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Stability of gaps for periodic potentials under variation of a magnetic field

J Avron† and B Simon‡§

† Department of Physics, Technion, Haifa, Israel
‡ Division of Physics, Mathematics and Astronomy, California Institute of Technology, Pasadena, CA 91125, USA

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Abstract. Let $H_0(B)$ denote the Hamiltonian of a free electron in a magnetic field B. Let V be a periodic potential. We show that if an interval $[a, b]$ is not in the spectrum of $H_0(B_0)+V$ for some B_0, then it is not in the spectrum for all B sufficiently close to B_0.

1. Introduction

Let us consider a two-dimensional electron in a magnetic field B and periodic potential $V(x, y)$. In Landau gauge

$$H(B) \equiv H_0(B)+V \qquad H_0(B)=\left(\frac{1}{\mathrm{i}}\frac{\partial}{\partial x}+By\right)^2+\left(\frac{1}{\mathrm{i}}\frac{\partial}{\partial y}\right)^2. \tag{1.1}$$

Because of the growth of the B^2y^2 term at infinity, the B dependence of $H(B)$ is rather singular. For this reason, the physically reasonable fact that gaps of $H(B)$ are stable under small perturbations of B is not mathematically trivial. It is this fact that we wish to prove in this paper. We were motivated by work of Dana *et al* (1984) who needed to know this stability during an argument. This stability is related to, but is nevertheless quite distinct from, the continuity of the integrated density of states which one of us proved (Simon 1982a). Mathematically, our result here is much more subtle than that of Simon.

To understand how we will attack the problem, fix B_0 and, given B near B_0, write $B=\lambda B_0$ with λ near 1. Consider the unitary scaling $(U_\lambda f)(x)=\lambda^{-1/2}f(\lambda^{-1/2}x)$. Then

$$U_\lambda H(B)U_\lambda^{-1}=\lambda^{-1}(H_0(B_0)+\lambda V_\lambda)$$

where

$$V_\lambda(x)=V(\lambda^{-1/2}x).$$

Thus, the stability of gaps for $H(B)$ in B is equivalent to stability with B fixed for small changes in the period of V (the stability over the overall change of energy scale is trivial). Since it is well known that a magnetic field has a natural period associated with it (given by the area associated with one quantum of magnetic flux), this is

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analogous to the question of stability of gaps for an almost periodic potential over changes of the almost periods.

For the *one-dimensional* case, the stability of the gaps in almost periods has been proven by Elliott (1982) and by us (Avron and Simon 1983). Our proof relied heavily on a theorem of Johnson (1982), who (exploiting the stability theory of certain ODE due to Sacher and Sell (1974, 1976)) formalised an intrinsically one-dimensional proof. However, at the same time, we (see Avron and Simon 1983) gave a sketch of a proof of Johnson's theorem which, while one dimensional, can be extended to higher dimensions. After some preliminary results in § 2, we prove in § 3 the higher-dimensional analogue of Johnson's theorem in cases of both almost periodic and magnetic fields. In § 4 we use our argument (Avron and Simon 1983) to prove results on stability of gaps for both cases.

The basic reason for the stability of gaps is that, if they were otherwise, eigenfunctions would have to disappear at spatial infinity as B changed. This, however, cannot happen because the periodicity of the potentials implies a kind of compactness which prevents a disaster at spatial infinity.

We will discuss the case of a magnetic field in two dimensions, although our results easily extend to higher dimensions. While we will suppose all our potentials are bounded, it suffices that they lie in the class K_ν discussed by Simon (1982b).

2. Some local regularity results on eigenfunctions

Here we will need to recall (and prove for the magnetic field case) some estimates on eigenfunctions discussed by Simon (1982b). While the results are proven there for a class of unbounded potentials K_ν, we state them here for bounded potentials for simplicity.

Theorem 2.1. Suppose that $V \in L^\infty(R^\nu)$ and that

$$(-\Delta + V)u = Eu$$

(u may not be in L^2; only a local distributional solution is assumed); then u is continuous and

$$|u(x)| \leq C \int_{|y-x|\leq 1} |u(y)|\, \mathrm{d}^\nu y \tag{2.1}$$

where C only depends on E and $\|V\|_\infty$.

This is a special case of theorem 6.1 of Aizenman and Simon (1980); it is proven by showing regularity properties of the Poisson kernel for $-\Delta + V - E$. The proof of this result uses a path integral representation of the Poisson kernel. That representation and the Feynman–Kac–Ito formula for path integrals in a magnetic field (see § 15 of Simon (1979)) imply that the Poisson kernel in a magnetic field is bounded by the Poisson kernel without the field; so the method of proof of theorem 2.1 therefore implies:

Theorem 2.2. Suppose that $V \in L^\infty(R^\nu)$, that $a \in C^1(R^\nu)$ and that (in a distributional sense)

$$[(\mathrm{i}\nabla + a)^2 + V]u = Eu. \tag{2.2}$$

Then u is continuous and

$$|u(x)| \leq C \int_{|y-x|\leq 1} |u(y)| \, d^\nu y$$

where C only depends on E and $\|V\|_\infty$.

Note. C is independent of a; a need not be bounded.

Simon (1982b, C2) proved L^2_{loc} estimates on gradients of solutions of Schrödinger's equation. We use similar ideas to prove such an estimate in the case of a magnetic field.

Theorem 2.3. Suppose that $V \in L^\infty$, that $a \in L^2_{\text{loc}}(R^\nu)$ and that equation (2.2) holds. Then

$$\int_{|y-x|\leq 1} |(\nabla - \mathrm{i}a)u|^2 \, d^\nu y \leq C \int_{|y-x|\leq 2} |u(y)|^2 \, d^\nu y \tag{2.3}$$

where C only depends on $\|V - E\|_\infty$.

Remark. Actually, only the *local* K_ν norm of V need enter in C.

Proof. Without loss, suppose $x = 0$. Pick a smooth function $\eta(x)$ which is 0 if $|x| \geq 2$ and 1 if $|x| \leq 1$. Let $\pi = \nabla - \mathrm{i}a$. Then, integrating by parts and using equation (2.2),

$$\begin{aligned}\int \eta \overline{(\pi u)}(\pi u) &= -\int \eta \bar{u} \pi^2 u - \int (\nabla \eta)(\bar{u} \pi u) \\ &= \int \eta (E - V)|u|^2 - \int (\nabla \eta)(\bar{u} \pi u).\end{aligned}$$

Since the terms on the left-hand side and the first term on the right are real, we can replace $\bar{u}\pi u$ by $\text{Re}(\bar{u}\pi u) = \frac{1}{2}\nabla|u|^2$. Integrating by parts again

$$\int \eta |\pi u|^2 = \int (\eta(E - V) + \tfrac{1}{2}\Delta \eta)|u|^2$$

from which (2.3) follows.

In the next two sections, we will need a compactness result. It is well known; we provide a proof for the reader's convenience.

Theorem 2.4. Let u_n be a sequence of functions on R^ν so that for any $C < \infty$ we have that

$$\sup_n \int_{|x|<C} (|\nabla u_n|^2 + |u_n|^2) \, d^\nu x < \infty. \tag{2.4}$$

Then we can find a subsequence $u_{n(i)}$ and a function $u_\infty \in L^2_{\text{loc}}$ so that, for any C

$$\lim_{i\to\infty} \int_{|x|<C} |u_{n(i)} - u_\infty|^2 \, d^\nu x = 0. \tag{2.5}$$

Proof. By a standard subsequence argument (see § I.5 of Reed and Simon (1972)) it suffices to prove (2.5) for C fixed. Pick $\eta \in C_0^\infty$ so that $\eta(x)=1$ if $|x| \leqslant C$. Let $\phi_n(x)=\eta(x)u_n(x)$. Since $\nabla\phi_n = \eta\nabla u_n + (\nabla\eta)u_n$ and η, $\nabla\eta$ have compact support, (2.4) implies that

$$\sup_n \int (|\nabla\phi_n|^2 + x^2|\phi_n(x)|^2)\, \mathrm{d}^\nu x < \infty.$$

The compactness of the resolvent of the harmonic oscillator implies that ϕ_n lies in a compact subset of $L^2(R^\nu)$, so we can pick ϕ_∞ so that $\int |\phi_n - \phi_\infty|^2\, \mathrm{d}^\nu x = 0$. Set $u_\infty(x) = \phi_\infty(x)$ if $|x| < C$.

3. Multidimensional version of Johnson's theorem

We will first prove a result in the case of a magnetic field.

Theorem 3.1. Let V be periodic on R^2 (say $V(x_1+n_1a_1, x_2+n_2a_2) = V(x_1, x_2)$, when $n_i = 0, \pm1, \pm\ldots$) and let $E \in \mathrm{spec}(H(B))$ (H is given by equation (1)). There then exists a solution u of (2.2) obeying

$$\sup_{n_i} \int_{\substack{|x_1-n_1a_1|<\frac{1}{2}a_1 \\ |y_2-n_2a_2|<\frac{1}{2}a_2}} |u(x)|^2\, \mathrm{d}^2x \leqslant 1 \qquad (3.1a)$$

$$\int_{\substack{|x_1|<\frac{1}{2}a_1 \\ |y_2|<\frac{1}{2}a_2}} |u(x)|^2\, \mathrm{d}^2x \geqslant \tfrac{1}{2}. \qquad (3.1b)$$

In particular, u is bounded. Conversely, if (2.2) has a bounded eigenfunction, then $E \in \mathrm{spec}(H(B))$.

Lemma 3.2. If (2.2) has a bounded solution, then $E \in \mathrm{spec}(H(B))$.

Proof. Given theorem 2.3, and using $\pi^2(\eta u) = \eta\pi^2 u + 2\nabla\eta\cdot\pi u + u\Delta\eta$, this is essentially a result of Sch'nol (1957), who dealt with the case where there is no magnetic field (see § C4 of Simon (1982b)).

Lemma 3.3. Given n_1, $n_2 = 0, \pm1, \ldots$, let

$$(U_{\boldsymbol{n}}f)(x) = \exp(\mathrm{i}n_2a_2xB)f(x+n_1a_1, y+n_2a_2).$$

Then $U_{\boldsymbol{n}}$ commutes with $H(B)$.

Proof. This is a straightforward and well known calculation.

Lemma 3.4. Theorem 3.1 holds if $(2\pi)^{-1}Ba_1a_2$ is rational.

Proof. In that case, $H(B)$ can be analysed by a Bloch wave analysis with a unit cell of size a_1q by a_2 (if $(2\pi)^{-1}Ba_1a_2 = p/q$) (see, for example, Zak 1964). Thus, (2.2) has solutions u with $|u(x_1+a_1qn_1, x_2+a_2n_2)|$ periodic, so (3.1a) certainly holds. Among the q $a_1\times a_2$ cells in the basic $a_1q\times a_2$ cells, there is one, C_α, where $\int|u|^2$ is largest. Let $u' = U_{\boldsymbol{n}}u$ where $\boldsymbol{n}$ is chosen to translate C_α to the cell about the origin.

Lemma 3.5. If $B_n \to B$ and $\phi \in L^2$, then $\|(H(B_n)+\mathrm{i})^{-1}\phi - (H(B)+\mathrm{i})^{-1}\phi\| \to 0$.

Proof. This need only be proven for a dense set of ϕ. By the essential self-adjointness of $H(B)$ on C_0^∞ (see, for example, Ikebe and Kato 1962), the set of $\phi = (H(B)+\mathrm{i})g$ with $g \in C_0^\infty$ is dense. For such a ϕ:

$$(H(B_n)+\mathrm{i})^{-1}\phi - (H(B)+\mathrm{i})^{-1}\phi = (H(B_n)+\mathrm{i})^{-1}(H(B)-H(B_n))g$$

which is easily seen to converge to zero in norm.

Lemma 3.6. If $B_n \to B$ and $E \in \mathrm{spec}(H(B))$, then there exists $E_n \in \mathrm{spec}(H(B_n))$ so that $E_n \to E$.

Proof. This is an abstract functional analytic consequence of lemma 3.5; see VIII.24 of Reed and Simon (1972).

Remark. Under strong resolvent convergence (lemma 3.5), the spectrum cannot suddenly appear (lemma 3.6) but, in general, it can disappear (think, for example, of the Stark Hamiltonian as the external field goes to zero). Effectively, the compactness associated with periodicity (lemma 3.3) and gradient bounds (theorem 2.3) implies that in this special case, the spectrum cannot suddenly disappear either.

Proof of theorem 3.1. We have already proven the final statement (lemma 3.2). Thus, suppose $E \in \mathrm{spec}(H(B))$. Find B_n rational (i.e. $(2\pi)^{-1}B_n a_1 a_2$ rational) so that $B_n \to B$. Then, by lemma 3.6, find $E_n \to E$ with $E_n \in \mathrm{spec}(H(B_n))$. By lemma 3.4, find u_n obeying (2.2) (for B_n) and obeying equations (3.1*a*) and (3.1*b*) of theorem 3.1. By theorem 2.3, the functions u_n obey the hypotheses of theorem 2.4; so by that theorem, there exists a function u obeying

$$\int_{|x|<C} |u_n(x) - u(x)|^2 \, \mathrm{d}^2x \to 0.$$

Since the functions u_n obey (3.1*a*) and (3.1*b*), so will u. Moreover, if $g \in C_0^\infty$ (where, for h of compact support and $w \in L^\infty$, $\langle h, w\rangle = \int hw \, \mathrm{d}^2x$):

$$\langle (H(B)-E)g, u\rangle = \langle (H(B_n)-E_n)g, u_n\rangle + \langle (H(B_n)-E_n)g, (u-u_n)\rangle$$
$$+ \langle (H(B)-H(B_n)+E_n-E)g, u\rangle.$$

The first term is zero, since $g \in C_0^\infty$ and u_n is an eigenfunction. The second term goes to zero, since the supports and L^2 norms of $(H(B_n)-E_n)g$ are bounded and $u_n - u \to 0$ in L^2_{loc}. The final term goes to zero, since $u \in L^2_{\mathrm{loc}}$ and $(H(B)-H(B_n))g$, $(E-E_n)g \to 0$ in L^2 of a bounded set. Thus u is a distributional solution of (2.2) and so a classical solution by theorem 2.2.

There is also a result in the almost periodic case. For simplicity of exposition, we state it in the quasi-periodic case, although it is true in the more general almost periodic case. We state it in the continuum case; the same proof (actually it is easier since § 2 is not needed) works for the discrete (tight-binding) case. Let F be a function from $R^k \to R$; $k > \nu$ which is periodic with period one in each variable. Let A be a linear transformation from $R^\nu \to R^k$ and let $\theta \in T^k = \{(\theta_1, \ldots, \theta_k) \in R^k, 0 \leq \theta_i \leq 1\}$ with $\theta_i = 1$

and $\theta_i = 0$. Let

$$V_{A;\theta}(x) = F(Ax + \theta).$$

F is almost periodic and, if ran A/Z^k is dense in T^k (as is generally the case), then $\{V_{A;\theta}(\cdot) \mid \theta \in T^k\}$ is the hull of $V_{A,\theta}$. In that case, $\mathrm{spec}(-\Delta + V_{A,\theta}) \equiv S(A, \theta)$ is independent of θ. In any event, define

$$S(A) = \bigcup_{\theta \in T} S(A, \theta).$$

Theorem 3.7. Fix A. Then there exists a $\theta \in T^k$ so that

$$(-\Delta + V_{A,\theta})u = Eu$$

has a solution obeying

$$\sup_x \int_{|y-x| \leq 1} |u(x)|^2 \, \mathrm{d}^\nu x \leq 1 \tag{3.2a}$$

$$\int_{|y| \leq 1} |u(x)|^2 \, \mathrm{d}^\nu x \geq \tfrac{1}{2} \tag{3.2b}$$

if and only if $E \in S(A)$.

Proof. Since it is so similar to theorem 3.1, we only sketch the details. The converse part is Sch'nol's theorem (1957). If A has rational components, $V_{A,\theta}$ is periodic, so the theorem follows from a Bloch wave analysis. Since we have the translation (θ) freedom, we can arrange for (3.2b) to be true. As $A_n \to A$, $-\Delta + V_{A_n,\theta}$ converges to $-\Delta + V_{A,\theta}$ in the strong resolvent sense, so if $E \in S(A)$, we can find A_n rational with $A_n \to A$ and $E_n \in S(A_n)$ with $E_n \to E$. Using theorems 2.1, 2.3 and 2.4 together with the compactness of T^k, we can construct the necessary u as we did in the proof of theorem 3.1.

4. Stability of gaps

Our main result is:

Theorem 4.1. Suppose $[a, b] \cap \mathrm{spec}(H(B)) = \phi$. There then exists δ so that if $|B' - B| < \delta$, then $[a, b] \cap \mathrm{spec}(H(B')) = \phi$.

Proof. If the above is not so, we can find $B_n \to B$ and $E_n \in \mathrm{spec}(H(B_n)) \cap [a, b]$. By compactness of $[a, b]$, we can pass to a subsequence and suppose that $E_n \to E \in [a, b]$. By theorem 3.1, we can find u_n solving (2.2) for B_n, E_n with properties (3.1a), (3.1b) of theorem 3.1. By theorems 2.3 and 2.4, we can find a subsequence so $u_n \to u$ in L^2_{loc}. By property (3.1b), $u \neq 0$. By the argument in the proof of theorem 3.1, u is a solution of (2.2) for B, E. Thus, by (3.1a), $E \in \mathrm{spec}(H(B))$. This contradiction establishes the theorem.

Given theorem 3.7, the same argument yields:

Theorem 4.2. In the context of quasi-periodic Schrödinger operators, if $[a, b] \cap S(A) = \phi$, there exists a δ so that if $|A' - A| < \delta$, then $[a, b] \cap S(A') = \phi$.

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