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ON THE SELFADJOINTNESS OF DIRAC OPERATORS WITH ANOMALOUS MAGNETIC MOMENT

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ABSTRACT. We provide a new proof of Behncke's remarkable result that the Coulombic Dirac equation with nonzero anomalous magnetic moment is essentially selfadjoint (on $C_{00}^{\infty}(R^3)^4$) for *any* value of the Coulomb charge.

In this note we shall consider Dirac operators. In the simplest version, these have the form

$$H = \vec{\alpha} \cdot \vec{p} + m\beta + V \tag{1}$$

where $\vec{p} = -i\vec{\nabla}$ on $L^2(R^3)$ and H acts on $L^2(R^3, d^3x; C^4)$. α, β are 4×4 matrices, written in terms of the conventional 2×2 Pauli sigma matrices, $\vec{\sigma}$, as 2×2 blocks of 2×2 matrices:

$$\beta = \left(\begin{array}{c|c} \mathbf{1} & 0 \\ \hline 0 & -\mathbf{1} \end{array}\right), \qquad \vec{\alpha} = \left(\begin{array}{c|c} 0 & \vec{\sigma} \\ \hline \vec{\sigma} & 0 \end{array}\right).$$

It is well known (see e.g. [**1, 8, 9**]) that for $V = e|\vec{x}|^{-1}$, (1) is essentially selfadjoint on $C_0^{\infty}(R^3)^4$ if and only if $e \leqslant \sqrt{3/4}$, and strange spectral properties occur if $e > 1$ [**11**]. Indeed, it has been speculated that these difficulties have physical significance for the stability of the world if superheavy nucleii with charge $Z > 137$ exist (written back in conventional units $Z = e\alpha^{-1}$ with α the fine structure constant); see [**6, 10**] and the references therein. We feel that these speculations are ill founded for a number of reasons including the theme of this note.

Equation (1) corresponds to the equation for an electron with magnetic moment 1 (in units of Bohr magnetons), but it is known that the actual value is $1 + \mu$ where $\mu = 0.001159$ is called the anomalous magnetic moment [**5**] (understood from the point of view of quantum electrodynamics). Equation (1) in the presence of such an anomalous moment should be replaced by [**13**]

$$H = \vec{\alpha} \cdot \vec{p} + m\beta + V - \frac{\mu}{2m}\vec{\tau} \cdot \vec{\nabla} V \tag{2}$$

with

$$\vec{\tau} = \left(\begin{array}{c|c} 0 & i\vec{\sigma} \\ \hline -i\vec{\sigma} & 0 \end{array}\right).$$

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Recently, Behncke [3] (see also his papers [**4**, **16**] and the earlier work of Barut and Kraus [**2**]), discovered the remarkable result that if $\mu \neq 0$ and $V = e|\vec{x}|^{-1}$, then (2) is essentially selfadjoint on $C_{00}^{\infty}(R^3)^4$ (here $C_{00}^{\infty} = C_0^{\infty}(R^3 \setminus \{0\})$ for *any* $e \neq 0$). Behncke's analysis depends essentially on the central symmetry of $V(\vec{x}) = e|\vec{x}|^{-1}$ which allows one to write (2) as a direct sum of (two-component) ODE's. He analyzes these new operators by the well-developed selfadjointness techniques of such ODE's (see e.g. Weidman [**15**]). Our goal here is to prove Behncke's result using operator theoretic methods.

Absorbing $\mu/2m$ into V, (2) finally becomes

$$H = \vec{\alpha} \cdot \vec{p} + m\beta + \frac{2m}{\mu} V - \vec{\tau} \cdot \vec{\nabla} V. \tag{2'}$$

In order to study (2′), it turns out to be useful to introduce operators of the type

$$C = \begin{pmatrix} 0 & A_- \\ A_+ & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} S_+ & 0 \\ 0 & S_- \end{pmatrix}. \tag{3}$$

We first state (see Kato [**7**] for discussions of A-boundedness)

PROPOSITION 1. *S is C-bounded with relative bound zero if and only if S_+ is A_+-bounded and S_- is A_--bounded, each with relative bound zero.*

PROOF. $\|C(\varphi_+, \varphi_-)\|^2 = \|A_+\varphi_+\|^2 + \|A_-\varphi_-\|^2$ and $\|S(\varphi_+, \varphi_-)\|^2 = \|S_+\varphi_+\|^2 + \|S_-\varphi_-\|^2$. □

This leaves the question of essential selfadjointness of operators of the form C.

PROPOSITION 2. *An operator of the form C is essentially selfadjoint if and only if*
(i) $(A_-)^* = \overline{A}_+$,
(ii) $(A_+)^* = \overline{A}_-$.
Moreover, (i) *is equivalent to* (ii).

PROOF. The equivalence of (i) and (ii) follows from $X^{**} = \overline{X}$ and $\overline{X^*} = X^*$. The first assertion follows from the observation that $\overline{C} = \left(\begin{smallmatrix} 0 & \overline{A}_- \\ \overline{A}_+ & 0 \end{smallmatrix}\right)$ and secondly from the easy calculation that $C^* = \left(\begin{smallmatrix} 0 & A_+^* \\ A_-^* & 0 \end{smallmatrix}\right)$. □

REMARK. Nelson has noticed [**12**] that the fact that $C = \left(\begin{smallmatrix} 0 & A \\ A^* & 0 \end{smallmatrix}\right)$ is selfadjoint if A is closed provides a trivial proof of von Neumann's theorem that A^*A is densely defined as selfadjoint: Just notice that by the spectral theorem, if C is self-adjoint, then C^2 is densely defined and selfadjoint.

The final abstract result that we require is:

PROPOSITION 3. *Let A, B be operators so that $B \subset A^*$, $A[D(A)] \subset D(\overline{B})$ and BA is essentially selfadjoint on $D(A)$. Then $\overline{B} = A^*$.*

PROOF. By the last proposition, it is sufficient to prove $C = \left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$ is essentially selfadjoint on $D(B) \oplus D(A)$. Now $C^* = \left(\begin{smallmatrix} 0 & B^* \\ A^* & 0 \end{smallmatrix}\right)$ on $D(A^*) \oplus D(B^*)$. Suppose that $(u, v) \in D(A^*) \oplus D(B^*)$ with $A^*u = \pm iv$, $B^*v = \pm iu$. Let $\varphi \in D(A)$. Then

$$(\overline{B}A\varphi, v) = (A\varphi, B^*v) = \pm i(A\varphi, u) = \pm i(\varphi, A^*u) = -(\varphi, v)$$

so $v \perp (\bar{B}A + 1)[D(A)]$, violating the assumed essential selfadjointness of the positive operator $\bar{B}A$. Thus v and so u equal zero. It follows that C has zero deficiency indices so it is essentially selfadjoint. □

The standard angular momentum decomposition of Dirac operators when $V(x) = V(|x|)$, shows that the operator of (2′) is unitarily equivalent (under a transformation taking $C_{00}^{\infty}(R^3)$ to $C_0^{\infty}(0, \infty)$) to a direct sum of operators $H_{j,\sigma}$ indexed by $j = \frac{1}{2}, \frac{3}{2}, \ldots$ and a sign $\sigma = \pm$ (and with the space indexed by j occurring $2j + 1$ times) with $H_{j,\sigma}$ acting on $L^2((0, \infty), dr; C^2)$ by [2, 3]

$$H_{j,\sigma} = \begin{pmatrix} S_{+,j,\sigma} & A_{-,j,\sigma} \\ A_{+,j,\sigma} & S_{-,j,\sigma} \end{pmatrix}, \tag{4}$$

where $S_{\pm} = 2mV(r)/\mu \pm m$ and

$$A_{\pm} = \pm \frac{d}{dr} - \frac{\kappa(j, \sigma)}{r} - \frac{dV}{dr}$$

where $\kappa(j, \sigma) = \sigma(j + 1/2)$. Define

$$W(r) = -\frac{\kappa(j, \sigma)}{r} - \frac{dV}{dr}. \tag{5}$$

Then, on $C_0^{\infty}(0, \infty)$, $A_{+}^{*} \supset A_{-}$ and

$$B_{\pm} = \bar{A}_{\pm} A_{\mp} = \frac{-d^2}{dr^2} + W^2 \pm \frac{dW}{dr} \equiv \frac{-d^2}{dr^2} + V_{\text{eff}}^{\pm} \tag{6}$$

where $V_{\text{eff}}^{\pm} = (V')^2 + 2\kappa r^{-1} V' \mp V'' - (\kappa^2 \pm \kappa) r^{-2}$. If $V = e|\vec{x}|^{-1}$, then $W \sim e|\vec{x}|^{-2}$ for small $|\vec{x}|$ and

$$V_{\text{eff}}^{\pm} \sim e^2 |\vec{x}|^{-4} \tag{7}$$

for small $|\vec{x}|$; so (6) is essentially selfadjoint on $C_0^{\infty}(0, \infty)$ by well-known results (the usual argument is one-dimensional—see Reed-Simon [**14**, Theorem X.10], but there are multidimensional arguments which apply also—see Reed-Simon [**14**, Theorem X.30]). Moreover, by (7)

$$V^2 \leqslant \varepsilon V_{\text{eff}}^{\pm} + C(\varepsilon) \tag{8}$$

so V^2 is a form bounded perturbation of $B_{\pm}$. Thus, by Proposition 1.3, we have proven Behncke's theorem:

THEOREM 5. *If* $\mu \neq 0$ *and* $V = e|\vec{x}|^{-1}$ $(e \neq 0)$, *then* (2) *is essentially selfadjoint on* $C_{00}^{\infty}(R^3)^4$.

The difficulties with noncentral potentials is shown by the fact that while (7), (8) hold for each value of j, they are *not* uniform in j.

We end by noting that one can compare our proof with that of Behncke [**3**] by noting the conditions under which our respective arguments apply. Behncke requires:

(B1) $V, V' \in L_{\text{loc}}^2(0, \infty)$, $\mu \neq 0$;

(B2) Sgn V' constant near 0;

(B3) $\lim_{r\to 0^+}(rV'(r)) = \infty = \lim_{r\to 0^+}|V'(r)/V(r)|$.

Our proof requires (if one uses Wüst's Theorem [**14**, Theorem X.14] in borderline cases to add on S)

(GST1) $V'' \in L^2_{\mathrm{loc}}(0,\infty)$, $\mu \neq 0$,

(GST2) $V^+_{\mathrm{eff}}(r) \geqslant 3/4r^2 - d$ for some d and r small *or* the same result for V^-_{eff},

(GST3) $V^{\pm}_{\mathrm{eff}}(r) - V^2(r) \geqslant -(4r^{-2}) - d$ for some d where $V^{\pm}_{\mathrm{eff}}$ are defined by (5), (6), i.e.,

$$V^{\pm}_{\mathrm{eff}} = (V')^2 + 2\kappa r^{-1}V' \mp V'' + (\kappa^2 \pm \kappa)r^{-2}.$$

$V'' \in L^2_{\mathrm{loc}}$ in (GST1) can be replaced by $V'' \in L^1_{\mathrm{loc}}(0,\infty)$ using standard techniques.

Since Behncke has no conditions on V'', his results are, in a sense, stronger. But curiously, he does not allow $V = 0$ or $V = [\ln(|r|^{-1} + 2)]^\alpha$ with $\alpha < 1$ while we do.

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