;g}\leqslant C\,\|f\|_{2;g}.$$

Proof. By general principles, $E_0(g)$ is bounded. Thus, by Rosen's lemma and Theorems 4.2, 4.6 it suffices to prove that

$$-\log\psi_g(x)\leqslant c_1\,gV(g^{-1/2}x)+c_2 \tag{6.8}$$

for constants c_1, c_2 independent of g.

Define $\eta_g(x)=\psi_g(g^{1/2}x)$ so (6.8) is equivalent to

$$-\log\eta_g(x)\leqslant c_1\,gV(x)+c_2. \tag{6.9}$$

We also note that η_g obeys

$$[-\tfrac{1}{2}\Delta+g^2V(x)]\,\eta_g=gE_0(g)\,\eta_g. \tag{6.10}$$

We also note two additional preliminaries which follow from [35]:

(i) On any compact set K we have

$$\eta_g(x) \geqslant C_1 e^{-C_2 g} \qquad \text{all} \quad x \in K,\ g \geqslant 1. \tag{6.11}$$

(ii) For $|x - x_a| \leqslant g^{-1/2}$

$$\eta_g(x) \geqslant C_3. \tag{6.12}$$

(6.11) is proven in [35], where $\lim_{g \to \infty}(1/g) \ln \eta_g(x)$ is computed uniformly in $x \in K$; actually, the simple lower bound (6.11) can be obtained by a simple estimate without recourse to the theory of large deviations (just modify our argument below controlling the region $|x - x_a| \leqslant \delta$). (6.12) is proven [35] by showing that ψ_g converges in L^∞-norm to a combination of Gaussians centered at the points $x^{(a)}$.

From (6.11) on the ball of radius R, the hypothesis (6.7) and a comparison argument using (6.10), we see that for $|x| \geqslant R$,

$$\eta_g(x) \geqslant C_1 e^{-C_2 g} e^{-C_4 g |x|^\beta}, \qquad \beta = \tfrac{1}{2}a + 1.$$

Thus, by the hypothesis (6.7) again, (6.9) holds in the region $|x| \geqslant R$.

By (6.11) again, we can be sure that (6.9) holds in the region $\{x \mid |x| \leqslant R, |x - x^{(a)}| \geqslant \delta\}$ for a fixed small δ. We can choose δ so that

$$A(x - x^{(a)})^2 \leqslant V(x) \leqslant B(x - x^{(a)})^2 \qquad \text{if} \quad |x - x^{(a)}| \leqslant \delta. \tag{6.13}$$

Using (6.12) and (6.13), we want to prove that if $|x - x^{(a)}| \leqslant \delta$,

$$\eta_g(x) \geqslant C_5 \exp(-C_6\, g(x - x^{(a)})^2), \tag{6.14}$$

in which case we have (6.9) in the last region needed. Let $f(s) = (1 - T^{-1}s)x + T^{-1}sx_a$. By the Feynman–Kac formula and (6.10), since $E_0(g) > 0$,

$$\begin{aligned} \eta_g(x) &\geqslant E_x(e^{-g^2 \int_0^T V(b(s))\, ds}\, \eta_g(b(T))) \\ &\geqslant d e^{-d_2 g^2 (x - x_a)^2 T} P_x(|b(s) - f(s)| \leqslant g^{-1/2}; 0 \leqslant s \leqslant T), \end{aligned} \tag{6.15}$$

where E_x (resp. P_x) is expectation (resp. probability) for Brownian motion starting at x. We get the second inequality (following an idea of [7]) by looking at the contribution of paths with $|b(s) - f(s)| \leqslant g^{-1/2}$, using (6.13) to get an upper bound on $\int_0^T V(b(s))\, ds$ and (6.12) to estimate $\eta_g(b(T))$ from below.

By the Cameron–Martin formula and Jensen's inequality (see [7]):

$$\begin{aligned} &P_x(|b(s) - f(s)| \leqslant g^{-1/2}; 0 \leqslant s \leqslant T) \\ &\qquad \geqslant e^{-(1/2)\int_0^T \dot{f}(s)^2 ds}\, P_0(|b(s)| \leqslant g^{-1/2}; 0 \leqslant s \leqslant T) \\ &\qquad = e^{-d_3 T^{-1}(x - x^{(a)})^2} P_0(|b(s)| \leqslant g^{-1/2}; 0 \leqslant s \leqslant T\}. \end{aligned} \tag{6.16}$$

Choosing $T = g^{-1}$ and noting that the scaling

$$P_0(|b(s)| \leqslant g^{-1/2}; 0 \leqslant s \leqslant g^{-1})$$

is a positive constant, we obtain (6.14) from (6.15), (6.16). ∎

Remarks. (1) (6.7) is a natural condition, given Theorem 1.3, if we want $e^{-T\tilde{H}(g)}$ to be hypercontractive even for fixed g.

(2) Because, in the limit, $H(g)$ looks like a harmonic oscillator near each $x^{(a)}$, we cannot hope to have anything better than hypercontractivity uniformly in g.

(3) For simple cases, like the one-dimensional double well, one can use the theory of large deviations to prove this theorem directly.

7. Dirichlet Laplacians and Second Order Elliptic Operators: Lower Bounds on ψ_0 and Interior Conditions

In the last three sections of this paper, we want to discuss intrinsic contractive properties of Dirichlet Laplacians. The key will be to verify (5.1) in two steps. In this section, we will concentrate on obtaining lower bounds on $\psi_0(x)$ in terms of the distance to the boundary, and in the next, we will see when suitable functions of this distance can be bounded by the Dirichlet Laplacian. As we already emphasized in the Introduction, there are bounded regions for which the Dirichlet Laplacian fails to be intrinsically ultracontractive so one must expect various regularity conditions on the boundary to be relevant.

We begin with the case of smooth boundary. This case can be easily analyzed for more general objects than Laplacians. Let X be a bounded open connected subset of R^ν with C^∞ boundary, ∂X (i.e., there is a C^∞ function, $F(x)$, so that $X = \{x \mid F(x) > 0\}$ and so that $(\nabla F)(x)$ does not vanish for $x \in \partial X$). Let $a(x)$ be a C^∞-$(\nu \times \nu)$-matrix valued function on $\bar{X}$ with values in the real symmetric matrices. Assume for some $\lambda, \mu > 0$ one has the matrix inequality

$$\lambda 1 \leqslant a(x) \leqslant \mu 1 \tag{7.1}$$

and define

$$H = -\sum_{i,j=1}^{\nu} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} \tag{7.2}$$

with Dirichlet boundary conditions, i.e., H is the form closure of the obvious form on $C_0^\infty(\Omega)$. The relevant lower bound on ψ_0 follows from a modification of the Hopf boundary point lemma [15]:

THEOREM 7.1. *Let* $\rho(x) = \mathrm{dist}(x, R^\nu \backslash X)$. *Then, for a suitable constant* $a > 0$,

$$\psi_0(x) \geqslant a\rho(x) \qquad \textit{for all} \quad x \in \Omega. \tag{7.3}$$

Proof. It is not hard to show that $\rho(x)$ is C^∞ in a neighborhood of ∂X and that $|\nabla\rho| = 1$ there. Thus, we can find a C^∞ function f on $\bar{X}$ with (i) $f \upharpoonright \partial X = 0$, (ii) $|\nabla f| = 1$ near ∂X, and (iii) $f \geqslant \rho$ on $\bar{X}$. Clearly $f^2 \in D(H)$ and by a direct computation,

$$H(f^2)(x) = -2f(x)\,\alpha(x) - 2\sum_{i,j} a_{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}, \tag{7.4}$$

where

$$\alpha(x) = -\sum_{i,j}\left(\frac{\partial a_{ij}}{\partial x_i}\frac{\partial f}{\partial x_j} + a_{ij}\frac{\partial^2 f}{\partial x_i\,\partial x_j}\right).$$

Since α is bounded, the first term in (7.4) goes to zero as $x \to \partial X$ while the second is bounded by 2λ by (7.1) and the fact that $|\nabla f| = 1$ near ∂X. Thus, there exists ε with

$$[H(f^2)](x) \leqslant -\lambda$$

if $\rho(x) < \varepsilon$ and thus a (large) number b so that if $\rho(x) < \varepsilon$, then

$$g(x) \equiv [H(f + b^2 f^2)](x) \leqslant -1. \tag{7.5}$$

By (7.5) and the fact (Harnack's inequality) that ψ_0 is strictly positive on $\{x \mid \rho(x) \geqslant \varepsilon\}$, we see that

$$\psi_0(x) \geqslant cg(x) \tag{7.6}$$

for some $c > 0$. Thus, since $H\psi_0 = E_0\psi_0$ and H^{-1} is positivity preserving

$$\begin{aligned}\psi_0 &= E_0 H^{-1}\psi_0 \geqslant cE_0 H^{-1} g\\ &= E_0 c(f + b^2 f) \geqslant E_0 cf \geqslant E_0 c\rho\end{aligned}$$

as desired. ∎

An estimate like (7.3) is definitely false when X has corners. For example, $-\Delta$ on a ν-dimensional hypercube has a ψ_0 which vanishes as ρ^ν at a "corner." There has been extensive study of the behavior of eigenfunctions of

$-\Delta$ on polyhedra, see, e.g., Grisvard [17]. For example, for X the interior of a polygon in R^2, the best one can do in $\psi_0 \geqslant c\rho^\beta$ is to take

$$\beta = \pi/\alpha, \tag{7.7}$$

where α is the minimum angle in ∂X. For studies of boundary behavior in more general regions, see Oddson [47] and Miller [45, 46].

Let $A \subset S^{\nu-1}$ be an open set. Define, for $x \in R^\nu$ and $\varepsilon > 0$,

$$C(x, A, \varepsilon) = \{ y \mid 0 < |x-y| < \varepsilon;\ y - x/|y-x| \in A \}$$

the (truncated cone at x with base A. Given $A \subset S^{\nu-1}$, we can define the Dirichlet Laplace–Beltrami operator L_A on A by restricting the Laplace–Beltrami operator on $S^{\nu-1}$ to $C_0^\infty(A)$ and closing the form. Let $\lambda(A)$ be the smallest eigenvalue of L_A and define $\alpha(A)$ by $\alpha > 0$ and $\alpha(\alpha + \nu - 2) = \lambda(A)$.

DEFINITION. Let A be an open subset of $S^{\nu-1}$. We say that X, an open subset of R^ν obeys an A-interior cone condition if there exists an $\varepsilon > \delta > 0$, $\beta > 0$, and for each $x \in X$ with $\rho(x) < \delta$, a point $y(x) \in \partial X$ and a rotation R_x so that (see Fig. 1),

$$x \in C(y(x), R_x(A), \varepsilon) \subset X$$
$$\operatorname{dist}(x - y(x)/|x - y(x)|, S^{\nu-1} \backslash R_x(A)) > \beta. \tag{7.8}$$

THEOREM 7.2. *Let X obey an A-interior cone condition. Then for some C*

$$\psi_0(x) \geqslant C\rho(x)^{\alpha(A)}, \tag{7.9}$$

where ψ_0 is the lowest eigenfunction of the Dirichlet Laplacian in X.

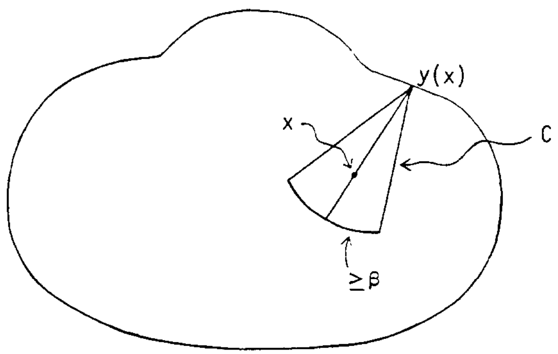

FIGURE 1

Proof. We have $D(\mu) \equiv \inf(\psi_0(x) \mid \rho(x) \geqslant \mu) > 0$ if $\mu > 0$ since ψ_0 is continuous and cannot vanish in X by Harnack's inequality. Let Ω_A be defined on $S^{\nu-1}$ by $L_A \Omega_A = \lambda(A)\, \Omega_A$, $\Omega_A > 0$ and $\sup_A \Omega_A(\omega) = 1$. Given x_0 with $\rho(x_0) < \delta$ (δ given by the A-interior condition), consider

$$\eta_{x_0}(x) = f(|x - y(x_0)|)\, \Omega_A(R_{x_0}[x - y(x_0)/|x - y(x_0)|])$$

defined on $C(y(x), R_x(A), \varepsilon)$, where f obeys

$$-f'' - (\nu - 1)\, r^{-1} f' + \lambda(A)\, r^{-2} f = E_0 f,$$

where E_0 is the lowest eigenvalue of $-\Delta_X$ and, where f obeys, where f obeys $f(0) = 0$, $f(\varepsilon) = D(\varepsilon)$. Then ψ_0 and η both obey $-\Delta u = E_0 u$ on $C(y(x), R_x(A), \varepsilon)$ and $\psi_0 \geqslant \eta$ on ∂C. By a maximum principle argument (see, e.g., [12]), $\psi_0 \geqslant \eta$ on all of C. It is easy to prove that $f(r)$ is asymptotic to $r^{\alpha(A)}$ for r small, so by (7.8) (which lets us bound $\Omega_A(R_{x_0}[x_0 - y(x_0)/|x - y(x_0)|])$ away from zero), (7.9) holds if $\rho(x) \leqslant \delta$. The bound when $\rho(x) \geqslant \delta$ (for suitable (c) is trivial since $D(\delta) > 0$. ▮

EXAMPLE. If $\nu = 2$, and A is the subset of the circle $\{\theta \mid 0 < \theta < \theta_0\}$, then $\Omega_A = \sin(\pi\theta/\theta_0)$, $\lambda(A) = (\pi/\theta_0)^2$, and $\alpha(A) = \pi/\theta_0$. In particular, this yields the correct lower bound for polygons [17].

The final issue we want to discuss in this section concerns lower bounds on the ψ_0 associated to the Dirichlet Laplacians of certain unbounded regions. For simplicity, we restrict ourselves to two dimensions and take X to have the form

$$X = \{(x, y) \mid |y| \leqslant F(x) \text{ and } x \geqslant 0\}, \tag{7.10}$$

where F obeys: (i) $F_{(x)}$ is bounded and C^1 on $[0, \infty)$, (ii) $F(x) \to 0$ as $x \to \infty$, (iii) $F'(x) \leqslant 0$ for all large x, (iv) $F'F^{-1} \to 0$ as $x \to \infty$. The examples to keep in mind are $F(x) = (x+1)^{-1} \ln(x+2)^{-\beta}$.

THEOREM 7.3. *Let ψ_0 be the ground state eigenfunction for the Dirichlet Laplacian in a region X given by* (7.10) *with F obeying* (i)–(iv) *above. Let $\rho(x, y) \equiv \operatorname{dist}((x, y), R^2 \backslash X)$. Then for suitable C, $D > 0$,*

$$\psi_0(x, y) \geqslant C\rho(x, y)^2 \exp(-DxF(x)^{-1}). \tag{7.11}$$

Remarks. (1) It is probable that with some additional estimates, we could replace $\rho(x, y)^2$ by $\rho(x, y)$.

(2) In the next section, we will obtain an upper bound on ψ_0 by $\exp(-c \int_0^x F(z)^{-1}\, dz)$ which, in typical cases, looks like $\exp(-\tilde{c}xF(x)^{-1})$ so (7.10) captures the leading crude correct rate of falloff along the axis $y = 0$.

(3) Our method of proof follows that of Agmon for decay of N-body eigenfunctions (Agmon's proof is unpublished but is described in Simon [35]). Alternatively, one could use path integral estimates.

Proof. By the argument in Theorem 7.1, (7.11) holds in any region $\{(x, y) \in X \mid x \leqslant \alpha\}$ with α fixed, so we need only check it if x is sufficiently large. We thus consider $\alpha > 1$ so large that $F(\alpha) < 1$ and that if $x > \alpha$,

$$F(x+1) \geqslant \tfrac{1}{2}F(x) \tag{7.12a}$$

$$-F'(x) \leqslant 1 \tag{7.12b}$$

which is possible by (ii), (iv).

Suppose now that $\alpha < a < \infty$ and $0 \leqslant b < F(A)$ and set $c = F(a) - b$. Let R be the rectangle

$$R = \left\{(x, y) \,\middle|\, |y| \leqslant b + \frac{c}{2}, \frac{c}{2} \leqslant x \leqslant a + \frac{c}{2}\right\}.$$

We claim that $R \subset X$. This is equivalent to

$$G(\beta) \equiv F(a + \tfrac{1}{2}(F(a) - \beta)) - \tfrac{1}{2}(\beta + F(a)) \geqslant 0 \tag{7.13}$$

for $\beta = b$. (7.13) holds because $G(\beta = F(a)) = 0$ and $dG/d\beta \leqslant 0$ by (7.12).

Let G be the function

$$g(x, y) = \cos\left(\frac{\pi}{2}\frac{y}{b + 1/2c}\right) \sinh\left(\frac{\pi}{2}\left(\frac{\pi}{2}\frac{a + c/2 - x}{b + c/2}\right)\right) \Big/ \sinh\left(\frac{\pi}{2}\frac{a}{b + c/2}\right).$$

Then $\Delta g = 0$ and $g = 0$ on three sides of ∂R. On the fourth side ($x = c/2$, $|y| < b + c/2$), $g \leqslant 1$ while ψ_0 is bounded below there (since $c \leqslant F(1)$ is bounded). Thus $\psi_0 \geqslant dg$ on R for a d independent of a, b. In particular,

$$\begin{aligned}\psi_0(a, b) &\geqslant d \sin\left(\frac{\pi}{2}\frac{c/2}{b + 1/2c}\right) \sinh\left(\frac{\pi}{2}\frac{c/2}{b + c/2}\right) \Big/ \sinh\left(\frac{\pi}{2}\frac{a}{b + c/2}\right) \\ &\geqslant \tilde{d}\left(\frac{F(a) - b}{F(a) + b}\right)^2 e^{-\pi a/2F(a)}.\end{aligned}$$

Since $\frac{1}{2}c \leqslant \rho(a, b) \leqslant c$, this implies (7.11). ∎

8. Dirichlet Laplacians: Quasiradii and Exterior Conditions

In the last section, we found upper bounds on $-\ln \psi_0$ in terms of the geometry of X, explicitly by functions connected to $\rho(x) \equiv \operatorname{dist}(x, R^{\nu} \backslash X)$. In this section, we want to bound functions of ρ by H so that in the next section we can prove that (5.1) holds. Our basic input will be an inequality of Davies [11].

DEFINITION. Given an open set X in R^ν, $x \in X$, and a unit vector e, let $d(x, e) = \inf\{|r| \mid x + re \notin X\}$ (so $\rho(x) \equiv \inf_e d(x, e)$). Define the *quasidistance* $q(x)$ by

$$q(x)^{-2} = \int_{S^{\nu-1}} d\mu_0(e)\, d(x, e)^{-2}, \tag{8.1}$$

where $d\mu_0$ is the usual normalized invariant measure on $S^{\nu-1}$.

Then Davies [11] proves

THEOREM 8.1. *For any X,*

$$q^{-2} \leqslant 4\nu^{-1} H_X, \tag{8.2}$$

where H_X is the Dirichlet Laplacian on $L^2(X, d^\nu x)$.

While (8.2) is powerful, its proof is not hard; by a one dimensional argument, one proves that $d(x, e)^{-2} \leqslant 4(e \cdot \nabla)^2$ and then averages over e. Since $d(x, e) \geqslant \rho(x)$ we immediately have

$$q(x) \geqslant \rho(x). \tag{8.3}$$

In order to go from (8.2) to useful bounds on $-\ln \psi_0$. we will need an estimate like $q(x) \leqslant c\rho(x)$. This is certainly not universally true as a ball in R^2 with a line segment removed easily shows. The following definition and the idea of Theorem 8.2 are also due to Davies [11]; we provide them here for the reader's convenience.

DEFINITION. We say that X obeys an *exterior cone condition* with angle θ and size r if and only if for any $x_0 \in \partial X$ there exists a unit vector $e(x_0)$ so that

$$\{x \mid 0 < |x - x_0| \leqslant r;\ (x - x_0) \cdot e(x_0) \geqslant \cos(\theta)\, |x - x_0|\}$$

is disjoint from X.

Given θ, let $a_\nu(\theta, \alpha)$ be twice the μ_0 measure of those $e \in S^{\nu-1}$ with $|e - (1, 0, \ldots)| < \alpha/(1 + \alpha) \sin \theta$. Up to inessential details, the following is Theorem 18 of [11]:

THEOREM 8.2. *If X obeys an exterior cone condition with angle θ and size r, then for all x with $\rho(x) < \alpha^{-1}r(1+\sin\theta)^{-1}$ we have that*

$$q(x) \leqslant (1+\alpha)\, a_\nu(\theta, \alpha)^{-1/2}\rho(x). \tag{8.4}$$

Theorems 8.1 and 8.2 immediately imply

COROLLARY 8.3. *Let X be a region which obeys an exterior cone condition so that either* (i) $r=\infty$, *or* (ii) $\rho(x)$ *is bounded on X. Then for a suitable constant c,*

$$\rho^{-2} \leqslant cH_X. \tag{8.5}$$

Remark. For the applications in the next section, it suffices that $\rho^{-\alpha} \leqslant cH_X$ for some $\alpha > 0$ (although the power in the rate of divergence of $\|e^{-tH}\|_{2,\infty}$ will depend on α). We say that X obeys an *exterior trumpet condition* of degree $\beta \geqslant 1$ if and only if there is a C and r so that for any $x_0 \in \partial X$ there is a rotation $R(x_0)$ with $R(x_0)\, T(C, r, \beta)$ disjoint from X, where T is the trumpet $T = \{(x_1,..., x_0) \mid 0 < x_\nu \leqslant r,\ |(x_1,..., x_{\nu-1}, 0)| \leqslant Cx_\nu^\beta\}$. The proof of Theorem 18 of [11] shows that $q(x)^{-2} \geqslant \tilde{C}\rho^{-2}\rho^{(\beta-1)(\nu-1)}$, so we have a bound of the form $\rho^{-\alpha} \leqslant cH_X$ so long as $\beta < 1 + 2/(\nu-1)$. By working slightly harder, one can probably obtain an estimate for β up to $1 + 2/(\nu-3)$ but if $\beta \geqslant 1 + 2(\nu-3)^{-1}$, (and $\nu > 3$), and X is the exterior of such a trumpet, then the tip of the trumpet is a point with $q(x) \neq 0$ and $\rho(x) = 0$, so one can not obtain a $\rho^{-\alpha} \leqslant cH_X$ estimate, at least via Theorem 8.1.

We can specialize this last corollary to horn-shaped regions of the form (7.9):

COROLLARY 8.4. *Let X be a horn-shaped region of the form* (7.10), *where F obeys* (i), (ii). *Then*

(a) $\rho^{-2} \leqslant c_1 H_X$,

(b) $F(x)^{-2} \leqslant c_2 H_X$.

Proof. X obeys an exterior cone condition with $\theta = \pi/4$, $r = \infty$, so (a) follows from Corollary 8.3. For (b), we need only note $\rho(x, y) \leqslant F(x)$. ∎

While the bounds in the last few results interest us here primarily because of their relevance to (5.1), we note they have implications for decay of ψ_0. This is because of the following result, essentially in Agmon [1].

THEOREM 8.5. *Suppose that H_X is a Dirichlet Laplacian in an open region X and let W obey $W(x) \leqslant H_X$ as an operator inequality, and let E be* *any eigenvalue of H_X with eigenvector ψ. Suppose that $\overline{\{x \mid W(x) \leqslant E\}}$ is compact (in the* open *set X). Let*

$$A(x) = \inf \left(\int_0^1 \sqrt{\max(0, W(\gamma(s)) - E)}\, |\dot\gamma(s)|\, ds \mid \gamma(0) = x, \gamma(s) = 0 \right).$$

Then for all $\varepsilon > 0$, $e^{(1-\varepsilon)A}\psi \in L^2$.

In most cases, one can go from an L^2 to an L^∞ bound; see, e.g., [37].

Example 1 (Bounded X with smooth boundary). Theorem 8.1 implies that a result for the W determined by $v/4q^{-2}$. In the case of smooth boundary, it is easy to see that $q(x) \sim v^{+1/2}\rho(x)$ for x near ∂X so $v/4q^{-2}$ diverges as $\frac{1}{4}\rho^{-2}$ near ∂X and thus $A(x) \sim \frac{1}{2} \ln \rho(x)^{-1}$ as X approaches ∂X. Then from Theorem 8.1 and 8.5, one obtains a bound by $C_\varepsilon \rho^{1/2-1/2\varepsilon}$. Since the true behavior is linear ρ, this is not so impressive but we still find it remarkable that these rather indirect methods yield a result that is so good. We note that even in one dimension, where on $L^2(0, a)$, $H_0 \geqslant \frac{1}{4}x^{-2}$ has the optimal constant, Agmon's method only yields to $x^{1/2}$ decay.

Example 2 (Bounded X with exterior conditions). For exterior cone conditions, one gets vanishing at least as fast as ρ^α for some α and with exterior trumpet conditions no vanishing whatsoever by this method! It is known (see Appendix C) that some exterior trumpet condition already implies ψ vanishes on ∂X.

Example 3 (Horn regions). From Corollary 8.4(b) and Theorem 8.5, one obtains an upper bound of the form $\exp(-\int_0^X F(x)^{-1}\, dx)$ on $\psi_0(x, y)$ for horn-shaped regions. As we already explained, this yields for "reasonable" F upper and lower bounds on $\psi_0(x, 0)$ which are qualitatively the same, i.e.,

$$C_1 \exp(-D_1 x F(x)^{-1}) \leqslant \psi_0(x, 0) \leqslant C_2 \exp(-D_2 x F(x)^{-1}). \tag{8.6}$$

We remark both arguments apply if we take instead $X \subset R^\nu$, and $X = \{(x_1, ..., x_\nu\} \mid |(x_1, ..., x_{\nu-1}, 0)| \leqslant F(x_\nu)\}$. One obtains (8.6) for $\psi_0(0, ..., 0, x_\nu = x)$.

9. Sobolev Estimates for Dirichlet Laplacians

In this section, we combine the preliminaries in Sections 5, 7, and 8 to obtain intrinsic ultracontractive estimates for Dirichlet Laplacians. Often our results will be good enough to obtain Sobolev estimates up to the boundary. We will require some geometric restrictions on X, so we begin with an example which shows that intrinsic ultracontractivity does not hold for the Dirichlet Laplacians of all bounded (open, connected) regions.

EXAMPLE 1. This will be a region in R^2 obeying an exterior cone condition (with angle θ arbitrarily close to $\pi/8$). The region will depend on three decreasing sequences of positive numbers $\{R_n\}_{n=0}^{\infty}\{d_n\}_{n=1}^{\infty}\{l_n\}_{n=1}^{\infty}$ with $\sum R_n < \infty$, $\sum l_n < \infty$, and $d_n < R_n$. Basically, the region will be shaped like a fir tree consisting of diamonds of size $2R_n$ connected by corridors of height l_n and width $2d_n$ (see Fig. 2). Define y_n inductively by $y_0 = 0$, $y_n = y_{n-1} + l_n + (R_n + R_{n-1})$. Let

$$Q_n = \{(x, y) \mid |x| + |y - y_n| < R_n\},$$
$$C_n = \{(x, y) \mid |x| < d_n,\ y_{n-1} < y < y_n\},$$

and $X = \bigcup_{n=0}^{\infty} Q_n \cup \bigcup_{n=1}^{\infty} C_n$. We will show if the d_n are sufficiently small one does not even have intrinsic hypercontractivity. We will do this by obtaining upper bounds on ψ_0 and lower bounds on b_t, the diagonal part of the integral kernel e^{-tH_0}, which show that $\int b_t^4 \psi_0^{-2}\, d^\nu x = \infty$ for all t, so e^{-tA_0} cannot map $L^2(X, \psi_0^2\, d^\nu x)$ to $L^4(X, \psi_0^2\, d^\nu x)$ for any t by the arguments in Theorem 3.1.

Let $p(y_0)$ be the width of the slice $\Omega \cap \{(x, y_0)\}$ so $\rho(x, y) \leqslant \frac{1}{2} p(y)$ and thus, since we have an exterior cone condition, $p(y)^{-2} \leqslant cH_X$ by Corollary 8.3. Thus by Theorem 8.5, $e^{(1-\varepsilon)A}\psi_0 \in L^2$. Therefore, if we define

$$\tilde{A}(x, y) = \sum_{1}^{n} l_j d_j^{-1} \equiv A_n \qquad \text{if} \quad y_n < y < y_{n+1}, \tag{9.1}$$

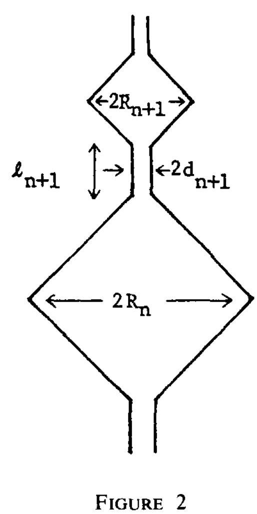

FIGURE 2

we have that

$$e^{\delta \tilde{A}}\psi_0 \in L^2 \qquad \text{for some} \quad \delta > 0. \tag{9.2}$$

Now let $S(t, R)$ denote the minimum value of $e^{-tH(R)}(x, x)$ for $|x| < \frac{1}{4}R$, and $H(R)$ the Dirichlet Laplacian in the square of side $R\sqrt{2}$. By scaling $S(t, R) = R^{-2}S(t/R^2, 1)$ and it is easy to see that $S(t, 1) \geqslant C_1 e^{-C_2 t}$ so

$$S(t, R) \geqslant C_1 R^{-2} \exp(-C_2 t/R^2). \tag{9.3}$$

By a comparison argument with X the region of Fig. 2,

$$b_t(x, y) \leqslant C_1 R_n^{-2} \exp(-C_2 t/R_n^2),$$

if $x^2 + (y - y_n)^2 \leqslant (\frac{1}{4}R_n)^2$. Thus by (9.1)–(9.3) and the arguments of Theorem 3.1, we have

THEOREM 9.1. *If for all $\delta > 0$, $\lambda > 0$ we have*

$$\sum_n e^{\delta A_n} \exp(-\lambda/R_n^2) = \infty$$

then the region of Fig. 2 *has a Laplacian which is* not *intrinsically hypercontractive.*

For example, if $\alpha, \beta, \gamma < 1$, $l_n = \alpha^n$, $d_n = \beta^n$, $R_n = \gamma^n$, then it suffices that $\beta < \alpha\gamma^2$.

We are struck that the fir tree region is precisely the sort of region for which the Neumann Laplacian has a resolvent which is *not* compact. We wonder if there is a general relation between compactness of the Neumann Laplacian and intrinsic contractivity of the Dirichlet Laplacian.

With the negative result in mind, we state the following positive results:

THEOREM 9.2. *Let X be a bounded open connected subset of R^ν with C^∞ boundary. Let H have the form* (7.2), *where a is L^∞ and obeys* (7.1). *Then e^{-tH} is intrinsically ultracontractive, indeed for $k > 1 + \nu/4$,*

$$|\varphi(x)| \leqslant \psi_0(x)\, \|(H + 1)^k \varphi\|_{L^2(X, d^\nu x)}. \tag{9.4}$$

THEOREM 9.3. *Let X be a bounded open connected subset of R^ν which obeys an A-interior cone condition and an exterior cone condition. Then e^{-tH} is intrinsically ultracontractive, indeed*

$$\|e^{-t\tilde{H}}\|_{\infty,2} \leqslant Ct^{-\beta}, \qquad t \leqslant 1, \tag{9.5}$$

with $\beta = \frac{1}{4}\nu + \frac{1}{2}\alpha(A)$.

Proof of Theorem 9.2. We give a direct proof which does not go through logarithmic Sobolev inequalities although there is an alternate proof along the lines we use to prove Theorem 9.2.

By a standard Sobolev estimate [33], one has

$$\|(\nabla\varphi)\|_\infty \leqslant C_1 \,\|(H+1)^k\varphi\|_{L^2(X,d^\nu x)}$$

so, since any $\varphi \in D(H^k)$ vanishes on ∂X, we have

$$|\varphi(x)| \leqslant c_2 \rho(x) \,\|(H+1)^k \varphi\|_{L^2(X,d^\nu x)}.$$

Theorem 7.1 completes the proof. ∎

Proof of Theorem 9.3. By Theorem 7.2,

$$-\ln \psi_0(x) \leqslant \alpha(A) \ln \rho(x)^{-1} + c$$

and by Corollary 8.3,

$$\rho(x)^{-2} \leqslant cH_X.$$

Obviously, for some C, $\ln y \leqslant y^2 + C$, so putting $y = \delta^{1/2} a$,

$$\ln a \leqslant \delta a^2 + C + \tfrac{1}{2} \ln \delta^{-1}$$

and thus

$$-\ln \psi_0 \leqslant \delta H_X + d + \frac{\alpha(A)}{2} \ln(\delta^{-1}).$$

Thus Theorem 5.2 implies (9.5). ∎

COROLLARY 9.4. *Under the hypotheses of Theorem* 9.3, *fix any real* $l > \frac{1}{4}\nu + \frac{1}{2}\alpha(A)$. *Then there is a constant* C *so that for any* $\varphi \in D_{L^2}(H_X^l)$ *we have*

$$|\varphi(x)| \leqslant C\psi_0(x) \,\|H^l \varphi\|_{L^2(X,d^\nu x)}. \tag{9.6}$$

Proof. By integrating the semigroup to get a power of resolvent, we get that if $l > \beta$ (to make the integral converge at $l = 0$),

$$\|\tilde{H}^{-l}\eta\|_\infty \leqslant C \,\|\eta\|_{L^2(X,\psi_0^2 d^\nu x)}$$

which easily yields (9.6) by letting $\eta = \psi_0^{-1} H^l \varphi = \tilde{H}^l \psi_0^{-1} \varphi$. ∎

Remark. For a square, we claim that (9.5) and (9.6) are optimal (modulo the fact that $l > \cdots$, might be able to be replaced by $l \geqslant \cdots$). For an estimate on $(\tilde{H})^{-k}$ from L^2 to L^∞ easily yields an estimate of $e^{-t\tilde{H}}$ by t^{-k} since

$$\|e^{-t\tilde{H}}\|_{\infty,2} \leqslant \|\tilde{H}^{-k}\|_{\infty,2} \|\tilde{H}^k e^{-t\tilde{H}}\|_{2,2}$$

so (9.5) is optimal if and only if (9.6) is optimal. Next we note that, by duality and interpolation,

$$\|e^{-2t\tilde{H}}\|_{\infty,1} \leqslant \|e^{-t\tilde{H}}\|_{\infty,2}^2$$

while by interpolation,

$$\|e^{-t\tilde{H}}\|_{\infty,2}^2 \leqslant \|e^{-t\tilde{H}}\|_{\infty,1},$$

so we need only show that

$$\|e^{-t\tilde{H}}\|_{\infty,1} \sim Ct^{-2\beta}$$

to conclude that (9.5) is optional. For a square, A is a sector of open angle $\pi/2$, so $\lambda(A) = 4$, $\alpha(A) = 2$, and $2\beta = 3$. But $\|e^{-t\tilde{H}}\|_{\infty,1}$ is the square of this quantity for an interval and is easy to show that $\sup_{x,y} e^{-tH}(x, y)\, \psi_0(x)^{-1} \psi_0(x)^{-1} \psi_0(y)^{-1}$ is $O(t^{-3/2})$ for an interval.

Various authors, e.g., Jerison & Kenig [44] discuss boundary vanishing in terms of vanishing of the Green's function $G_X(x, y)$, the integral kernel of H^{-1}. In this regard, the following result is interesting:

THEOREM 9.5. *Let X obey the hypotheses of Theorem 9.3. Fix $x \in X$, and a neighborhood N of x, with $\bar{N} \subset X$. Then for suitably nonzero constants c, d, depending on x, and N and all $y \notin N$,*

$$d(x)\, \psi_0(y) \leqslant G_X(x, y) \leqslant c(x)\, \psi_0(y),$$

c, d are bounded and bounded away from zero as x runs through a compact subset K of X, and y runs through the complement of a neighborhood of K.

Proof. We give the details for x fixed, leaving it to the reader to check that c, d obey the final uniformity statement. To get the lower bound, we note that

$$G_X(x, y) = \int_0^\infty (e^{-tH})(x, y)\, dt$$

$$\geqslant \int_1^2 (e^{-tH})(x, y) \geqslant c\psi_0(x)\, \psi_0(y),$$

by Theorems 9.3 and 3.2. To get the upper bound, pick $\eta \in C_0^\infty(X)$ with $\eta(y) = 1$ if $y \notin N$ and $\eta \equiv 0$ near x. Let $u(y) = \eta(y)\, G_X(x, y)$. Then

$u \in D(H)$ and Hu is a smooth function supported in N. Thus $u \in D(H^l)$ for all l, so by (9.6)

$$\eta(y)\, G_X(x, y) \leqslant c\psi_0(y)$$

which is the desired upper bound. ∎

As a next subject, we consider regions of the form (7.10).

THEOREM 9.6. *Let X be a region of the form* (7.10), *where F obeys the four conditions listed after* (7.10). *Then*

(a) *If $F(x) \leqslant Cx^{-1}$ for x large, then e^{-tH_X} is intrinsically hypercontractive.*

(b) *If $\lim_{x\to\infty} xF(x) = 0$, then e^{-tH_X} is intrinsically supercontractive.*

(c) *If $xF^{-1} \leqslant \varepsilon F^{-2} + \exp(\varepsilon^{-a})$ for some $c > 0$ and ε small, then e^{-tH_X} is intrinsically ultracontractive.*

Proof. (b), (c) follow from Theorem 5.2, Theorem 7.3, and Corollary 8.4. (a) replaces Theorem 5.2 by Rosen's result [30]. ∎

Remarks. (1) If we take $F(x) = (x+1)^{-\alpha} \ln(|x|+2)^{-\beta}$, we obtain intrinsic hypercontractivity if $\beta = 0$, $\alpha = 1$, intrinsic supercontractivity if $\alpha = 1$, $\alpha < \beta \leqslant 1$, and intrinsic ultracontractivity if $\alpha = 1$, $\beta > 1$, or $\alpha > 1$. We are struck by the fact that the borderline *of our method* for ultracontractivity is that for infinite volume.

(2) In higher dimensions, i.e., regions $\{(x, y) \mid |y| \leqslant F(x)\}$ with $y \in R^{\nu-1}$, one still the same borderlines as if $\nu = 2$, but the borderline for infinite volume is now $F(x) = x^{-1/\nu-1}$, i.e., all these higher dimensional examples with even hypercontractivity have $\mathrm{vol}(X) < \infty$. This leads us to

Question. Does intrinsic ultracontractivity of a Dirichlet Laplacian imply that $\mathrm{vol}(X) < \infty$?

As a final subject, we want to note intrinsic ultracontractive estimates for operators of the form $H_0 + V$, where H_0 is a Dirichlet Laplacian of a region X and $V \in K_\nu$. The first result we require shows how to use ultracontractivity to obtain information in perturbed ground states (cf. Theorem 3.3).

THEOREM 9.7. *Let X be a region in R^ν for which e^{-tH_0} is intrinsically ultracontractive. Let $V \in K_\nu$ and suppose that $\tilde{\psi}_0$ is the ground state of $H_0 + V$. Then, for any $\alpha > 1$,*

$$d_\alpha[\psi_0(x)]^{1/\alpha} \geqslant \tilde{\psi}_0(x) \geqslant C_\alpha[\psi_0(x)]^\alpha, \tag{9.7}$$

where ψ_0 is the ground state for H_0.

Proof. By Hölder's inequality in path space, if p and q are dual indices

$$(e^{-H_0}\tilde{\psi}_0)(x) \leqslant (e^{-(H_0+V)}\tilde{\psi}_0)^{1/p}(x)(e^{-(H_0-(q/p)V)}\,\tilde{\psi}_0)^{1/q}(x)$$
$$\leqslant c\tilde{\psi}_0^{1/p}(x),$$

where we have used the fact that $\tilde{\psi}_0$ is an eigenfunction of $H_0 + V$ and that $e^{-(H_0-(q/p)V)}$ is bounded from L^2 to L^∞ since $V \in K_\nu$ (see [37, p. 460]). By ultracontractivity and Theorem 3.2, $(e^{-H_0}\tilde{\psi}_0)(x) \geqslant c(\psi_0, \tilde{\psi}_0)\,\psi_0(x)$. Choosing $p = \alpha$, we get the second inequality in (9.7).

The other half is similar:

$$(e^{-(H_0+V)}\tilde{\psi}_0)(x) \leqslant (e^{-H_0}\tilde{\psi}_0)^{1/p}(x)(e^{-(H_0+qV)}\tilde{\psi}_0)^{1/q}(x)$$

and $e^{-H_0}\tilde{\psi}_0 \leqslant c(\psi_0, \tilde{\psi}_0)\,\psi_0$. ∎

Remark. For some X, using methods of Brossard [4] and Zhao [40], one should be able to prove (9.7) with $\alpha = 1$.

THEOREM 9.8. *The conclusions of Theorem* 9.3 *hold if* H_0 *is replaced by* $H_0 + V$ *with* $V \in K_\nu$ *with the sole change that one needs* $\beta > \frac{1}{4}\nu + \frac{1}{2}\alpha(A)$.

Proof. Since V is in K_ν, $H_0 + V \geqslant \frac{1}{2}H_0 - c$ for some constant c, so $\rho^{-2} \leqslant d(H_0 + V + c)$. By the last theorem, for any $\gamma > 1$, $\tilde{\psi}_0 \geqslant c\rho^{\gamma_\alpha}$. Given these estimates, we just follow the proof of Theorem 9.3. ∎

APPENDIX A: HYPERCONTRACTIVITY AND $\mathrm{tr}(e^{-Ht})$

In this appendix we investigate the status of the assumption

$$\mathrm{tr}[e^{-Ht}] < \infty \qquad \text{all} \quad t > 0,$$

in relation to hypercontractivity. For ultracontractive semigroups the situation is very simple.

LEMMA A.1. *If* $H = -\Delta + V$ *in* $L^2(\mathbb{R}^N)$, *where* $0 \leqslant V \in L^1_{\mathrm{loc}}$ *and* e^{-Ht} *is intrinsically ultracontractive, then*

$$\mathrm{tr}[e^{-Ht}] < \infty \tag{A.1}$$

for all $t > 0$. *Indeed, for any* $\alpha > 0$, *one has*

$$H \geqslant \alpha x^2 - \beta \tag{A.2}$$

as quadratic forms for some $\beta \in \mathbb{R}$.

Proof. The first part is abstract in nature [9]. By Theorem 3.2(iv) we see that $\tilde{a}_t(x, y)$ is a bounded integral kernel. But X has measure unity with respect to μ, so $\tilde{a}_t$ is a Hilbert–Schmidt kernel and

$$\begin{aligned}\operatorname{tr}[e^{-Ht}] &= \|e^{-Ht}\|_2^2 \\ &= \int_X \int_X |\tilde{a}_t(x, y)|^2 \, dx \, dy \\ &< \infty.\end{aligned}$$

The second part depends upon a use of the Trotter product formula to establish that

$$a_t(x, y) \leqslant (4\pi t)^{-N/2} e^{-(x-y)^2/4t}.$$

Using Theorem 3.2(vi) we see that

$$\begin{aligned}\varphi_0(x) &\leqslant c_5 \, a_t(x, 0) \, \varphi_0(0)^{-1} \\ &\leqslant ce^{-x^2/4t}\end{aligned}$$

for all $t > 0$. By varying t we conclude that

$$\gamma_\alpha \equiv \int_{\mathbb{R}^N} e^{2\alpha x^2} \varphi_0(x)^2 \, dx < \infty \qquad \text{all} \quad \alpha > 0.$$

By Segal's lemma [27, p. 260]

$$\begin{aligned}\|e^{-(H-\alpha x^2)}\| &\leqslant \|e^{\alpha x^2} e^{-H}\| \\ &= \|e^{\alpha x^2} e^{-\tilde{H}}\|\end{aligned}$$

and if $g \in L^2(\mathbb{R}^N, \varphi_0(x)^2 \, dx)$ then (the L^p norms are w.r.t. $\varphi_0^2 \, dx$)

$$\begin{aligned}\|e^{\alpha x^2} e^{-\tilde{H}} g\|_2 &\leqslant \|e^{\alpha x^2}\|_2 \, \|e^{-\tilde{H}} g\|_\infty \\ &\leqslant \|e^{-\tilde{H}}\|_{\infty,2} \, \|g\|_2 \, \gamma_\alpha^{1/2}.\end{aligned}$$

The finiteness of $\|e^{-(H-\alpha x^2)}\|$ is equivalent to (A.2) by standard arguments. ∎

Note. It may be shown, as in the proof of Theorem A.8, that (A.2) implies (A.1).

The situation for hypercontractive semigroups is as follows: Carmona has shown [5] (see Theorem A.8) that (A.2) holds for some $\alpha > 0$ for all hypercontractive Schrödinger semigroups, and (A.1) thus also follows. On the other hand, for the hypercontractive semigroups studied in quantum field theory (A.1) is not true, so its general status is not clear. This section is devoted to this question. We emphasize that several of the abstract theorems below are much easier to prove for Schrödinger semigroups by other methods, and many of the proofs are abstractions of those of Carmona [5].

Our standing hypotheses throughout this appendix are as follows. We let dx be a σ-finite measure on the Borel space X, and let $H \geqslant 0$ be a selfadjoint operator on $L^2(X)$ such that e^{-Ht} is a contraction semigroup on $L^p(X)$ for all $1 \leqslant p \leqslant \infty$. We assume that e^{-Ht} is positivity preserving and irreducible on $L^2(X)$. We assume that the bottom of the spectrum of H consists of an eigenvalue E_0 of multiplicity one and that the corresponding eigenfunction φ_0 is a.e. positive. We also assume that e^{-Ht} is a bounded operator from $L^2(X)$ to $L^\infty(X)$ for all $t > 0$. This last assumption is not related to intrinsic hypercontractivity and is satisfied for practically all Schrödinger semigroups [37]. It implies by duality that e^{-Ht} is bounded from L^1 to L^2 and hence from L^1 to L^∞. Thus e^{-Ht} has a bounded integral kernel $a_t(x, y)$ such that $\|a_t\|_\infty$ is a monotonically decreasing function of t.

LEMMA A.2. *Suppose that $f: X \to \mathbb{R}^+$ has the property that*

$$S_n = \{x: f(x) \leqslant n\}$$

has finite measure for all integers $n \geqslant 1$. Then the form $H + f$ has compact resolvent.

Proof. We write $f = f_1 - f_2$, where $f_2 = (n - f)\chi_{S_n}$ so that $0 \leqslant f_2 \leqslant n$ and $f_1 \geqslant n$. Then

$$e^{-(H+f)t} = e^{-(H+f_1)t} + \int_0^t e^{-(H+f_1)(t-s)} f_2 e^{-(H+f_1)s} \, ds$$

$$+ \int_{T_1 \cup T_2} e^{-(H+f_1)(t-s)} f_2 e^{-(H+f)(s-u)} f_2 e^{-(H+f_1)u} \, du \, ds,$$

where

$$T_1 = \left\{ (u, s): 0 \leqslant u \leqslant s \leqslant t \text{ and } s - u \leqslant \frac{t}{2} \right\},$$

$$T_2 = \left\{ (u, s): 0 \leqslant u \leqslant s \leqslant t \text{ and } s - u > \frac{t}{2} \right\}.$$

We see, by the Trotter product formula, that

$$\begin{aligned} \|f_2 e^{-(H+f)(s-u)} f_2\|_2^2 &\leqslant \iint f_2^2(x) \, a_{s-u}(x, y)^2 f_2(y)^2 \, dx \, dy \\ &\leqslant \|f_2\|_2^4 \, \|a_{s-u}\|_\infty^2 \\ &\leqslant n^4 \, |S_n|^2 \, \|a_{s-u}\|_\infty^2. \end{aligned}$$

Therefore

$$\|f_2 e^{-(H+f)(s-u)} f_2\|_2 \leqslant \eta^2 \, |S_n| \, \|a_{t/2}\|_\infty$$

if $(u, s) \in T_2$, whence

$$\left\| \int T_2 e^{-(H+f_1)(t-s)} f_2 e^{-(H+f)(s-u)} f_2 e^{-(H+f_1)u} \, du \, ds \right\|_2 \leqslant \frac{1}{2} t^2 n^2 \, |S_n| \, \|a_{t/2}\|_\infty$$
$$< \infty.$$

It therefore remains to prove that

$$\lim_{n \to \infty} \left\| e^{-(H+f_1)t} + \int_0^t e^{-(H+f_1)(t-s)} f_2 e^{-(H+f_1)s} \, ds \right.$$
$$\left. + \int_{T_1} e^{-(H+f_1)(t-s)} f_2 e^{-(H+f)(s-u)} f_2 e^{-(H+f_1)u} \, du \, ds \right\| = 0.$$

Using the estimate

$$\|e^{-(H+f_1)t}\| \leqslant e^{-nt},$$

we see that this lhs is dominated by

$$\lim_{n \to \infty} \left\{ e^{-nt} + \int_0^t e^{-nt} n \, ds + \int_{T_1} e^{-n(t-s+u)} n^2 \, du \, ds \right\}$$
$$\leqslant \lim_{n \to \infty} \{ e^{-nt} + e^{-nt} nt + e^{-nt/2} n^2 t^2 / 2 \}$$
$$= 0. \quad \blacksquare$$

We now assume intrinsic hypercontractivity.

LEMMA A.3. *If* $\|e^{-Ht}\|_{4,2} < \infty$ *for some* $t > 0$ *and* $f: X \to \mathbb{R}^+$ *satisfies*

$$\int_X e^{4f(x)} \varphi_0(x)^2 \, dx < \infty$$

then $(H - t^{-1} f)$ *is bounded below.*

Proof. By Segal's lemma

$$\|e^{-(tH-f)}\| \leqslant \|e^f e^{-Ht}\| = \|e^f e^{-Ht}\|.$$

If $g \in L^2(X, \varphi_0(x)^2\,dx)$ then

$$\begin{aligned}\|e^f e^{-\tilde{H}t} g\|_2 &\leqslant \|e^f\|_4 \, \|e^{-\tilde{H}t} g\|_4 \\ &\leqslant \|e^f\|_4 \, \|e^{-\tilde{H}t}\|_{4,2} \, \|g\|_2 \\ &= c\,\|g\|_2,\end{aligned}$$

where $c < \infty$ by the hypothesis. The boundedness of $e^{-(tH-f)}$ is equivalent to $(H - t^{-1}f)$ being bounded below as a quadratic form sum. ∎

THEOREM A.4. *Assume the standing hypotheses preceding Lemma* A.2. *If H is intrinsically hypercontractive, then it has compact resolvent on* L^2.

Proof. Since e^{-Ht} is bounded from L^2 to L^∞ we see that φ_0 is bounded. For $n \geqslant 1$, let

$$S_n = \left\{ x : \frac{\|\varphi_0\|_\infty}{n+1} < \varphi_0(x) \leqslant \frac{\|\varphi_0\|_\infty}{n} \right\}$$

so that $\{S_n\}$ is a partition of X into sets of finite measure. Let $\{c_n\}$ be a sequence of positive numbers converging to infinity slowly enough so that if

$$f(x) = \sum_{n=1}^{\infty} c_n \cdot \mathscr{X}_{S_n}(x)$$

then

$$\int_X e^{4f(x)} \varphi_0(x)^2 \, dx < \infty.$$

We deduce that $(H + t^{-1}f)$ has compact resolvent and $(H - t^{-1}f) \geqslant -c$ for some finite c, by Lemmas A.2 and A.3. Therefore

$$\begin{aligned} H &= \tfrac{1}{2}(H + t^{-1}f) + \tfrac{1}{2}(H - t^{-1}f) \\ &\geqslant \tfrac{1}{2}(H + t^{-1}f) - \frac{c}{2} \end{aligned}$$

and minimax implies that H has compact resolvent. ∎

Notes. (1) The conclusion of this theorem is false for the hypercontractive semigroups appearing in quantum field theory [38]. The reason is that the condition that e^{-Ht} has a bounded integral kernel is not then valid.

(2) The above theorem implies that H has pure point spectrum and that there is a gap in the spectrum between E_0 and the next eigenvalue E_1. However, it does not give any lower bound to $(E_1 - E_0)$ and consideration of

double well Schrödinger operators proves that no such lower bound can be found in terms of the information supplied.

In our next lemma, we prove a decay property of φ_0 in the abstract context. For Schrödinger semigroups, much more powerful results can be obtained by using other methods [1, 6,12].

LEMMA A.5. *Under the hypotheses of this appendix, if H has compact resolvent on $L^2(X)$, then $\varphi_0 \in L^p$ for all $1 < p < \infty$.*

Proof. Let us denote the operator H on $L^p(X)$ by H_p. Since $e^{-H_2 t}$ is compact and $e^{-H_p t}$ is bounded for all $1 \leqslant p \leqslant \infty$, it follows by interpolation [25; 3, p. 85] that $e^{-H_p t}$ is compact for all $1 < p < \infty$. Applying the spectral theory of compact operators to $e^{-H_2 t}$ and $e^{-H_p t}$ simultaneously as in [3], we find that H_2 and H_p have the same eigenvalues and eigenvectors. This, incidentally, proves the p-invariance of the spectrum of H_p (see [34]) under the condition that H_2 has compact resolvent. ∎

We now collect our results together into a single theorem, which is implicit in the results of Carmona [5] for the case of Schrödinger semigroups.

THEOREM A.6. *Suppose that e^{-Ht} is an irreducible positivity preserving semigroup on $L^2(X)$ which is a contraction semigroup on $L^p(X)$ for all $1 \leqslant p \leqslant \infty$, and suppose further that e^{-Ht} is bounded from L^2 to L^∞ for all $t > 0$. If H is intrinsically hypercontractive, then*

$$\operatorname{tr}[e^{-Ht}] < \infty$$

for all large enough $t > 0$.

Proof. Since e^{-Ht} is bounded from L^2 to L^∞, the ground state eigenfunction φ_0 is bounded. If we define $f: X \to \mathbb{R}$ by

$$f = -\log \varphi_0$$

then f is bounded below and

$$\int_X e^{4\alpha f(x)} \varphi_0(x)^2 \, dx = \int_X \varphi_0(x)^{2-4\alpha} \, dx = \infty$$

for $0 < \alpha < \frac{1}{4}$ by Theorem A.4 and Lemma A.5. It follows by Lemma A.3 that for small enough $\beta > 0$ there exists $\gamma \in \mathbb{R}$ such that

$$H - \beta f \geqslant -\gamma.$$

We next note that the Golden–Thompson inequality implies that

$$\begin{aligned} \operatorname{tr}[e^{-Ht-2f}] &\leqslant \operatorname{tr}[e^{-f}e^{-Ht}e^{-f}] \\ &= \int_X e^{-f(x)} a_t(x,x)\, e^{-f(x)}\, dx \\ &\leqslant c_t \int_X e^{-2f(x)}\, dx \\ &= c_t < \infty \end{aligned}$$

for all $t > 0$. Therefore $tH + 2f$ has pure point spectrum with eigenvalues μ_n satisfying

$$\sum_{n=0}^{\infty} e^{-\mu_n} = \operatorname{tr}[e^{-tH-2f}] < \infty.$$

Choosing $t = 2/\beta$ we have

$$\begin{aligned} 2tH &= (tH + 2f) + (tH - 2f) \\ &\geqslant (tH + 2f) - t\gamma. \end{aligned}$$

Therefore the eigenvalues E_n of H satisfy

$$2tE_n \geqslant \mu_n - t\gamma$$

and

$$\begin{aligned} \operatorname{tr}[e^{-2Ht}] &\leqslant e^{t\gamma} \sum_{n=0}^{\infty} e^{-\mu_n} \\ &< \infty. \quad \blacksquare \end{aligned}$$

For the sake of completeness, we finally prove a theorem which states roughly that the potential of an intrinsically hypercontractive Schrödinger operator must increase at least quadratically at infinity. This theorem may be found in Carmona [5], but its complete proof requires information in an unpublished letter of 1975 from I. Herbst to L. Gross. It will be clear that certain steps in the argument below can be written down abstractly.

We assume for the rest of the appendix that $H = -\Delta + V$ on $L^2(\mathbb{R}^N)$, where $V \in L^1_{\text{loc}}$ is bounded below. We assume that H is intrinsically hypercontractive with lowest eigenvalue zero and corresponding eigenfunction φ_0. The operator $\tilde{H}$ is then defined as a Dirichlet form by

$$\langle \tilde{H}f, f \rangle = \int_{\mathbb{R}^N} |\nabla f|^2\, d\mu,$$

where

$$d\mu(x) = \varphi_0(x)^2\, dx.$$

LEMMA A.7. *Under the above conditions, $e^{-\tilde{H}t}$ is a contraction from L^2 to L^4 for large enough $t > 0$. There exists $c > 0$ such that $\tilde{H}$ satisfies the logarithmic Sobolev inequality*

$$\int_{\mathbb{R}^N} |f|^2 \log |f|\, d\mu \leqslant c\langle \tilde{H}f, f\rangle + \|f\|_2^2 \log \|f\|_2 \tag{A.3}$$

for all $f \in L^2(\mathbb{R}^N, d\mu)$. Moreover,

$$\int_{\mathbb{R}^N} x^2\, d\mu(x) < \infty. \tag{A.4}$$

Proof. Theorem A.4 implies that 0 is an isolated eigenvalue which is of multiplicity one by irreducibility. The fact that $e^{-\tilde{H}t}$ is not only bounded from L^2 to L^4 for large t but actually a contraction for sufficiently large t is a result of Glimm [16]. If $0 < \lambda < 1$ and $p = 4/(2-\lambda)$ then conplex interpolation implies that

$$\|e^{-\tilde{H}\lambda t}\|_{p,2} \leqslant 1.$$

If $\|f\|_2 = 1$ and $f_\lambda = e^{-\tilde{H}\lambda t}|f|$ and

$$h(\lambda) = \int_{\mathbb{R}^N} f_\lambda(x)^{4/(2-\lambda)}\, dx$$

then $h(0) = 1$ and we have that

$$h(\lambda) \leqslant 1 \qquad \text{all} \quad 0 < \lambda < 1.$$

Thus

$$\begin{aligned} 0 \geqslant h'(0) &= \frac{d}{d\lambda} \int \exp\left(\frac{4}{2-\lambda} \log f_\lambda\right) d\mu \Bigg|_{\lambda=0} \\ &= \int \left\{\log f_0 + 2\frac{f_0'}{f_0}\right\} f_0^2\, d\mu \\ &= \int (\log f_0) f_0^2\, d\mu - 2t\langle \tilde{H}|f|, |f|\rangle. \end{aligned}$$

Therefore

$$\int_{\mathbb{R}^N} |f|^2 \log |f| \, d\mu \leqslant 2t\langle \tilde{H} |f|, |f| \rangle$$
$$\leqslant 2t\langle \tilde{H}f, f \rangle$$

provided $\|f\|_2 = 1$. The last inequality follows from the fact that $e^{-t\tilde{H}}$ is positivity preserving [29, p. 209]. The general bound (A.3) is now obtained by substituting $f/\|f\|_2$ into this inequality.

To prove (A.4) we put

$$\begin{aligned} f_n(x) &= x \qquad \text{if} \quad |x| \leqslant n, \\ &= n \qquad \text{if} \quad x > n, \\ &= -n \qquad \text{if} \quad x < -n, \end{aligned}$$

and suppose for contradiction that

$$\lim_{n\to\infty} \|f_n\|_2^2 = \infty.$$

Since $\tilde{H}$ is a Dirichlet form,

$$\lim_{n\to\infty} \langle \tilde{H}f_n, f_n \rangle = \lim_{n\to\infty} \int_{|x| \leqslant n} \varphi_0(x)^2 \, dx = 1,$$

so if $g_n = f_n/\|f_n\|_2$ we have $\|g_n\|_2 = 1$ and

$$\lim_{n\to\infty} \langle \tilde{H}g_n, g_n \rangle = 0.$$

Since the eigenvalue 0 of $\tilde{H}$ is isolated and of multiplicity one, it follows that

$$\lim_{n\to\infty} |\langle g_n, 1 \rangle| = 1.$$

Now

$$\begin{aligned} |\langle g_n, 1 \rangle|^2 &\leqslant \|f_n\|_2^{-2} \left\{ \int |f_n| \, d\mu \right\}^2 \\ &\leqslant \|f_n\|_2^{-2} \int \left(\frac{\varepsilon}{2} |f_n|^2 + \frac{1}{2\varepsilon} \right) d\mu \\ &= \frac{\varepsilon}{2} + (2\varepsilon \|f_n\|_2^2)^{-1}. \end{aligned}$$

Putting $\varepsilon = \|f_n\|_2^{-1}$ and letting $n \to \infty$, we obtain the required contradiction. ∎

Remark. In the above, we used Glimm's theorem that if $e^{-tB}1 = 1$ and $\|e^{-tB}f\|_4 \leqslant C_t \|f\|_2$ for t large, then $C_t = 1$ for t perhaps even larger. If 4 is replaced by ∞, then this result is false. For suppose $A1 = 1$ and $\|Af\|_\infty \leqslant \|f\|_2$. Let $\langle g, 1 \rangle = 0$ and pick $f_\varepsilon = 1 + \varepsilon g$. Then $\|f_\varepsilon\|_2 = 1 + O(\varepsilon^2)$ while $\max_\pm \|Af_{\pm\varepsilon}\|_\infty = 1 + |\varepsilon| \, \|Ag\|_\infty$ so $\|Af\|_\infty \leqslant \|f\|_2$ can only happen if $Af = (1, f)1$ and never for a semigroup.

In the following theorem we continue with the conventions written down above Lemma A.7. The proof is essentially that of Herbst (unpublished).

THEOREM A.8. *If H is an intrinsically hypercontractive Schrödinger operator on $L^2(\mathbb{R}^N)$, then*

$$H \geqslant \alpha x^2 - \beta \tag{A.5}$$

for some $\alpha > 0$ and $\beta \in \mathbb{R}$. Hence

$$\operatorname{tr}[e^{-Ht}] < \infty \qquad \text{all} \quad t > 0. \tag{A.6}$$

Proof. We put

$$h_n(a) = \int e^{2af_n^2} \, d\mu,$$

where f_n is defined as in Lemma A.7. Then

$$h_n'(a) = \int 2f_n^2 \, e^{2af_n^2} \, d\mu$$

and putting $g_n = e^{af_n^2}$ into A.3, we obtain

$$\begin{aligned}
\frac{a}{2} h_n'(a) &= \int g_n^2 \log g_n \, d\mu \\
&\leqslant c \int |\nabla g_n|^2 \, d\mu + \tfrac{1}{2} h_n(a) \log h_n(a) \\
&= c \int |2ag_n f_n \nabla f_n|^2 \, d\mu + \tfrac{1}{2} h_n(a) \log h_n(a) \\
&\leqslant 4a^2 c \int f_n^2 g_n^2 \, d\mu + \tfrac{1}{2} h_n(a) \log h_n(a) \\
&= 2a^2 c h_n'(a) + \tfrac{1}{2} h_n(a) \log h_n(a)
\end{aligned}$$

If $0 < a \leqslant 1/8c$ we deduce that

$$h_n'(a) \leqslant \frac{2}{a} h_n(a) \log h_n(a).$$

Solving the associated differential equation and inserting the initial conditions $h_n(0) = 1$ and

$$h_n'(0) \leqslant b \equiv 2 \int x^2 \mu(dx) < \infty,$$

we deduce that

$$h_n(a) \leqslant e^{ba}.$$

Letting $n \to \infty$ it finally follows that

$$\int e^{2ax^2} \mu(dx) \leqslant e^{ba}$$

for all $0 < a \leqslant 1/8c$. The bound (A.5) is now a straightforward application of Lemma A.3. From (A.5) we deduce that

$$\begin{aligned} H &\geqslant \tfrac{1}{2}(-\Delta + V) + \tfrac{1}{2}(\alpha x^2 - \beta) \\ &\geqslant \gamma(-\Delta + x^2) - \delta \end{aligned}$$

for some $\gamma > 0$ and $\delta > 0$. This yields lower bounds on the eigenvalues of H which imply (A.6). ∎

APPENDIX B: Ultracontractivity and Harmonic Comparison

It follows from Theorem 3.1 that intrinsic ultracontractivity is equivalent to a bound of the type

$$|\varphi_n(x)| \leqslant c_t e^{E_n t} \varphi_0(x) \tag{B.1}$$

for all $x \in X$, $n \geqslant 1$, and $t > 0$. Although the use of logarithmic Sobolev inequalities allows one to establish intrinsic ultracontractivity and hence (B.1) in great generality, one manages to achieve this without knowing the precise rate at which the two sides of (B.1) decay. It is natural to ask whether one can establish (B.1) by obtaining direct bounds on the two sides, and indeed this was the method by which one of us approached the whole problem in [9]. A little thought soon convinces one that eigenfunction bounds based on the Agmon metric are not (at present) strong enough to enable one to establish (B.1), but that some results should be obtainable by use of the subharmonic comparison theorem, as described, for example, in [12]. In this appendix, we apply the method to one test example. The computations can be adapted to other central potentials and to certain noncentral potentials, but one can also sometimes prove intrinsic ultracontractivity in the noncentral case by comparison with a suitable central case; see Theorem 3.4.

We now consider the Hamiltonian

$$H = -\Delta + |x|^{\alpha}$$

in $L^2(\mathbb{R}^N)$ for $2 < \alpha < \infty$. Let the eigenvalues $\{E_n\}_{n=0}^{\infty}$ be written in increasing order, and let the corresponding eigenfunctions $\{\varphi_n\}_{n=0}^{\infty}$ be normalized by $\|\varphi_n\|_2 = 1$ and $\varphi_0 > 0$. We obtain upper bounds on $|\varphi_n|$ and a lower bound on φ_0 by comparing them with the function

$$f_E(x) = r^{-\alpha/4+(N-1)/2} \exp\left[-\frac{2}{\alpha+2} r^{1+\alpha/2} - \frac{E}{\alpha-2} r^{1-\alpha/2}\right],$$

where $r = |x|$. This function was chosen by an old-fashioned JWKB expansion.

LEMMA B.1. *There exist constants $\gamma_1, \gamma_2, \gamma_3$ such that*

$$\Delta f_E / f_E = r^{\alpha} - E + \gamma_1 r^{-2} + \gamma_2 E r^{-\alpha/2-1} + \gamma_3 E^2 r^{-\alpha}$$

for all $r > 0$.

Proof. This is a direct computation. If

$$f(r) = r^{-\beta} e^{-g(r)}$$

then

$$\Delta f(r) = f''(r) + \frac{N-1}{r} f'(r),$$

so

$$\Delta f/f = \beta(\beta+1) r^{-2} + 2\beta g' r^{-1} + (g')^2 - g'' - (N-1)\beta r^{-2} - (N-1) g' r^{-1}.$$

Substituting

$$\beta = \frac{\alpha}{4} + \frac{N-1}{2}$$

and

$$g'(r) = r^{\alpha/2} - \frac{E}{2} r^{-\alpha/2}$$

yields

$$\Delta f/f = r^{\alpha} - E + \frac{\beta(\beta + 2 - N)}{r^2} + \frac{E^2}{4r^{\alpha}} - \frac{-E\alpha}{2r^{\alpha/2+1}}. \quad \blacksquare$$

We now use the subharmonic comparison theorem ([12] and references there) to obtain a lower bound on φ_0.

LEMMA B.2. *There exist positive constants c_1, c_2, c_3 such that*

$$\varphi_0(x) \geqslant c_1 \qquad \text{if} \quad |x| \equiv r \leqslant c_3, \tag{B.2}$$

$$\geqslant c_2 f_0(r) \qquad \text{if} \quad |x| > c_3. \tag{B.3}$$

Proof. Putting $E = 0$ in Lemma B.1 we obtain

$$\Delta f_0 = W f_0,$$

where

$$W(x) = r^{\alpha} + \gamma_1 / r^2$$

which has to be compared with

$$\Delta \varphi_0 = (r^{\alpha} - E)\, \varphi_0.$$

There exists $c_3 > 0$ such that

$$W(r) \geqslant r^{\alpha} - E$$

for all $r \leqslant c_3$, and then $c_2 > 0$ such that

$$\varphi_0(x) \geqslant c_2 f_0(x)$$

whenever $|x| = c_3$, by the continuity and strict positivity of φ_0. The bound (B.3) now follows by the subharmonic comparison theorem. The bound (B.2) is a consequence of the continuity and strict positivity of φ_0 on $\{x : |x| \leqslant c_3\}$. $\blacksquare$

THEOREM B.3. *There exist positive constants c_4 and c_5 such that*

$$|\varphi_n(x)| \leqslant c_5 \exp(c_4 E_n^{1/\alpha + 1/2})\, \varphi_0(x)$$

for all $x \in \mathbb{R}^N$ and all $n \geqslant 1$.

Proof. If $|x| \leqslant c_3$ this is an immediate consequence of the bound

$$|\varphi_n(x)| \leqslant c_6 E_n^{N/4} \tag{B.4}$$

valid for all $x \in \mathbb{R}^N$ by [12]. It is therefore sufficient by Lemma B.2 to prove that

$$|\varphi_n(x)| \leqslant c_7 \exp(c_4 E_n^{1/\alpha + 1/2}) f_0(x)$$

for all $|x| \geqslant c_3$. This bound is actually valid for all $x \in \mathbb{R}^N$.

We start by observing that there is a constant c_8 such that if $E > E_0$ and $r \geqslant c_8 E^{1/\alpha}$, then

$$|\gamma_1| r^{-2} + |\gamma_2| E r^{-\alpha/2-1} + |\gamma_3| E^2 r^{-\alpha} < \tfrac{1}{2}E.$$

It follows that if $n \geqslant 1$ and $r \geqslant c_8(2E_n)^{1/\alpha}$, then

$$|\Delta f_{2E_n}/f_{2E_n} - (r^\alpha - 2E_n)| \leqslant E_n$$

so

$$r^\alpha - 3E_n \leqslant \Delta f_{2E_n}/f_{2E_n} \leqslant r^\alpha - E_n.$$

We deduce that there exists c_9 such that if $n \geqslant 1$ and $r \geqslant c_9 E_n^{1/\alpha}$, then

$$0 \leqslant \Delta f_{2E_n} \leqslant (r^\alpha - E_n) f_{2E_n}.$$

We now treat the cases $r \geqslant c_9 E_n^{1/\alpha}$ and $r \leqslant c_9 E_n^{1/\alpha}$ separately.

If $r \geqslant c_9 E_n^{1/\alpha}$, then it follows from the subharmonic comparison theorem that

$$\frac{|\varphi_n(r)|}{f_{2E_n}(r)} \leqslant \frac{|\varphi_n(c_9 E_n^{1/\alpha})|}{f_{2E_n}(c_9 E_n^{1/\alpha})}.$$

Combining (B.4) with the identity

$$f_{2E_n}(c_9 E_n^{1/\alpha}) = c_{10} E_n^{-1/4-(N-1)/2\alpha} \exp[-c_{11} E_n^{1/\alpha+1/2}],$$

we deduce that

$$\begin{aligned} |\varphi_n(r)| &\leqslant c_6 E_n^{N/4} c_{10}^{-1} E_n^{1/4+(N-1)/2\alpha} \exp[c_{11} E_n^{1/\alpha+1/2}] f_{2E_n}(r) \\ &\leqslant c_{12} \exp[c_{13} E_n^{1/\alpha+1/2}] f_{2E_n}(r) \\ &\leqslant c_{12} \exp[c_{13} E_n^{1/\alpha+1/2}] f_0(r) \\ &\leqslant c_{14} \exp[c_{13} E_n^{1/\alpha+1/2}] \varphi_0(r). \end{aligned} \tag{B.5}$$

Now suppose that $0 \leqslant r \leqslant c_9 E_n^{1/\alpha}$. Since φ_0 is a monotonically decreasing function of $|x| = r$, we have

$$\begin{aligned}\frac{|\varphi_n(r)|}{\varphi_0(r)} &\leqslant \frac{c_6 E_n^{N/4}}{\varphi_0(c_9 E_n^{1/\alpha})} \\ &\leqslant \frac{c_6 E_n^{N/4}}{c_2 f_0(c_9 E_n^{1/\alpha})} \\ &= c_{15} E_n^{N/4 + \alpha/4 - (N-1)/2} \exp[c_{16} E_n^{1/\alpha + 1/2}] \\ &= c_{17} \exp[c_{18} E_n^{1/\alpha + 1/2}]. \end{aligned} \tag{B.6}$$

The theorem follows by combining (B.5) and (B.6). ∎

COROLLARY B.4. *For all $t > 0$ there exists c_t such that*

$$|\varphi_n(x)| \leqslant c_t e^{E_n t} \varphi_0(x)$$

for all $x \in \mathbb{R}^N$ and $n \geqslant 1$. Thus the Schrödinger operator $H = -\Delta + |x|^\alpha$ is intrinsically ultracontractive for all $\alpha > 2$.

Proof. Since $\alpha > 2$ we have

$$c_4 E_n^{1/\alpha + 1/2} \leqslant E_n t$$

for large enough n, depending upon t. We may now apply Theorem 3.1(i) with $p = \infty$.

Note. One can actually deduce from Theorem B.3 that the semigroup $e^{-H^\beta t}$ is ultracontractive for all β satisfying

$$\frac{1}{\alpha} + \frac{1}{2} < \beta < 1.$$

Similar methods to those above can be used to obtain rather sharp upper bounds on $\varphi_n(x)$ and lower bounds on $\varphi_0(x)$ when $0 < \alpha < 2$. We do not present them here since they cannot possibly to prove hypercontractivity of e^{-Ht} because of Theorem A.8.

APPENDIX C: THE FORM DOMAIN OF DIRICHLET FORMS

In Section 4, we proved that if A is the Dirichlet form associated to a Schrödinger operator, then $D(A)$ is precisely the set of $\varphi \in L^2(X, d\mu)$ with $\nabla\varphi \in L^2(X, d\mu)$. In the Dirichlet case, we only proved that $D(A)$ is contained in the set. Our goal in this appendix is to prove equality of $D(A)$ and this set under an additional regularity assumption.

THEOREM C.1. *Let ψ_0 be the ground state eigenfunction of the Dirichlet Laplacian H for the bounded region $U \subseteq \mathbb{R}^N$. Then ψ_0 is a C^∞ bounded function. Suppose the boundary ∂U of U is regular enough so that $x \to \partial U$ implies $\psi_0(x) \to 0$. Let L^2 stand for $L^2(U, \psi_0^2(x)\, d^N x)$ and let the closed quadratic form Q be defined by*

$$Q(f) = \int |\nabla f(x)|^2\, \psi_0^2(x)\, d^N x,$$

where $\mathrm{Dom}(Q) = \{f \in L^2 : \nabla f \in L^2\}$, ∇ *denoting the distributional derivative. Then $C_0^\infty(U)$ is a quadratic form core for Q, so Q is the quadratic form of A, the Dirichlet form associated to H.*

Proof. We first show that

$$\mathscr{D}_1 = \{f \in L^\infty : \nabla f \in L^2\}$$

is a form core of Q. Let $F_n : \mathbb{R} \to \mathbb{R}$ be a sequence of C^∞ functions such that

$$0 \leqslant |s| \leqslant n \Rightarrow F_n(s) = s,$$

$$n \leqslant |s| \leqslant 2n \Rightarrow n \leqslant F_n(s) \frac{s}{|s|} \leqslant n+1,$$

$$2n \leqslant |s| < \infty \Rightarrow F_n(s) \frac{s}{|s|} = n+1,$$

$$0 \leqslant F_n'(s) \leqslant 1 \qquad \text{all} \quad -\infty < s < \infty.$$

If $f \in \mathrm{Dom}(Q)$, let $f_n = F_n(f)$. Then

$$\|f_n - f\|_2 \to 0 \qquad \text{as} \quad n \to 0,$$

and

$$\begin{aligned} Q(f_n - f) &= \int_U |\nabla f - F_n'(f) \nabla f|^2\, \psi_0^2\, dx \\ &= \int_U |1 - F_n'(f)|^2\, |\nabla f|^2\, \psi_0^2\, dx \\ &\to 0 \end{aligned}$$

by the dominated convergence theorem.

We next show that

$$\mathscr{D}_2 = \{f \in L^\infty_{\text{comp}}: \nabla f \in L^2\}$$

is a form core of Q. Let G: $[0, \infty) \to [0, \infty)$ be a C^∞ function with

$$\begin{aligned} G(s) &= 0 \qquad \text{if} \quad 0 \leqslant s \leqslant 1, \\ 0 \leqslant G(s) &\leqslant 1 \qquad \text{if} \quad 1 \leqslant s \leqslant 2, \\ G(s) &= 1 \qquad \text{if} \quad 2 \leqslant s < \infty, \end{aligned}$$

and for $f \in \mathscr{D}_1$, let

$$f_n = G(n\psi_0) f.$$

Since ψ_0 is assumed to vanish as $x \to \partial U$, we see that $f_n \in L^\infty_{\text{comp}}$ and $\|f_n - f\|_2 \to 0$. Also

$$\begin{aligned} Q(f_n - f) &= \int_U |G(n\psi_0)\, \nabla f + fF'(n\psi_0)\, n\nabla\psi_0 - \nabla f|^2\, \psi_0^2\, d^N x \\ &\geqslant 2 \int_U |G(n\psi_0) - 1|^2\, |\nabla f|^2\, \psi_0^2\, d^N x \\ &\quad + 2 \int_U |f|^2\, |G'(n\psi_0)\, n\psi_0|^2\, |\nabla \psi_0|^2\, d^N x. \end{aligned}$$

The first integral vanishes by the dominated convergence theorem. The second also vanishes by the dominated convergence theorem once one notices that $f \in L^\infty$ and

$$\int_U |\nabla \psi_0|^2\, d^N x = \langle H\psi_0, \psi_0 \rangle < \infty.$$

To show that $C_0^\infty(U)$ is dense in $\mathscr{D}_2$ depends upon a standard mollifier argument.

The final statement of the Lemma depends upon the fact that we have proven (Proposition 4.4D) that C_0^∞ is a form core for A and that A obeys (4.5). ∎

We conclude by providing some well-known conditions which imply that ψ_0 vanishes on ∂U. Let P_X and E_X denote probability and expectation for Brownian motion starting at t. Recall [26]:

DEFINITION. Given a path $b(s)$ and U, an open set in R^ν, we define $T(b) = \inf(s > 0 \mid b(s) \notin U)$. $y \in \partial U$ is called *regular* if and only if $P_y(T > 0) = 0$ (note that $T(s)$ is defined with $s > 0$, *not* $s \geqslant 0$).

We quote the following results from [26] ((a) is Proposition 23.3 and (b) is Proposition 23.4):

LEMMA C.2. (a) *Let $y \in \partial U$. If there is an open cone C with vertex y so that $C \cap \{x \mid |x-y| < \delta\} \cap U = \phi$ for some $\delta > 0$, then y is a regular point of Ω.*

(b) *If $y \in \partial U$ is regular and $s, \varepsilon > 0$ are given, we can find $\delta > 0$ so that $P_x(T > s) < \varepsilon$ if $|x-y| \leqslant \delta$.*

Let H denote the Dirichlet Laplacian for U.

THEOREM C.3. *Let $k > \nu/4$ and let $g \in D(H^k)$ so g is continuous on U (by a Sobolev estimate). If $y \in \partial U$ is regular, then $\lim_{x \to y} g(x) = 0$.*

Given this theorem, a compactness argument yields

COROLLARY C.4. *If every point in ∂U is regular (e.g., if U obeys an exterior cone condition), then $\psi_0(x) \to 0$ as $x \to \partial U$.*

Proof of Theorem C.3. Let $f = (\frac{1}{2}H + 1)^k g$, so $f \in L^2$ and

$$g(x) = c_k \int_0^\infty e^{-s} s^{k-1} (e^{-(1/2)sH} f)(x)\, ds.$$

Now [26, p. 224], for a.e. s, x,

$$(e^{-(1/2)sH} f)(x) = E_x(f(b(s))\, \mathscr{X}_{\{b \mid T(b) > s\}}),$$

so for a.e. $x \in \Omega$,

$$|g(x)| \leqslant c_k \int_0^\infty e^{-s} s^{k-1} E_x(|f(b(s))|\, \mathscr{X}_{\{b \mid T(b) > s\}})\, ds.$$

Next, note that, by the Schwarz inequality,

$$\begin{aligned} E_x(|f(b(s))|\, \mathscr{X}_{\{b \mid T(b) > s\}}) &\leqslant E_x(|f(b(s))|^2)^{1/2} P_x(T(b\mathrm{W} > s)^{1/2} \\ &\leqslant (2\pi s)^{-\nu/4} \|f\|_2\, P_k(T(b) > s). \end{aligned}$$

The next result, initially a.e. and then for all x in Ω, is

$$|g(x)| \leqslant \tilde{C}_k \|(H+2)^k g\|_2 \int_0^\infty e^{-s} s^{k-1-\nu/4} P_x(T(b) > s)^{1/2}.$$

From this inequality and Lemma C.2(b), the theorem follows. ∎

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REFERENCES

1. S. AGMON, in preparation.
2. M. AIZENMAN AND B. SIMON, Brownian motion and Harnack's inequality for Schrödinger operators, *Comm. Pure Appl. Math.* **35** (1982), 209–273.
3. J. BERGH AND J. LÖFSTROM, "Interpolation Spaces: An Introduction," Springer, New York/Berlin, 1976.
4. J. BROSSARD, The problem of Dirichlet for the Schrödinger operator, *Ann. Inst. Four. (Grenoble)*, 1984.
5. R. CARMONA, Regularity properties of Schrödinger and Dirichlet semigroups, *J. Funct. Anal.* **17** (1974), 227–237.
6. R. CARMONA, Pointwise bounds for Schrödinger eigenstates, *Comm. Math. Phys.* **62** (1978), 97–106.
7. R. CARMONA AND B. SIMON, Pointwise bounds on eigenfunctions and wave packets in N-body quantum systems. V. Lower bounds and path integrals, *Comm. Math. Phys.* **80** (1981), 59–98.
8. E. B. DAVIES, "One Parameter Semigroups," Academic Press, New York, 1980.
9. E. B. DAVIES, Hypercontractive and related bounds for double well Schrödinger operators, *Quart. J. Math. Oxford Ser. (2)* **34** (1983), 407–421.
10. E. B. DAVIES, Spectral properties of metastable Markov semigroups, *J. Funct. Anal.* **52** (1983), 315–329.
11. E. B. DAVIES, Some norm bounds and quadratic form inequalities for Schrödinger operators II, *J. Operator Theory*, in press.
12. E. B. DAVIES, JWKB and related bounds on Schrödinger eigenfunctions, *Bull. London Math. Soc.* **14** (1982), 273–284.
13. E. B. DAVIES AND B. SIMON, Ultracontractive semigroups and some problems in analysis, *in* "Aspects of Mathematics and Its Applications," North-Holland, Amsterdam, 1984.
14. J. P. ECKMAN, Hypercontractivity for anharmonic oscillators, *J. Funct. Anal.* **16** (1974), 388–404.
15. D. GILBARG AND N. TRUDINGER, "Elliptic Partial Differential Equations of Second Order," Springer, New York/Berlin, 1977.
16. J. GLIMM, Boson fields with nonlinear self-interaction in two dimensions, *Comm. Math. Phys.* **8** (1968), 12–25.
17. P. GRISVARD, Singularities for the problem of limits in polyhedrons, Sem. Goul.-Schwartz, 1981–82, Ex. No. 8.
18. L. GROSS, Logarithmic Sobolev inequalities, *Amer. J. Math.* **97** (1976), 1061–1083.
19. L. GROSS, unpublished bibliography on hypercontractivity.
20. E. HARRELL, On the rate of asymptotic eigenvalue degeneracy, *Comm. Math. Phys.* **60** (1978), 73–95.
21. B. HELFFER AND J. SJOSTRAND, Multiple wells in the semi-classical limit I, *Comm. Partial Differential Equations* **9** (4), (1984), 337–408.

22. J. HOOTON, Dirichlet semigroups on bounded domains, Louisiana State Univ., preprint.
23. E. NELSON, A quartic interaction in two dimensions, *in* "Mathematical Theory of Elementary Particles," pp. 69–73, MIT Press, Cambridge, Mass., 1966.
24. E. NELSON, The free Markov field, *J. Funct. Anal.* **12** (1973), 211–227.
25. A. PERSSON, Compact linear mappings between interpolation spaces, *Ark. Mat.* **5** (1964), 215–219.
26. S. PORT AND C. STONE, "Brownian Motion and Classical Potential Theory," Academic Press, New York, 1978.
27. M. REED AND B. SIMON, "Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness," Academic Press, New York, 1975.
28. M. REED AND B. SIMON, "Methods of Modern Mathematical Physics III: Scattering Theory," Academic Press, New York, 1979.
29. M. REED AND B. SIMON, "Methods of Modern Mathematical Physics IV: Analysis of Operators," Academic Press, New York, 1978.
30. J. ROSEN, Sobolev inequalities for weighted spaces and supercontractive estimates, *Trans. Amer. Math. Soc.* **222** (1976), 367–376.
31. I. M. SIGAL, A generalized Weyl theorem and L^p spectra of Schrödinger operators, Weizmann preprint.
32. B. SIMON, Ergodic semigroups of positivity preserving self-adjoint operators, *J. Funct. Anal.* **12** (1973), 335–339.
33. B. SIMON, Pointwise bounds on eigenfunctions and wave packets in N-body quantum systems III, *Trans. Amer. Math. Soc.* **208** (1975), 317–329.
34. B. SIMON, Large time behavior of the L^p norm of Schrödinger semigroups, *J. Funct. Anal.* **40** (1981), 66–83.
35. B. SIMON, Semiclassical analysis of low lying eigenvalues II: Tunneling, *Ann. of Math. (2)*, in press.
36. B. SIMON, Some quantum operators with discrete spectrum but classically continuous spectrum, *Ann. Physics* **146** (1983), 209–220.
37. B. SIMON, Schrödinger semigroups, *Bull. Amer. Math. Soc. (N.S.)* **7** (1982), 447–526.
38. B. SIMON, The $P(\phi)_2$ Euclidean (quantum) field theory, Princeton Series in Physics, Princeton Univ. Press, Princeton, N.J., 1974.
39. B. SIMON AND R. HOEGH-KROHN, Hypercontractive semigroups and two-dimensional self-coupled Bose fields, *J. Funct. Anal.* **9** (1972), 121–180.
40. Z. ZHAO, Continuity of conditioned Feynman–Kac functional and integral kernel for Schrödinger equation, Stanford preprint.
41. J. ROSEN AND B. SIMON, Fluctuations of the paths in $P(\phi)_1$ Markov processes, *Ann. Probab.* **4** (1976), 155–174.
42. A. SOKAL, private communication.
43. B. SIMON, "Functional Integration and Quantum Physics," Academic Press, New York, 1979.
44. D. JERISON AND C. KENIG, Boundary behavior of harmonic functions in nontangentially accessible domains, *Adv. in Math.* **46** (1982), 80–147.
45. K. MILLER, Barriers on cones for uniformly elliptic operators, *Ann. Mat. Pura Appl. (4)* **76** (1976), 93–105.
46. K. MILLER, Extremal barriers on cones with Phragmèn–Lindlöf theorems and other applications, *Ann. Mat. Pura Appl. (4)* **90** (1971), 297–329.
47. J. ODDSON, On the boundary point principle for elliptic equations in the plane, *Bull. Amer. Math. Soc. (N.S.)* **74** (1978), 666–670.

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