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Absence of absolutely continuous spectrum for some one dimensional random but deterministic Schrödinger operators

by

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ABSTRACT. — We study $H_\omega = -d^2/dx^2 + V_\omega(x)$ and its discrete analog. For a large class of V_ω which are intuitively random but nevertheless deterministic, we prove that $\sigma_{ac}(H_\omega)$ is empty. Typical is $V_\omega(x) = \Sigma f(x - x_n(\omega))$ where f (and thus V) is analytic and $x_n(\omega)$ is a Poisson process. For example, if f decays exponentially but is not a reflectionless potential, we prove that $\sigma_{ac}(H_\omega)$ is empty.

RÉSUMÉ. — On étudie $H_\omega = -d^2/dx^2 + V_\omega(x)$ et son analogue discret. Pour une classe générale de V_ω qui sont intuitivement aléatoires mais néanmoins déterministes, on montre que $\sigma_{ac}(H_\omega)$ est vide. Un cas typique est $V_\omega(x) = \Sigma f(x - x_n(\omega))$ où f (donc V) est analytique et où $x_n(\omega)$ est un processus de Poisson. Par exemple, si f décroît exponentiellement mais n'est pas un potentiel sans réflexion, on montre que $\sigma_{ac}(H_\omega)$ est vide.

1. INTRODUCTION

This paper should be regarded as comment on [*14*]. To describe our goals we need little background. We will study stochastic Schrödinger operators (on $L^2(R)$)

$$H_\omega = -\frac{d^2}{dx^2} + V_\omega(x) \tag{1.1}$$

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and their discrete analogs, the stochastic Jacobi matrices

$$(h_\omega u)(n) = u(n+1) + u(n-1) + V_\omega(n)u(n). \qquad (1.2)$$

Here V_ω is an ergodic process, i. e. there is a probability measure space $(\Omega, d\mu)$, an ergodic family of measure processing transformations $T^{\#}$ ($\#$ runs through R in case (1.1) and through Z in case (1.2)) and a function, F, so that

$$V_\omega(\#) = F(T^{\#}\omega). \qquad (1.3)$$

It is often useful to slightly extend the notion to require only that $V_\omega(\#)$ be measurable functions on Ω, so that there is an invertible ergodic transformation T on Ω so that

$$V_\omega(\# + L) = V_{T\omega}(\#) \qquad (1.4)$$

for some L (required to be in Z in case (1.2)). There is a suspension procedure [*10*] for reducing this case to the fully invariant case.

Operators like the above H_ω and h_ω have been heavily studied in the solid state literature when V_ω has verv strong mixing properties (the example to think about is (1.2) where $\{ V_\omega(n) \}_{n=-\infty}^{\infty}$ are independent identically distributed random variables), under the rubric « random potentials ». The general belief is that this random one-dimensional case has only point spectrum and, in particular, no absolutely continuous spectrum [*17*].

For rather special V_ω (e. g. (1.1) when Ω is Brownian motion on a compact Riemannian manifold and F rather specific [*7*] or (1.2) in the i. i. d. case with absolutely continuous density [*15*]) it is known that H_ω or h_ω have dense point spectrum.

One might hope to prove that $\sigma_{ac} = \phi$ in great generality. The generality *cannot* be all ergodic processes since periodic potentials are included in the above formalism and they have only a. c. spectrum. Moreover, there are almost periodic potentials with some a. c. spectrum [*6*] [*2*]. In this regard Kotani [*13*] (see Simon [*21*] for the discrete case (1.2) and Kirsch [*10*] for the case (1.4)) proved a very general result:

DEFINITION. — A stochastic process $V_\omega(\#)$ indexed by $\#$ in R or Z is called *deterministic* if $V_\omega(0)$ is a measurable function of $\{ V_\omega(\zeta) \}_{\zeta \leq L}$ for all L. It is called *non-deterministic* if it is not deterministic.

THEOREM 1.1. — If $V_\omega(x)$ is a bounded non-deterministic process, then

$$\sigma_{ac}(H_\omega) = \phi.$$

Remark. — While [*13*] [*21*] require V_ω to be bounded, one can get away with much weaker assumptions (see the second appendix): in the discrete case (1.2), it suffices that $E(ln(|V_\omega(0)| + 1)) < \infty$; in the continuum case, (1.1) a bound

$$E(|V_\omega(0)|^2) < \infty \tag{1.5}$$

will suffice.

Theorem 1.1 would appear to close the case: Surely any reasonable « random » potential is non-deterministic. In fact, there are some elementary examples which are intuitively random but at the same time deterministic !

EXAMPLE 1. — (Introduced by Herbst-Howland [8].) Let $\{ x_n(\omega) \}_{n=-\infty}^{\infty}$ denote the position of a Poisson process (indexed, e. g. so $x_{-1} < 0 < x_0$ and $x_n < x_{n+1}$). Let f be a function analytic in the strip $\{ f \mid |\operatorname{Im} z| < \delta \}$ for some δ, obeying

$$|f(z)| \leq C(1 + |z|)^{-2-\varepsilon} \tag{1.6}$$

there. Let

$$V_\omega(x) = \sum_n f(x - x_n(\omega)). \tag{1.7}$$

Then intuitively V_ω is « random », but since V_ω is analytic for a. e. ω, $V_\omega(0)$ is determined by $V_\omega(x)$ for x in any half line $x < -L$; i. e. V_ω is deterministic.

EXAMPLE 2. — (Introduced by Kirsch-Martinelli [11] [12].) This is a continuum example obeying (1.4). Let f be a fixed function, which we demand at least obey

$$f \in l_1(L^2) = \left\{ f \,\middle|\, \sum_n \left(\int_n^{n+1} |f(x)|^2 dx \right)^{\frac{1}{2}} < \infty \right\}.$$

Let $q_n(\omega)$ be bounded i. i. d's. Let

$$V_\omega(x) = \sum_n q_n(\omega) f(x - n). \tag{1.8}$$

If f has compact support, it is easy to see that V_ω is non-deterministic. However, it is not hard to construct lots of examples where V_ω is deterministic, e. g. f analytic. Indeed, in the first appendix, we prove that for almost all f (i. e. for a dense G_δ in $l_1(L^2)$), the process is deterministic.

EXAMPLE 3. — Here is a discrete example. Let $\alpha(n)$ be a bijection from Z to $Z^+ = \{ n \in Z \mid n \geq 0 \}$, e. g. $\alpha(0) = 0$, $\alpha(1) = 1$, $\alpha(-1) = 2$, $\alpha(2) = 3$, $\alpha(-2) = 4 \ldots$ Let

$$f(n) = 3^{-\alpha(n)}.$$

Let $q_n(\omega)$ be i. i. d's with $q_n(0) = 0, 1$ with probability p and $1 - p$ respectively. Fix $\lambda > 0$. Let

$$V_\omega(n) = \lambda \sum_m q_m(\omega) f(n - m). \tag{1.9}$$

This is clearly random in an intuitive sense but one can read the value of q_m off the base 3 decimal expansion of $\lambda^{-1}V_\omega(n)$ at any fixed n, so the process is clearly deterministic.

In this paper we will prove that each of these examples includes (for a suitable large choice of f and/or q_n) many deterministic potentials for which σ_{ac} is empty. There will be two parts to our proofs. The first is a recent result of Kotani [*14*] which has the following consequence: if W is a periodic potential in the support (see Section 2 for a precise definition of this) of the process $V_\omega(\#)$ and $E_0 \notin \sigma(H_0 + W)$, then $E_0 \notin \sigma_{ac}(H_0 + V_\omega)$. In certain cases, and, in particular for the Herbst-Howland example with $f \geq 0$, one will be able to just use this first step to prove that $\sigma_{ac} = \phi$.

To explain the second part of the proof, consider the Herbst-Howland example. It is not hard to see that if x_n is an arbitrary periodic sequence (i. e. $x_{n+\kappa m} = x_n + \kappa L$ for some m, L) then

$$W = \Sigma f(x - x_n)$$

lies in the support of the Herbst-Howland process. In particular, the spectral gaps of $-\dfrac{d^2}{dx^2} + W$ are not in $\sigma_{ac}(H_\omega)$. It would obviously be useful to compute gap positions in some limit and thus in Section 3, we discuss the gaps for

$$\frac{-d^2}{dx^2} + \sum_n f(x - nL)$$

in the limit $L \to \infty$. When (1.6) holds we will be able to show that a gap occurs at some fixed energy $E > 0$ for suitable large L so long as the reflection coefficient at energy E for $\dfrac{-d^2}{dx^2} + f(x)$ is non-zero. In Section 4, we return to the above examples armed with the ideas of Sections 2-3.

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2. SUPPORT AND ABSOLUTELY CONTINUOUS SPECTRUM

In this section we recall Kotani's basic result in [*14*], sketch briefly its proof and indicate how to use it to prove $\sigma_{ac} = \phi$ in some simple cases. We fix Ω, a class of potentials $V_\omega(\#)$. In the discrete case, Ω will be $\{ V_\omega(n) \mid |V_\omega(n)| < \infty \text{ all } n \}$, i. e. R^Z. We will topologize Ω in this case by putting in the (metric) topology of pointwise convergence. In the continuum case, Ω will be the set of L^2_{loc} functions obeying

$$\left(\int_n^{n+1} |V(x)|^2 dx\right)^{\frac{1}{2}} = C_{(V)}(n^2+1)^{\frac{1}{2}-\delta} \tag{2.1}$$

for some $\delta > 0$ and we put on the (metric) topology of L^2_{loc} convergence. The key fact of these topologies is that, as we prove in Appendix 2, the basic m (and h) functions (and thus the function, α, discussed below) evaluated at a fixed E with Im E > 0 are continuous in V.

We note that Ω is large enough so that if $V \in \Omega$, then $\tau_x V = V(\cdot + x)$ is also in Ω (where $x \in R$, resp. Z in the continuum, resp. discrete cases). By an ergodic measure on Ω, we mean a probability measure, μ, on Ω which is invariant and ergodic for the translation τ_x. In the continuum case we will demand that $E(|V(0)|^2) < \infty$. This will imply that the h function has finite expectation, a technical fact needed in the theory of [*13*] [*14*] [*21*]. We point out that if $V_\omega(x)$ is any ergodic process jointly measurable in ω in some $\tilde{\Omega}$ and x in R with a probability measure on $\tilde{\Omega}$ with $E(|V_\omega(0)|^2) < \infty$, then one can realize it in an essentially equivalent way on Ω (see Appendix 2). In the discrete case we will require the weaker condition $E[ln(|V(0)|+1)] < \infty$.

It is a consequence of the ergodic theorem [*15*] [*12*] that if μ is ergodic, then $\sigma_{ac}(H_\omega) \equiv \sigma_{ac}^{(\mu)}$ is a. e. constant. Indeed, if γ is the Lyaponov exponent (see e. g. [*1*]) for H_ω, then ([*13*] [*21*])

$$\sigma_{ac}(H_\omega) = \overline{\{E \mid \gamma(E) = 0\}}^{\text{ess}}$$

where the closure is an « essential closure », i. e. closure up to sets of measure zero. Since Ω has a topology, the support of μ is well defined, i. e. $V \in \operatorname{supp} \mu \Leftrightarrow \mu(N) > 0$ for any neighborhood, N, of V.

Kotani's basic result in [*14*] is:

THEOREM 2.1. — (Kotani [*14*]). *If v is an ergodic measure on Ω with $V_\infty \in \operatorname{supp} v$, and $\sup_n \left(\int_n^{n+1} |V_\infty(x)|^2 dx\right) + E_v(|V(0)|^2) < \infty$ in the continuum case (resp. $\sup_n |V_\infty(n)| + E_v(ln(1+|V(0)|)) < \infty$ in the discrete case) then $\sigma_{ac}^{(v)} \subset \sigma_{ac}\left(-\frac{d^2}{dx^2} + V_\infty(x)\right)$.*

SKETCH 1. — The Green's function, i. e. integral kernel $G(x, y; E, \omega)$ of $(H_\omega - E)^{-1}$ is well defined for all E with Im E > 0. Let $\alpha_\omega(E) = G(0, 0; E, \omega)$. It is a fundamental result of Kotani [*13*] (see the second appendix for the generality we allow here) that for a. e. ω
(w. r. t μ) (a) $\alpha_\omega(E)$ has finite non-tangential boundary values ($\alpha_\omega(E + i0)$) for a. e. $E \in R$ and

(*b*) $\{ E \mid \gamma(E) = 0 \} = \{ E \mid \text{Re } \alpha_\omega(E + i0) = 0 \}$ up to sets of measure zero.

2. α_ω is a Herglotz function, i. e. $\text{Im } \alpha_\omega(E) > 0$ if $\text{Im } E > 0$. It is not hard to prove the following fact about such functions (see Kotani [*14*]): if $\alpha^{(n)}$ and $\alpha^{(\infty)}$ are Herglotz functions and $\alpha^{(n)}(E) \to \alpha^{(\infty)}(E)$ for all E in the upper half plane and if $A \subset \mathbb{R}$ is such that $\text{Re } \alpha^{(n)}(E + i0) = 0$ for all n and all $E \in A$, then $\text{Re } \alpha^{(\infty)}(E + i0) = 0$ for a. e. E in A.

3. Let $V_n \to V_\infty$ in the topology of Ω; let $\text{Im } E > 0$ and let $\alpha^{(n)}(E)$ denote the integral kernel of $(H_0 + V_n - E)^{-1}$ evaluated at (0, 0). Then $\alpha^{(n)}(E) \to \alpha^{(\infty)}(E)$ as $n \to \infty$. This is trivial in the discrete case since we have strong resolvent convergence and $\alpha^{(n)}(E) = (\delta_0, (H_0 + V_n - E)^{-1}\delta_0)$. It is a simple m-function argument in the continuum case (see appendix 2).

4. For any V_∞, the essential closure of $\{ E \mid \text{Re } \alpha^{(\infty)}(E + i0) = 0 \}$ is contained in $\sigma_{\text{ac}}\left(-\dfrac{d^2}{dx^2} + V_\infty(x) \right)$ since $\text{Im } \alpha^{(\infty)}(E + i0) > 0$ there.

5. Since $V_\infty \in \text{supp } \nu$, we can find $V_n \in \text{supp } \nu$ so that $V_n \to V_\infty$ and so that there is a set A with $\overline{A} = \sigma_{\text{ac}}^{(\nu)}$ and $\text{Re } \alpha^{(n)}(E + i0) = 0$, all $E \in A$ and n. By steps 2 and 3, $\text{Re } \alpha^{(\infty)}(E + i0)$ for a. e. $E \in A$ so $\overline{A} \subset \sigma_{\text{ac}}\left(-\dfrac{d^2}{dx^2} + V_\infty(x) \right)$. ∎

REMARKS. — 1. Even though the proof [*13*] that

$$\sigma_{\text{ac}} = \{ E \mid \text{Re } \alpha_\omega(E + i0) = 0 \}$$

for a. e. ω is one dimensional, one might hope that a result like this holds for the discrete case (when $G(0, 0) < \infty$) in higher dimensions which would imply that Thm. 2.1 holds in higher dimension. Unfortunately, this is false even for the case $V = 0$. For in that case $\text{Im } G(0, 0) = dk/dE$, the derivative of the integrated density of states, so in the discrete one dimensional case with $V = 0$, $\text{Re } G(0, 0) = \dfrac{d\gamma}{de}$. Taking limits in the Thouless formula for the strip [*3*] one sees that when $V = 0$ and $\nu = 2$:

$$\text{Re } G(0, 0; E + i0) = \frac{d\Gamma}{dE}$$

$$\Gamma(E) = \frac{1}{2\pi} \int_0^{2\pi} dk \gamma_0(E + 2 \cos k)$$

where $\gamma_0(E)$ is the free one dimensional Lyaponov exponent, i. e.,

$$\begin{aligned} \gamma_0(E) &= 0 \qquad |E| \leq 2 \\ &= \text{Arccosh}\left(\frac{|E|}{2} \right) \qquad |E| \leq 2 . \end{aligned}$$

From this, one can see that $d\Gamma/dE$ is not zero, on $[-4, 4] = \sigma_{\text{ac}}(h_0)$.

2. The same proof shows that up to sets of E of Lebesque measure zero, if $\mu\left(\left\{\text{V periodic} \,\middle|\, \text{E} \in \text{spec}\left(\frac{-d^2}{dx^2} + \text{V}\right)\right\}\right) > 0$, then $\text{E} \in \sigma_{\text{ac}}^{(\mu)}$. This may be useful in the study of limit periodic potentials.

In Kotani's paper [*14*], an example is given of a deterministic process where one can prove that $\sigma_{\text{ac}} = \phi$ using Thm 2.1. We close this section with two more.

THEOREM 2.2. — Let f be any non-negative continuous function, not identically zero obeying (1.6) and let V_ω be given by (1.7) where $x_n(\omega)$ is a Poisson process. Then $\sigma_{\text{ac}}(\text{H}_\omega) = \phi$.

REMARKS. — 1. In particular, $\sigma_{\text{ac}}(\text{H}_\omega) = \phi$ for the Herbst-Howland potential if $f \geqslant 0$.

2. As this proof shows, the result holds for a *very* large class of processes $x_n(\omega)$ much larger than just the Poisson Process. For example, if $\zeta_n(\omega)$ are i. i. *d*'s with Gaussian distribution and $x_n(\omega) = n + \zeta_n(\omega)$ the result still holds (this V_ω is not ergodic directly but obeys (1.4) and so is related to an ergodic example via a suspension construction). It is not essential that the ζ_n be Gaussian, any distribution with support all of R and with sufficient decay of $\text{Prob}(|\zeta_0| > b)$ for the sum (1.7) to converge will do. In fact, all that is needed is that the support of $\{x_n(\omega)\}$ includes periodic sequences of arbitrarily high density.

Proof. — Let x_n be a sequence of points obeying $\#\{n \mid |x_n| \leq \text{L}\} \leq \text{CL}$ for some C. Then, we claim that

$$\text{W} = \Sigma f(x - x_n)$$

is in the support of the measure associated to the Herbst-Howland process. For, given L_1 and ε, we can first find L_2 so that

$$\sup_{|x| \leq \text{L}_1} \left| \text{W}(x) - \sum_{|x_n| \leq \text{L}_2} f(x - x_n) \right| \leq \varepsilon/3 .$$

Next find (by continuity of f and (1.5)) δ so that

$$\sup_x \sum_{|x_n| \leq \text{L}_2} | f(x - x'_n) - f(x - x_n) | \leq \varepsilon/3$$

if

$$\sup_{|x_n| \leq \text{L}_2} | x'_n - x_n | < \delta .$$

Finally, find L_3, so that for a set of probability bigger than 0 for the Poisson process

$$\sup_{|x| \leq L_1} \sum_{|x_n(\omega)| \geq L_3} | f(x - x_n(\omega)) | \leq \varepsilon/3 .$$

If $\tilde{x}_n(\omega)$ is a set of points which is in this typical configuration for $| x_n | \geq L_3$ and has points near (within δ) of $\{ x_n \}_{|x_n| \leq L_2}$ then

$$\sup_{|x| \leq L_1} | W(x) - \Sigma f(x - \tilde{x}_n(\omega)) | \leq \varepsilon$$

so $W \in \operatorname{supp} \mu$.

Now by translating, we can suppose that $f(0) > 0$ so $f(x) \geq \varepsilon > 0$ if $| x | \leq \delta$. Fix λ and let $x_n = n\delta/2k$ then $W(x) \geq k\varepsilon$ and thus

$$\sigma\left(\frac{-d^2}{dx^2} + W(x)\right) \subset [k\varepsilon, \infty) .$$

By Theorem 2.1

$$\sigma_{ac}(H_\omega) \subset [k\varepsilon, \infty) .$$

Since k is arbitrary, $\sigma_{ac}(H_\omega) = \phi$. ■

THEOREM 2.3. — Let V_ω be the potential (1.9) of Example 3. Then

$$\sigma_{ac}(H_\omega) \subset [-2, 2] \cap \left[\frac{3}{2}\lambda - 2, \frac{3}{2}\lambda + 2\right].$$

In particular, if $\lambda > 8/3$, $\sigma_{ac}(H_\omega) = \phi$.

Remarks. — 1. It is not hard to show that

$$\sigma(H_\omega) \supset [-2, 2] \cup \left[\frac{3}{2}\lambda - 2, \frac{3}{2}\lambda + 2\right]$$

so by this theorem, there is considerable non-ac spectrum for any non-zero λ.

2. By using the idea of the next two sections, one should be able to show that $\sigma_{ac}(H_\omega) = \phi$ for any λ.

Proof. — The cases $V = 0$ (all $q_m = 0$) and $V = \frac{3}{2}\lambda$ (all $q_m = 1$) lie in $\operatorname{supp} \mu$ but $\sigma(H_0 + c) = [c - 2, c + 2]$ so the result follows from Theorem 2.1. ■

3. GAPS FOR SPARCE PERIODIC POTENTIALS

Let f be a function obeying

$$| f(x) | \leq C(1 + | x |)^{-1-\varepsilon} \tag{3.1}$$

for some C, $\varepsilon > 0$. Let

$$V_L(x) = \sum_{n=-\infty}^{\infty} f(x - nL) \tag{3.2}$$

so for L large V_L is a periodic potential which is mainly zero with one « bump » per period. Our goal here is to analyze where the gaps of the spectrum of $\frac{-d^2}{dx^2} + V_L(x)$ lie for large L. We deal only with the continuum case here although a similar analysis is possible for the discrete case.

A crucial role is played by the transmission and reflection coefficients for the operator $\frac{-d^2}{dx^2} + f(x)$ so we begin by recalling their definition and basic properties [4] [5]. Intuitively t and r are defined by seeking a solution of $-u'' + fu = k^2u$ which as $x \to -\infty$ has an input and a reflected wave and as $x \to +\infty$ a transmitted wave, i. e.

$$\begin{aligned} u(x) &\sim e^{ikx} + re^{-ikx} \qquad (x \to -\infty) \\ &\sim te^{ikx} \qquad (x \to +\infty) \end{aligned}$$

u is thus singled out by its asymptotics at $+\infty$, which suggests we instead seek $\psi \sim e^{ikx}$ at $+\infty$ and to $t^{-1}e^{ikx} + rt^{-1}e^{-ikx}$ at $-\infty$. For this analysis, it is useful to fix a W obeying (3.1) and to allow $f(x) = \lambda W(x)$ with λ a coupling constant. One can construct ψ by solving the Jost integral equation:

$$\psi(x; \lambda, k) = e^{ikx} + \frac{\lambda}{k}\int_x^{\infty} \sin(k(y-x))W(y)\psi(y; \lambda, k)dy\,. \tag{3.3 a}$$

This is a Volterra integral equation so it is easy to see that the iteration of (3.3) (Neumann series) is entire in λ and thus ψ (and ψ') for x fixed are analytic in λ and by (3.1), continuous in k. Two distinct expressions for t^{-1} and r/t are useful. First of all, since every solution of $-\psi'' + V\psi = k^2\psi$ is bounded at $-\infty$, we can read the $x \to -\infty$ asymptotics off (3.3) and obtain

$$t^{-1} = 1 - \frac{\lambda}{2ik}\int_{-\infty}^{\infty} e^{-iky}W(y)\psi(y; \lambda, k)dy \tag{3.4 a}$$

$$rt^{-1} = \frac{\lambda}{2ik}\int_{-\infty}^{\infty} e^{iky}W(y)\psi(y; \lambda, k)dy\,. \tag{3.4 b}$$

Secondly, we can construct a solution $\psi^{\#}(x, \lambda, k)$ obeying Jost boundary conditions at $-\infty$, i. e.,

$$\psi^{\#}(x; \lambda, k) = e^{ikx} - \frac{\lambda}{k}\int_{-\infty}^{x} \sin(k(y-x))W(y)\psi^{\#}(y; \lambda, k) \tag{3.3 b}$$

from which (where $W(f, g) = f(0)g'(0) - f'(0)g(0)$)

$$t^{-1} = (-2ik)^{-1} W(\psi, \overline{\psi}^{\#}) \tag{3.5 a}$$
$$rt^{-1} = (2ik)^{-1} W(\psi, \psi^{\#}). \tag{3.5 b}$$

From (3.5) we can immediately deduce that t^{-1} and rt^{-1} have the same regularity properties in λ, k as ψ does. In addition, since ψ, ψ', $\psi^{\#}$, $(\psi^{\#})'$ are given by convergent series, we can prove that if $|W_n(x)| \leq C(1 + |x|)^{-1-\varepsilon}$ and $W_n(x) - W(x) \to 0$ pointwise, then $r_r/t_n \to r/t$, $t_n^{-1} \to t^{-1}$. In particular, given R, define

$$W_R(x) = W(x) \qquad |x| < R \tag{3.6 a}$$
$$= 0 \qquad |x| > R \tag{3.6 b}$$

and let r_R, t_R be the corresponding reflection and transmission coefficients. Then $|r_R - r| + |t_R - t| \to 0$ as $R \to \infty$ if W obeys (3.1).

From (3.4) we can obtain an explicit expression for the Taylor coefficient of t^{-1} and rt^{-1} as a function of λ and in particular for the $0(\lambda)$ term (Born term). We summarize with

THEOREM 3.1. — Let W be a potential obeying (3.1). Let $t(k, \lambda)$, $r(k, \lambda)$ be its transmission and reflection coefficient for $\lambda \in (-\infty, \infty)$, $k \in (0, \infty)$ and let t_R, r_R be the corresponding objects for W_R (given by (3.6)). Then:

i) t, r are jointly real analytic in λ and continuous in k.

ii) For k, λ fixed, $|r_R(\lambda, k) - r(\lambda, k)| + |t_R(\lambda, k) - t(\lambda, k)| \to 0$ with convergence uniform in λ, k as λ, k run through compacts of $R \times (0, \infty)$,

iii)
$$r(\lambda, k) = \lambda(2ik)^{-1} \int_{-\infty}^{\infty} e^{2iky} W(y) dy + 0(\lambda^2).$$

Now, we want to try to study gaps for the operator $-d^2/dx^2 + V_L(x)$ where V_L is given by (3.2). It will be easier to initially study a related potential

$$\tilde{V}_L(x) = \sum_{n=-\infty}^{\infty} f_{L/2}(x - nL) \tag{3.7}$$

i. e. $\tilde{V}_L(x) = f(x)$ if $|x| < \frac{L}{2}$ and $\tilde{V}_L(x)$ has period L. From (3.1) one easily concludes that

LEMMA 3.2. — $\sup_x |V_L(x) - \tilde{V}_L(x)| \leq C_1 L^{-1-\varepsilon}$.

Proof. — By (3.1), for $|x| \leq \frac{L}{2}$

$$|\tilde{V}_L(x) - V_L(x)| \leq \sum_{n \neq 0} \sup_{|x| \geq \left(|n| - \frac{1}{2}\right)L} |f(x)|$$

$$\leq \tilde{C} \sum_{n \neq 0} (|n| L)^{-1-\varepsilon} = L^{-1-\varepsilon}\left[\tilde{C} \sum_{n \neq 0} |n|^{-1-\varepsilon}\right]. \quad \blacksquare$$

Now we compute the discriminant [*16*] [*20*] for the periodic potential $\tilde{V}_L$. Pick any basis, u_1, u_2 for the solutions of $-u'' + \tilde{V}_L u = Eu$.

Let $\Phi_i(x)$ be the two component vector $(u_i'(x), u_i(x))$ and write $\Phi_i(L/2) = \sum_i M_{ij}\Phi_j(-L/2)$. The discriminant $\tilde{\Delta}_L(E)$ is given by $\tilde{\Delta}_L(E) = M_{11} + M_{22}$ (as the trace of M, this is independent of the choice of u_1, u_2). Now set $E = k^2$ and let $\tilde{\psi}^{\#}$ be the solution of (3.3 *b*) with $W = f_{L/2}$. $\tilde{\psi}^{\#}$ obeys

$$-u'' + \tilde{V}_L u = Eu$$

on $(-L/2, L/2)$ and so we can let u_1 be the unique extension of this solution to $(-\infty, \infty)$. We let $u_2 = \bar{u}_1$. By construction, $\tilde{\psi}^{\#}(x) = e^{ikx}$ on $(-\infty, -L/2)]$ so $\Phi_1(-L/2) = (ike^{-ikL/2}, e^{-ikL/2})$. On the other hand, by (3.5), on $[L/2, \infty)$

$$\tilde{\psi}^{\#}(x) = \bar{t}_{L/2}^{-1}e^{ikx} + r_{L/2}/t_{L/2}e^{-ikx}$$

which allows us to compute $\Phi_1(L/2)$ and find that

$$M_{11} = \bar{t}_{L/2}^{-1}e^{+ikL}; \qquad M_{22} = (r_{\frac{1}{2}L}/t_{\frac{1}{2}L})e^{-ikL}.$$

Since $u_2 = \bar{u}_1$, $M_{22} = \overline{M}_{11}$, $M_{21} = \overline{M}_{12}$. The net result of this calculation is the following result, essentially due to Keller [*9*]:

LEMMA 3.3. — Let $E = k^2 > 0$, let $\tilde{\Delta}_L(E)$ be the discriminant for $-d^2/dx + \tilde{V}_L$ and let $t_{L/2}$ be the transmission coefficient for $f_{L/2}$. Then

$$\tilde{\Delta}_L(E) = 2\,\mathrm{Re}\,(t_{L/2}(k)^{-1}e^{-ikL}). \tag{3.8}$$

We recall [*16*] [*20*] that the gaps for $-d^2/dx^2 + \tilde{V}_L$ are given by the points where $|\tilde{\Delta}_L(E)| > 2$. This allows us to prove the main result of this section:

THEOREM 3.4. — Fix a function f obeying (3.1) and let V_L be given by (3.2). Suppose that $E_0 = k_0^2 > 0$ and that the reflection coefficient $r(k_0)$ for f is non-zero. Then there exists $\alpha > 0$ and an infinite sequence $L_l \to \infty$ so that $-d^2/dx^2 + V_{L_l}(x)$ has a gap in its spectrum including the interval $(E_0 - \alpha L_l^{-1}, E_0 + \alpha L_l^{-1})$. Moreover, if $k_0/2\pi$ is not rational, we can choose the L_l to be integral.

Remark. — The proof also shows that if $r(k) = 0$, then either E is not in any gap for L large or the gap which it lies inside shrinks faster than $0(L^{-1})$.

Proof. — If $-d^2/dx^2 + \tilde{V}_L$ has a gap (β, γ) with $|\gamma - \beta| > 2 \| V_L - \tilde{V}_L \|_\infty$, then $-d^2/dx^2 + V_L$ has a gap containing $(\beta + \| V_L - \tilde{V}_L \|_\infty, \gamma - \| V_L - \tilde{V}_L \|_\infty)$. Since Lemma 3.2 says that $\| V_L - \tilde{V}_L \|_\infty = 0(L^{-1-\varepsilon})$, we need only prove the theorem for V_L replaced by $\tilde{V}_L$. For any transmission coefficient, t, write $t = Te^{i\theta}$ where T, θ are real. Then, by (3.8):

$$\tilde{\Delta}_L(k^2) = 2T_{L/2}(k)^{-1} \cos(kL + \theta_L(k)).$$

By the continuity of t in k and as $L \to \infty$, we get that so long as $T_{L/2}(k_0) < 1$, if L_l is defined by $k_0 L_l + \theta_\infty(k_0) = 2\pi l$, then $|\tilde{\Delta}_{L_l}(k^2)| > 2$ in a symmetric neighbourhood of k_0 of size $0(L_l^{-1})$. So long as $k_0/2\pi$ is irrational, we can take integral L_l so that the fractional part of $k_0 L_l + \theta_\infty(k_0)$ goes to zero. The theorem results, if one remarks that $|r|^2 + |t|^2 = 1$ (this is proven by computing $W(\psi, \bar{\psi})$ near $+\infty$ and $-\infty$) so that $T < 1$ is equivalent to $r \neq 0$. ■

While we do not require the analysis of the negative spectrum of $-d^2/dx^2 + V_L$ in this paper, we want to sketch it for completeness sake. Let f_L^b be the potential obtained by restricting V_L to $(-L/2, L/2)$ so $f_L^b \to f$ and the reflection and transmission coefficients converge.

By writing out the perturbation series for ψ (resp. $\psi^\#$) one sees that ψ can be analytically continued to be region $\operatorname{Im} k \geq 0$ (resp. $\operatorname{Im} k \leq 0$) (i. e. analytic in the uper half plane, continuous up to boundary with $k = 0$ removed) and is real if $k = i\kappa$ (resp. $-i\kappa$) with $\kappa > 0$. Thus, by (3.5 *a*), t^{-1} has an analytic continuation to the region $\operatorname{Im} k > 0$ and t^{-1} is real if $k = i\kappa$. Moreover, since ψ decays at $+\infty$ and $\psi^\#$ at $-\infty$, $t^{-1}(i\kappa) = 0$ if and only if $E = -\kappa^2$ is an eigenvalue of $-d^2/dx^2 + W(x)$.

Define $\tau(\kappa) = t(i\kappa)$ and $\tau_L(\kappa) = t_L^b(i\kappa)$ with t_L^b the transmission coefficient for f_L^b. Since $t_L^b \to t$ in the region of analyticity τ_L and its derivative converge to τ.

To analytically continue the relation (3.8), we note that for k real, $\overline{\psi(x;k)} = \psi(x; -k)$ so $\overline{t_{L/2}(k)} = t_{L/2}(-k)$ and thus (3.8) can be rewritten

$$\tilde{\Delta}(E) = t_{L/2}(k)^{-1} e^{-ikL} + t_{L/2}(-k)^{-1} e^{ikL}. \tag{3.9}$$

We, therefore, need to be able to analytically continue $t_{L/2}(k)^{-1}$ to the region $\operatorname{Im} k < 0$. If

$$\int_{-\infty}^{\infty} | f(x) | e^{2a|x|} < \infty \tag{3.10}$$

then [4] [19], t and $t_{L/2}$ can be continued to $\{ k \mid \operatorname{Im} k > - a \}$ and $t_{L/2}$ converges to t in that region. We clearly have by analytically continuing (3.9):

LEMMA 3.5. — Let (3.10) hold. Then in the region $E > - a^2$, we have

$$\tilde{\Delta}_L(-\kappa^2) = \tau_{L/2}(\kappa)^{-1}e^{\kappa L} + \tau_{L/2}(-\kappa)^{-1}e^{-\kappa L}. \tag{3.11}$$

From this we obtain

THEOREM 3.6. — Let f obey (3.10) for an a with $-a^2 < \inf \operatorname{spec}\left(\frac{-d^2}{dx^2}+f\right)$. Let the negative eigenvalues of $\frac{-d^2}{dx^2} + f$ be $-\kappa_1^2, \ldots, -\kappa_l^2$. Fix $\varepsilon > 0$. Then for L large, the only spectrum that $\frac{-d^2}{dx^2} + V_L(x)$ has in $(-\infty, -\varepsilon)$ is l bands $B_1, \ldots, B_l$ whose centers converge to $-\kappa_j^2$ and whose widths obey

$$\lim_{L\to\infty} e^{\kappa_j L} | B_j | = 8\kappa_j \left[\left| \frac{d\tau^{-1}}{d\kappa} \right| (\kappa_j) \right]^{-1}.$$

Proof. — Define $\kappa_j^{(L)}$ by $\tau_{L/2}(\kappa_j^{(L)})^{-1} = 0$, with this the zero near κ_j. Then $\kappa_j^{(L)} \to \kappa_j$. The edges of the bands are the determined by $\tau_{L/2}(\kappa_j^{(L)} \pm \Delta^{\pm}\kappa_j^{(L)}) = \pm 2$. Because the $e^{-\kappa L}$ term in (3.11) is exponentially small and the first term is exponentially large, it is easy to see that

$$| \Delta^{\pm}\kappa_j^{(L)} | e^{\kappa_j^{(L)}L} \frac{d\tau_{\frac{1}{2}L}^{-1}(\kappa_j^{(L)})}{d\kappa} = 2 + 0((\Delta^{\pm}\kappa_j^{(L)})^2).$$

Thus, since

$$| (\kappa_j^{(L)} + \Delta^{+}\kappa_j^{(L)})^2 - (\kappa_j^{(L)} - \Delta^{-}\kappa_j^{(L)})^2 | = 2\kappa_j^{(L)}(\Delta^{+}\kappa_j^{(L)} - \Delta^{-}\kappa_j^{(L)}) + 0((\Delta\kappa_j)^2)$$

we find that

$$| B_j | = 2(2)(2)\kappa_j^{(L)}e^{-\kappa_j^{(L)}L} \left[\left| \frac{d\tau_{\frac{1}{2}L}^{-1}}{d\kappa} \right| (\kappa_j^{(L)}) \right]^{-1}$$

which yield the required result, if we note that $\kappa_j^{(L)} - \kappa_j = 0(e^{-cL})$ since f decays exponentially. ■

We will also need to analyse the following situation: let W_0 be a periodic potential with period 1, and let V_L be given by (3.2). We want to know about gaps for the potential $W_0 + V_L$ with L an integer and large. This can be studied by a straight-forward extension of our analysis above using the scattering theory for $W_0 + f$ [*18*].

We begin by recalling the eigenfunction expansion $-d^2/dx^2 + W_0$ [*20*]. There exist analytic functions $\varepsilon_n(k)$ on $[0, \pi]$ with $\operatorname{Ran} \varepsilon_n \restriction (0, \pi)$ disjoint for distinct n and for each $k \in (0, \pi)$ and n, a function $\psi_0^{(n)}(x; k)$ on $(-\infty, \infty)$ obeying

$$\left(\frac{-d^2}{dx^2} + W(x)\right)\psi_0^{(n)}(x; k) = \varepsilon_n(k)\psi_0^{(n)}(x; k) \tag{3.11}$$

$$\psi_0^{(n)}(x+1; k) = e^{ik}\psi_0^{(n)}(x; k). \tag{3.12}$$

Since $k \in (0, \pi)$, $\psi_0^{(n)}$ must be complex valued, so $\overline{\psi_0^{(n)}}$ is linearly independent. We set $\overline{\psi_0^{(n)}(x; k)} \equiv \psi_0^{(n)}(x; -k)$. There is a generalized eigenfunction expansion for $\dfrac{-d^2}{dx^2} + W(x)$ using the $\{ \psi_0^{(n)}(x; k) \}_{n=1,\ldots,k\in\pm(0,\pi)}$. Define

$$G(x, y; k) = W(\psi_0^{(n)}(k), \psi_0^{(n)}(-k))^{-1} \{ \psi_0^{(n)}(x; k)\psi_0^{(n)}(y; -k) - \psi_0^{(n)}(y, k)\psi_0^{(n)}(x; -k) \} \tag{3.12}$$

Then, given f obeying (3.1) one can solve the Volterra integral equation

$$\psi^{(n)}(x; \lambda, k) = \psi_0^{(n)}(x; k) + \lambda \int_x^\infty G(x, y; k) f(y) \psi^{(n)}(y; \lambda, k) dy$$

and define $r^{(n)}(\lambda, k)$, $t^{(n)}(\lambda, k)$ by

$$\psi^{(n)}(x; \lambda, k) \underset{x \to -\infty}{\simeq} t^{-1}\psi_0^{(n)}(x; k) + rt^{-1}\psi_0^{(n)}(x; -k). \tag{3.13}$$

In just the way that Theorems 3.1 and 3.4 were proven, we obtain:

THEOREM 3.7. — Let W_0 be a periodic potential of period 1, let f obey (3.1) and let $t^{(n)}(k, \lambda)$, $r^{(n)}(k, \lambda)$ be the transmission and reflection coefficients for $\lambda \in (-\infty, \infty)$, $k \in (0, \infty)$ and let $t_R^{(n)}$, $r_R^{(n)}$ be the corresponding objects for f_R given by (3.6). Then:

i) t, r are jointly real analytic in λ and continuous in κ.

ii) For n, k, λ fixed, $|r_R^{(n)}(\lambda, k) - r^{(n)}(\lambda, k)| + |t_R^{(n)}(\lambda, k) - t^{(n)}(\lambda, k)| \to 0$ with convergence uniform in λ, k as λ, k run through compact subsets of $R \times (0, \infty)$.

iii) $$r^{(n)}(\lambda, k) = \lambda W(\psi_0^{(n)}(k), \psi_0^{(n)}(-k))^{-1} \int_{-\infty}^{\infty} \psi_0^{(n)}(x; k)^2 f(x) dx + 0(\lambda^2). \tag{3.14}$$

THEOREM 3.8. — Fix a function f obeying (3.1) and a periodic function W_0 with period 1. Let V_L be given by (3.2) with L integral, suppose that $E_0 = \varepsilon_n(k_0)$ $\left(\varepsilon_n \text{ associated to } \dfrac{-d^2}{dx^2} + W_0(x)\right)$ for some $k_0 \in (0, \pi)$ with

k_0/π non-rational. Suppose that the reflection coefficient $r^{(n)}(k_0)$ for f $\left(\text{relative to } \frac{-d^2}{dx^2} + W_0(x)\right)$ is non-zero. Then there exists $\alpha > 0$ and an infinite sequence of integers $L_l \to W$ so that $\frac{-d^2}{dx^2} + W_0 + V_{L_l}$ has a gap in its spectrum including the interval $(E_0 - \alpha L_l^{-1}, E_0 + \alpha L_l^{-1})$.

4. ABSENCE OF σ_{ac}

We will analyze when $\sigma_{ac} = \phi$ for Examples 1 and 2 of Section 2. As in the second example following Theorem 2.3, one can treat potentials of the type (1.7) for much more general $\{ x_n(\omega) \}$ than the Poisson process. In addition, one can mix the two examples and discuss $\sum_n q_n(\omega) f(x - x_n(\omega))$. We will leave these applications to the reader. One can also discuss Example 3 by extending the analysis of Section 3 to the discrete case; we will not give details.

We begin, with the Herbst-Howland example:

THEOREM 4.1. — Let f be a function so that the reflection coefficient, $r(k)$, for $\frac{-d^2}{dx^2} + f(x)$ is non-zero except on a set, M, of measure zero. Then the potential V_ω of (1.7) with x_n the Poisson process has $\sigma_{ac}(H_\omega) = \phi$.

Proof. — As in the proof of theorem 2.2, any V of the form $\Sigma f(x - x_n)$ with x_n obeying $\# \{ n \| x_n | \leq L \} \leq CL$ lies in supp μ. Taking $\{ x_n \}$ to be the empty set, we see that $0 \in \text{supp}\, \mu$ so $\sigma_{ac} \cap (-\infty, 0) = \phi$. Thus, we need only show that $(0, \infty) \cap \sigma_{ac} = \phi$. Let $E_0 = k_0^2 > 0$ with $r(k_0) \neq 0$. By Theorem 3.4 for some L, $E_0 \notin \sigma_{ac}\left(\frac{-d^2}{dx^2} + \sum_n f(x - nL)\right)$ so $E_0 \notin \sigma_{ac}(H_\omega)$ by Theorem 2.1. Thus $\sigma_{ac} \subset \{ 0 \} \cup \{ k^2 \mid r(k) = 0 \}$. Since this set has Lebesgue measure zero $\sigma_{ac} = \phi$. ■

If V decays exponentially, $r(k)$ is real analytic and thus, either identically zero or $\{ k \mid r(k) = 0 \}$ is discrete. Potentials with $r(k) \equiv 0$ are called *reflectionless* and they can all be written down explicitly (if V is reflectionless and has l negative eigenvalues, then [5] V lies in a $2l$ dimensional manifold of potentials). In particular, if V decays faster than any exponential, it cannot be reflectionless. Thus:

COROLLARY 4.2. — If f decays exponentially, either the potential V_ω of (1.7) obeys $\sigma_{ac}(H_\omega) = \phi$ or else f is reflectionless.

COROLLARY 4.3. — If f decays faster than exponentially, the potential V_ω of (1.7) obeys $\sigma_{ac}(H_\omega) = \phi$.

Remark. — If f is reflectionless, we still expect that $\sigma_{ac}(H_\omega) = \phi$ but our proof does not work. Presumably, there are still gaps whose size decreases faster than $0(L^{-1})$.

Now we turn to the Kirsch-Martinelli class of examples. As a warmup, it is easier to treat the case where 0 is a non-isolated point of supp $q_0(\omega)$. We will suppose that f has the property that its Fourier transform, $\hat{f}$, is a. e. non-vanishing. It is possible that one could weaken this by going to higher order perturbation theory. In particular if f decays exponentially, then $\hat{f}$ is analytic and so non-vanishing a. e.

THEOREM 4.4. — Let f obey (3.1) and suppose that $\{ k \mid \hat{f}(k) = 0 \}$ has Lebesgue measure zero. Suppose that $q_n(\omega)$ are i. i. d's with the property that 0 is a non-isolated point in supp $q_0(\omega)$. Let V_ω be given by (1.8). Then $\sigma_{ac}(H_\omega) = \phi$.

Proof. — It is easy to see that if q_n is any bounded sequence of numbers in supp $q_0(\omega)$, then $\sum_n q_n f(x-n)$ lies in supp μ. Thus $0 \in$ supp μ so $\sigma_{ac} \subset (0, \infty)$. Thus, if $\lambda \in$ supp $q_0(\omega)$ (since zero lies in supp q_0 by hypothesis) for any integer L,

$$\lambda \sum_n f(x - nL)$$

lies in supp μ. By hypothesis, λ can be taken arbitrarily small and by Theorem 3.1 (*iii*), if $\hat{f}(2k) \neq 0$, then $r(\lambda, k) \neq 0$ for λ small. It follows that if $\hat{f}(2k) \neq 0$ and k/π is irrational, then k^2 lies in a gap for some V in supp μ and thus $\sigma_{ac}(H_\omega) \subset \{ k \mid \hat{f}(2k) = 0 \} \cup \{ k \mid k\pi^{-1} \text{ is rational} \}$. Since this set has measure zero, $\sigma_{ac}(H_\omega) = \phi$. ■

The general case is slightly more involved:

THEOREM 4.5. — Let f obey (3.1) and suppose that $\{ k \mid \hat{f}(k) = 0 \}$ has Lebesgue measure zero. Suppose that $q_n(\omega)$ are i. i. d's with distribution $d\nu$, and that $d\nu$ has a continuous component. Let V_ω be given by (1.8). Then $\sigma_{ac}(H_\omega) = \phi$.

Proof. — As above, if $q_n \in \operatorname{supp}(d\nu)$, then $\Sigma q_n f(x-n) \in \operatorname{supp} \mu$. Suppose that $\alpha, \beta \in \operatorname{supp}(d\nu)$ and set $\alpha - \beta = \lambda$. Let

$$W_0^{(\beta)} = \beta \sum_n f(x-n)$$

$$V_L^{(\lambda)} = \lambda \Sigma f(x - nL).$$

Then $W_0^{(\beta)} + V_L^{(\lambda)}$ is a periodic potential in supp μ. It follows, by the arguments used above that we need only find for a. e. E_0, either a β with $E_0 \notin \sigma\left(\frac{-d^2}{dx^2} + W_0^{(\beta)}\right)$ or some λ, β, k, n such that $E_0 = \varepsilon_n(k)$ for $W_0^{(\beta)}$ and the reflection coefficient for λf relative to $\frac{-d^2}{dx^2} + W_0^{(\beta)}$ is non-zero.

Now, if $\beta \in \operatorname{supp}(d\nu_{\text{cont.}})$, then λ can be taken arbitrarily small, so by (3.14), we need only show that there is a $\beta \in \operatorname{supp}(d\nu_{\text{cont.}})$ so that for all n and a. e. k

$$\tilde{r}^{(n)}(k, \beta) = W(\psi_0^{(n)}(\beta, k), \psi_0^{(n)}(\beta, k))^{-1} \int_{-\infty}^{\infty} \psi_0^{(n)}(x; k, \beta)^2 f(x) dx \quad (4.1)$$

is non-zero. In (4.1) we have made the β dependence of $\psi_0^{(n)}$ explicit. Perturbation theory on the fixed k Hamiltonian [*20*] easily shows that for k fixed in $(0, \pi)$, $\tilde{r}^{(n)}(k, \beta)$ is analytic in β and at $\beta = 0$ it is a multiple of $\hat{f}(2k + 2n\pi)$. Thus, by hypothesis for a. e. k, $\tilde{r}^{(n)}(k, \beta) \neq 0$ for β near zero and so by analyticity for a. e. k, $\tilde{r}^{(n)}(k, \beta) \neq 0$ except for a discrete set of β. Thus, by Fubini's theorem, for a. e. $\beta\, d\nu_{\text{con.}}$, $\tilde{r}^{(n)}(k, \beta) \neq 0$ for a. e. k. So we fix $\beta_0 \in \operatorname{supp} d\nu_{\text{cont.}}$ with $\tilde{r}^{(n)}(k, \beta_0) \neq 0$ for a. e. k, n. Now, by

$$\Delta(\varepsilon_n(k)) = 2 \cos k$$

one can see that $\frac{d\varepsilon}{dk} \neq 0$ on $(0, \pi)$ and thus, $\{ E \mid E = \varepsilon_n(k, \beta_0), \tilde{r}^{(n)}(k, \beta_0) = 0 \}$ has Lebesgue measure zero. By the argument in the last paragraph, this proves the theorem. ■

Remark. — It is easy to see that the same proof applies when $d\nu_{\text{cont.}} = 0$, so long as the set of accumulation points of supp ν has an accumulation point.

APPENDIX 1

GENERICITY OF DETERMINISM

Fix a measure $d\mu$ with bounded support and let $q_n(\omega)$ be i. i. d's with distribution $d\nu$. Fix $f \in l_1(\mathrm{L}^2)$ and

$$\mathrm{V}_\omega(x) = \sum_{n=-\infty}^{\infty} q_n(\omega) f(x-n).$$

Our goal here is to prove:

THEOREM A.1.1. — There is a dense G_δ, $\mathrm{S} \subset l_1(\mathrm{L}^2)$ so that V_ω is deterministic if $f \in \mathrm{S}$.

Proof. — An element f in $l_1(\mathrm{L}^2)$ is equivalent to a sequence of functions $f_n \in \mathrm{L}^2(0, 1)$ so that

$$|||\{ f_n \}||| = \Sigma \| f_n \|_2 < \infty \tag{A.1.1}$$

and

$$f(x) = \sum_n f_n(x+n). \tag{A.1.2}$$

Thus, if $x \in (0, 1)$ and $l \in \mathrm{Z}$:

$$\begin{aligned} \mathrm{V}_\omega(l+x) &= \sum_{-\infty}^{\infty} q_n(\omega) f(x+l-n) \\ &= \sum_{-\infty}^{\infty} q_{n-l}(\omega) f(x-n) \\ &= \sum_{-\infty}^{\infty} q_{n-l}(\omega) f_n(x). \end{aligned}$$

Thus, if $\operatorname{supp}(d\nu) \subset [-\mathrm{L}, \mathrm{L}]$, it suffices to find S so that if $\{ f_n \} \in \mathrm{S}$ and $|q_n| \leq 2\mathrm{L}$ and if $\Sigma q_n f_n \doteq 0$ in $\mathrm{L}^2(0, 1)$, then $q_n \equiv 0$. For this says that $\mathrm{V}_\omega(l+x)$ with $x \in [0, 1]$ uniquely determines the q_n's and so V_ω for all x.

Let

$$\mathrm{S}_{n,m} = \left\{ \{ f_j \} \mid \operatorname{dist}\left(f_n, \left\{ \sum_{j \neq n} q_j f_j \mid |q_j| \leq m \right\} \right) > 0 \right\}$$

where dist. is in the $\mathrm{L}^2(0, 1)$ norm. This set is open, since if $\{ f_j^{(0)} \} \subset \mathrm{S}_{n,m}$ and

$$|||\{ f_j - f_j^{(0)} \}||| \leq 1/2(m+1)^{-1} \operatorname{dist}(f_n^{(0)}, \{ \dots \}),$$

then $\{ f_j \} \subset \mathrm{S}_{n,m}$. Moreover, $\mathrm{S}_{n,m}$ is dense, for given any $\{ f_j \}$ and ε, we can find $f_j^{(1)}$ so that $||| f^{(1)} - f ||| < \frac{\varepsilon}{2}$ and so that only finitely many $f_j^{(1)}$ are non-zero. Then we can change $f_n^{(1)}$ by at most $\frac{\varepsilon}{2}$ so that $f_n^{(1)}$ does not lie in the finite dimensional subspace $\sum_{j \neq n} q_j f_j^{(1)}$.

Let $\mathrm{S} = \bigcap_n \bigcap_m \mathrm{S}_{n,m}$ which is a dense G_δ by the Baire category theorem. We claim that if $f \in S$ and if $\Sigma q_n f_n = 0$ with $|q_n| \leq 2L$, then all q_n are non-zero. For suppose some $q_n \neq 0$. Then

$$f_n = \sum_{j \neq n} q_n^{-1} q_j f_j$$

and $\tilde{q}_j = -q_n^{-1} q_j$ where $|\tilde{q}_j| \leq m$ for some m so $f \notin S_{n,m}$. Thus S has the required property. ■

While we gave this proof for $l^1(L^2)$, the same method clearly works for $l^1(L^1)$ or for $\{f \mid \|f\|_{(\varepsilon)} = \|(1 + x)^{1+\varepsilon} f\|_2 < \infty\}$ and should extend to Schwartz spaces and other classes of smooth potentials.

Remark. — The independence of the q_n is not needed anywhere in the above, i. e. if $f \in S$, $\Sigma q_n(\omega) f(x - n)$ is deterministic for any bounded sequence q_n of random variables.

APPENDIX 2

m-FUNCTIONS FOR UNBOUNDED POTENTIALS

In the basic theory of Kotani [13] which underlies the work in this paper and in its discrete analog [21] it was assumed that potentials are bounded for technical reasons. In this appendix we want to show how to extend this theory to certain classes of unbounded potentials. We will also show how to prove continuity of m in a suitable topology. All our results will be « deterministic », i. e. bounds on and control of m for a fixed potential; once one has this the theory of [13] [14] [21] will apply.

We begin with the discrete case which is technically easier. For a fix real-valued $V(n)$ with $|V(n)| < \infty$ for all $n \in Z$ and a fixed E with Im E > 0, we will show that

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n) \tag{A.2.1}$$

has a unique solution u_+ which is l_2 at $+\infty$ and a unique solution u_- l_2 at $-\infty$. m is defined by

$$m_\pm(E) = -u_\pm(\pm 1)/u_\pm(0). \tag{A.2.2}$$

The operators $H^\pm$ on $l^2(\pm 1, \pm\infty)$ with $u(0) = 0$ boundary conditions are related to m by

$$m_\pm(E) = (\delta_{\pm 1}, (H^\pm - E)^{-1}\delta_{\pm 1}). \tag{A.2.3}$$

THEOREM A.2.1. — Let V be a real-valued function on Z. Then

a) For Im E > 0, (A.2.1) has a unique solution, u_+, which is l_2 at $+\infty$ and $u_+(0) \neq 0$

b) m defined by (A.2.2) obeys (A.2.3)

c) $2(\text{Im E})(|E| + 1 + V(\pm 1)|)^{-2} \leq \text{Im}\, m_\pm(E) \leq |m_\pm(E)| \leq (\text{Im E})^{-1}$

d) If $V_k \to V$ pointwise, then $m_\pm^{(k)}(E) \to m_\pm(E)$ for all E with Im E > 0 uniformly on compact subsets.

Proof. — Since the off diagonal part of $H^\pm$ is bounded and any diagonal matrix is self-adjoint, $H^\pm$ define self-adjoint operators. As we will prove below,

$$\text{Im}\,(\delta_{\pm 1}, (H^\pm - E)^{-1}\delta_{\pm 1}) \geq 2(\text{Im E})(|E| + 1 + V(\pm 1)|)^{-2},$$

so $(H^\pm - E)^{-1}\delta_\pm \neq 0$ and thus it, continued from $[1, \infty) + 0\,(-\infty, \infty)$, yields a solution of (A.2.1) which is l_2 at infinity. Constancy of the Wronskian shows there cannot be more than one solution which is l_2 at $+\infty$. If $u_+(0) = 0$, then u_+ is an eigenfunction (violating self-adjointness), so $u_+(0) \neq 0$. That (A.2.3) holds is a simple argument using the explicit formula for $(H^\pm - E)^{-1}(i, j)$ built out of u_+ and the solution of (A.2.1) obeying $u(0) = 0$. If $V_k \to V$ pointwise, it is easy to see that $H_k^\pm \to H^\pm$ in strong resolvent sense since $H_k^\pm\varphi \to H^\pm\varphi$ for φ of finite support, which is a core of $H^\pm$ since it is a core for V. Thus by (A.2.3), the convergence in (*d*) holds. That leaves us with the proof of the bounds in (*c*). Since $\|(H^\pm - E)^{-1}\| \leq (\text{Im E})^{-1}$, the upper bound follows from (A.2.3). As for the lower bound, we note first that since

$$[(H^\pm - E)^{-1} - (H^\pm - \bar{E})^{-1}] = 2(\text{Im E})(H^\pm - E)^{-1}(H^\pm - \bar{E})^{-1}$$

we see that

$$\text{Im}\, m_\pm(E) \geq 2(\text{Im E})\,\|(H^\pm - E)^{-1}\delta_{\pm 1}\|^2 .$$

But for any operator, A

$$\|\varphi\|^2 = \langle A\varphi, A^{-1}\varphi\rangle \leq \|A\varphi\|\;\|A^{-1}\varphi\|$$

so

$$\|(H^\pm - E)^{-1}\delta_{\pm 1}\|^2 \geq \|(H^\pm - E)\delta_{\pm 1}\|^{-2} = \{ | V(\pm 1) - E|^2 + 1 \}^{-1}$$
$$\geq [| V(\pm 1)| + 1 + | E |]^{-2}.$$

Remark. — The key condition on random potentials $E(ln(1 + | V_\omega(0) |)) < \infty$ first allows us to apply the subadditive ergodic theorem to show $\gamma(E) < \infty$ and then by (*c*) above implies $E(| ln | m_+ \|)$ and $E(| ln(\mathrm{Im}\ m_+) |) < \infty$ which is needed for the proofs of [*21*].

We now turn to the continuum case. We begin by proving the following:

THEOREM A.2.2. — Let μ be an ergodic measure on $L^2_{loc}(R^\nu)$ so that $E_\mu(| V(0) |^2) < \infty$. Then for a. e. V, we have

$$\left(\int^{n+1} | V(x) |^2 dx\right)^{\frac{1}{2}} \leq C_{(V,\varepsilon)}(| n | + 1)^{\frac{1}{2}+\varepsilon} \qquad (A.2.4)$$

for all $\varepsilon > 0$.

Proof. — Let $g_n(V)$ denote the left side of (A.2.4). Then

$$E_\mu(\Sigma(n + 1)^{-1-\varepsilon} | g_n(V) |^2) < \infty$$

so, by Fubini's theorems $C_{(V),\varepsilon} \equiv [\Sigma(n + 1)^{-1+2\varepsilon} | g_n(V) |^2]^{\frac{1}{2}}$ is finite a. e. Clearly

$$g_n(V) \leq C_{(V,\varepsilon)}(| n | + 1)^{-\frac{1}{2}-\varepsilon}. \quad \blacksquare$$

The point of singling out $(| n | + 1)^{1-\delta}$ as we do in Section 2 is that one has the following corollary of a result of Kato [*24*]:

THEOREM A.2.3. — If V is an L^2_{loc} potential on $(-\infty, \infty)$ obeying

$$\left(\int_n^{n+1} | V_-(x) |^2 dx\right)^{\frac{1}{2}} \leq C(n^2 + 1)^{\frac{1}{2}-\delta} \qquad (A.2.5)$$

for some $\delta > 0$, then $\dfrac{-d^2}{dx^2} + V(x)$ is essentially self-adjoint on $C_0^\infty(-\infty, \infty)$.

REMARKS 1. — This result does not seem to appear explicitly in the literature, but we have been informed by I. Knowles and T. Read that it can be deduced from a number of distinct limit point criteria. Indeed, $1/2 - \delta$ can be replaced by $1 - \delta$. Self-adjointness when $E(V(0)^2) < \infty$ is discussed also in [*14*].

2. The argument is related to one in [*25*].

Proof. — Let $V_0(x) = -(1 + | x |^{2-\delta}$, $V_1(x) = V(x)$ (if $V(x) \geq V_0(x)$); $= 0$ (if $V(x) \leq V_0(x)$), $V_2(x) = V(x) - V_1(x)$. If we can prove that

i) $\displaystyle\int_{|x| \leq R} | V_2(x) |^2 dx \leq KR^{2s}$; $1 \leq R \leq \infty$, some $s > 0$

ii) $\displaystyle\lim_{\varepsilon\ 0} \left\{ \sup \left[\int_{|y-x| \leq \varepsilon} | V_2(y) | \, dy \right| \right\} = 0$

then the Theorem of Kato in [*24*] implies essential self-adjointness.

Since $| V_2 | \leq | V |$, (A.2.5) trivially implies (*i*). As for (*ii*), since $| V_2(x) | \geq (1 + | x |)^{2-\delta}$ on the set where it is non-zero we have that $| V_2(x) | \leq (1 + | x |)^{-(2-\delta)} | V_2(x) |^2$ so for $n \geq 0$

$$\int_n^{n+1} | V_2(x) | \, dx \leq C(1 + | n |)^{-2+\delta}(1 + n^2)^{1-2\delta} = 0(n^{-3\delta})$$

and for $n < 0$

$$\int_n^{n+1} | V_2(x) | \, dx \leq C | n |^{-2+\delta}(1 + n^2)^{1-2\delta} = 0(n^{-3\delta}).$$

Given this and (A.2.5), one easily sees that (*ii*) holds. ■

The theory of limit point-limit circle [*22*] [*23*] then implies that for any E with Im E > 0, there is a unique (up to constants) solution $u_\pm$ of

$$- u'' + Vu = Eu \tag{A.2.6}$$

which is L^2 at $\pm \infty$. It also implies that the operator $H^\pm$ on $L^2(0, \pm \infty)$ defined as the closure of $\frac{-d^2}{dx^2} + V$ on $\{ u \in C_0^\infty [0, \pm \infty) \mid u(0) = 0 \}$ is self-adjoint. Moreover, since $H^\pm$ has no non-real eigenvalues $u_\pm(0) \neq 0$, so

$$h^\pm(V, E, x) = \pm \frac{u'_\pm(x, \omega, E)}{u_\pm(x, \omega, E)} \tag{A.2.7}$$

is well defined. We will let $h_\pm(E) = h_\pm(V, E, x = 0)$.

By the Wronskian formula for $(H^\pm - E)^{-1}(x, y) \equiv G^\pm(x, y, E)$ one sees that

$$h_+(E) = \lim_{\substack{y > x > 0 \\ y \downarrow 0}} \frac{\partial^2}{\partial x \partial y} G^+(x, y; E). \tag{A.2.8}$$

Since we have already remarked on the analog, of Theorem A.2.1 (*a*), (*b*), we state only:

THEOREM A.2.4. — Let V, V_k obey (A.2.4). Then

a) There are universal constant C_1, C_2 so that for all E with Im E > 0 (and all V):

$$(\mathrm{Im}\, E) f(E)^{-1} \leq \mathrm{Im}\, h_+(E) \leq | h_+(E) | \leq f(E) \tag{A.2.9}$$

where

$$f(E) = C_1 + C_2 [(| E |^2 + 1)(1 + (\mathrm{Im}\, E)^{-1})] \left\{ 1 + \int_0^\infty e^{-2x} | V(x) |^2 dx \right\} \tag{A.2.10}$$

b) If $V_k \to V$ in L^2_{loc}, then $h^k_\pm(E) \to h_\pm(E)$ uniformly on compact subsets of the upper half-plane.

Proof. — *a*) Let $W = E - V + 1$ and $H_0^+ = \frac{-d^2}{dx^2}$ on $L^2(0, \infty)$ with $u(0) = 0$ boundary conditions. Then

$$(H^+ - E)^{-1} = (H_0^+ + 1)^{-1} + (H_0^+ + 1)^{-1} W (H_0^+ + 1)^{-1} + (H_0^+ + 1)^{-1} W (H^+ - E)^{-1} W (H_0^+ + 1)^{-1}.$$

From this and (A.2.8), we see that

$$h_+(E) = h_{+,0}(-1) + (g, Wg) + (g, W(H - E)^{-1} Wg)$$

where

$$g(y) = \lim_{x \downarrow 0} \frac{\partial}{\partial x} (H_0^+ + 1)^{-1}(x, y) = e^{-y}$$

and $h_{+,0}(E) = \sqrt{-E}$ is the h function for V = 0. Since

$$\| Wg \|_2^2 \leq C(1 + | E |) + \tilde{C} \int_0^\infty e^{-2x} | V(x) |^2 dx \qquad \text{and} \qquad \| (H - E)^{-1} \| \leq (\mathrm{Im}\, E)^{-1},$$

we see that $h_+(E) \leq f(E)$ for suitable c_1, c_2.

To control Im h_+, we note first that Im $(u'_+(x)\overline{u_+(x)})$ is up to a factor of $2i$ the Wronskian of u_+ and $\bar{u}_+$ so that

$$\frac{d}{dx}\,\mathrm{Im}\,(u'_+\bar{u}_+) = -\,|\,u_+(x)\,|^2(\mathrm{Im}\ \mathrm{E})$$

and thus since u_+ is L^2 at $+\infty$,

$$\mathrm{Im}\ u'_+(0)\overline{u_+(0)} = (\mathrm{Im}\ \mathrm{E})\int_0^\infty |\,u_+(x)\,|^2 dx$$

or

$$\mathrm{Im}\ h_+(\mathrm{E}) = \mathrm{Im}\ \mathrm{E}\int_0^\infty |\,\mathrm{R}_+(\mathrm{V}, \mathrm{E};\ x)\,|^2 dx \geq \mathrm{Im}\ \mathrm{E}\int_0^1 |\,\mathrm{R}_+(\mathrm{V}, \mathrm{E};\ x)\,|^2 dx \qquad (\mathrm{A}.2.11)$$

where

$$\mathrm{R}_+(\mathrm{V}, \mathrm{E};\ x) = u_+(x)/u_+(0)\,.$$

Next, note that

$$\frac{d}{dx}\,ln\mathrm{R}_+(\mathrm{V}, \mathrm{E};\ x) = h_+(\mathrm{V}, \mathrm{E};\ x)$$

and

$$|\,h_+(\mathrm{V}, \mathrm{E};\ x)\,| \leq e^2 f(\mathrm{E}) \qquad \text{for} \qquad x \in (0, 1)$$

by the upper bound we just proved $\left(\int_0^\infty e^{-2y}\,|\,\mathrm{V}(x+y)\,|^2 \leq e^{2x}\int_0^\infty e^{-2y}\,|\,\mathrm{V}(y)\,|^2 dy \text{ if } x \geq 0\right)$.
Thus, since $ln\mathrm{R}_+(x=0) = 0$, for $0 < x < 1$

$$\mathrm{R}_+(\mathrm{V}, \mathrm{E};\ x) \geq \exp(-\,xe^2 f(\mathrm{E}))\,.$$

Since $f(\mathrm{E}) \geq \mathrm{C}_1$, we have then that

$$\int_0^1 |\,\mathrm{R}_+(\mathrm{V}, \mathrm{E};\ x)\,|^2 dx \geq \mathrm{C}_3 f(\mathrm{E})^{-1}$$

which yields (A.2.9) if we redefine C_1, C_2.

b) Fix V and E. By finding $f \in \mathrm{C}_0^\infty(-\infty, -1)$ so that $\int f u_-(\mathrm{V}, \mathrm{E}, x)dx \neq 0$ we can find f with $(\mathrm{H}-\mathrm{E})^{-1}f \neq 0$ near $+\infty$ so that one may choose $u_+(\mathrm{V}, \mathrm{E};\ x) \equiv ((\mathrm{H}-\mathrm{E})^{-1}f)(x)$. Since $\mathrm{V}\varphi - \mathrm{V}_k\varphi \to 0$ for $\varphi \in \mathrm{C}_0^\infty$, a core for H, we have that

$$u_+(\mathrm{V}_k, \mathrm{E};\ x) \equiv ((\mathrm{H}_k - \mathrm{E})^{-1}f)(x)$$

converges in $\mathrm{L}^2(-1, \infty)$ to $u_+(\mathrm{V}, \mathrm{E};\ x)$. By the Schrödinger equation, $-\,u''_{+,k} \to -\,u''_+$ in $\mathrm{L}^1_{\mathrm{loc}}$. Standard analysis shows that convergence of u and u'' in $\mathrm{L}^1_{\mathrm{loc}}$ implies pointwise convergence of u and u' so $h = u'/u$ converges. Pointwise (in E) convergence plus analyticity and the bounds in (*a*) imply convergence uniform in E on compact subsets of Im E > 0.

The bounds (A.2.9) and the hypothesis $\mathrm{E}(\,|\,\mathrm{V}(0)\,|^2) < \infty$ imply that $\mathrm{E}(\,|\,h_+\,|\,)$ and $\mathrm{E}((\mathrm{Im}\ h_+)^{-1})$ are finite and given this, the theory of [*13*] [*21*] goes through. ■

REFERENCES

[*1*] J. Avron and B. Simon, Almost periodic Schrödinger operators, II. The integrated density of states, *Duke Math. J.*, t. **50**, 1983, p. 369-391.

[*2*] J. Bellissard, R. Lima and D. Testard, A metal-insulator transition for the almost Mathieu model, *Commun. Math. Phys.*, t. **88**, 1983, p. 207-234.

[*3*] W. Craig and B. Simon, Subharmonicity of the Lyaponov index, *Duke Math. J.*, t. **50**, 1983, p. 551-560.

[4] V. DE ALFARO and T. REGGE, *Potential Scattering*, North Holland, Amsterdam, 1965.

[5] P. DEIFT and E. TRUBOWITZ, Inverse Scattering on the Line, *Comm. Pure Appl. Math.*, t. 32, 1979, p. 121-251.

[6] E. DINABURG and Y. SINAI, On the one dimensional Schrödinger equation with quasiperiodic potential, *Funk Anal. i Pril.*, t. **9**, 1975, p. 8-21.

[7] I. GOLDSHEID, S. MOLCHANOV and L. PASTUR, A pure point spectrum of the stochastic and one dimensional Schrödinger equation, *Funct. Anal. Appl.*, t. **11**, 1977, p. 1-10.

[8] I. HERBST and J. HOWLAND, The Stark ladder and other one-dimensional external field problems, *Commun. Math. Phys.*, t. **80**, 1981, p. 23.

[9] J. KELLER, Discriminant, transmission coefficients and stability bands of Hill's equation, *J. Math. Phys.*, to appear.

[10] W. KIRSCH, On a class of random Schrödinger operators, to appear in *Adv. Appl. Math.*

[11] W. KIRSCH, F. MARTINELLI, On the spectrum of Schrödinger operators with a random potential: *Commun. Math. Phys.*, t. **85**, 1982, p. 329-350.

[12] W. KIRSCH, F. MARTINELLI, On the ergodic properties of the spectrum of general random operators: *J. Reine Angew. Math.*, t. **334**, 1982, p. 141-156.

[13] S. KOTANI, *Lyaponov Indices Determine Absolutely Continuous Spectra of Stationary Random One-Dimensional Schrödinger Operators*, *Proc. Stoch. Anal.*, Kyoto, 1982.

[14] S. KOTANI, Support Theorems for Random Schrödinger Operators, *Commun. Math. Phys.*, to appear.

[15] H. KUNZ and B. SOUILLARD, Sur le spectre des opérateurs aux différences finies aléatoires, *Commun. Math. Phys.*, t. **78**, 1980, p. 201-246.

[16] W. MAGNUS and S. WINKLER, *Hill's Equation*, Interscience, 1966; Dover Edition available.

[17] N. MOTT and W. TWOSE, The theory of impurity conduction, *Adv. in Physics*, t. **10**, 1961, p. 107-155.

[18] R. NEWTON, Inverse scattering by a local impurity in a periodic potential in one dimension. *J. Math. Phys.*, t. **24**, 1983, p. 2152.

[19] M. REED and B. SIMON, *Methods of modern mathematical physics. III. Scattering theory*, Academic Press, New York, 1979.

[20] M. REED and B. SIMON, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press, New York, 1978.

[21] B. SIMON, Kotani theory for one dimensional stochastic Jacobi matrices. *Comm. Math. Phys.*, t. **89**, 1983, p. 227.

[22] E. CODDINGTON and N. LEVINSON, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[23] N. DUNFORD and J. SCHWARTZ, *Linear Operators*, Vol. II, Wiley, New York, 1963.

[24] T. KATO, Schrödinger operators with singular potentials. *Israel J. Math.*, t. **13**, 1973, p. 135-148.

[25] W. KIRSCH and F. MARTINELLI, On the essential selfadjointness of stochastic Schrödinger operators, *Duke Math. J.*, t. **50**, 1983, p. 1255-1260.

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