Some Rigorous Results for the Anderson Model

Barry Simon
Division of Physics, Mathematics, and Astronomy, California Institute of Technology, Pasadena, California 91125

and

Michael Taylor
Department of Mathematics, State University of New York, Stony Brook, New York 11794

and

Tom Wolff
Division of Physics, Mathematics, and Astronomy, California Institute of Technoloogy, Pasadena, California 91125
(Received 10 January 1985)

We discuss two results for the Anderson model of random quantum Hamiltonians: (1) smoothness of the density of states in the one-dimensional model, even in many cases where the potential distribution is not smooth; and (2) a criterion for localization which, among other consequences, implies that certain estimates of Fröhlich and Spencer yield a dense point spectrum for the multidimensional model at large randomness or large energies.

PACS numbers: 71.55.Jv, 71.20.+c

In this Letter, we want to announce some rigorous results, and discuss their consequences on the regularity of the density of states in the Anderson model obtained by two of us[1], and on the localization in this model obtained by two of us.[2] Full details will appear elsewhere.[1,2]

The Anderson model of random impurities[3] is the random Hamiltonian $H_\omega = H_0 + V_\omega$ on $l^2(Z^\nu)$, where

$$(H_0 u)(n) = \sum_{|j|=1} u(n+j)$$

and V_ω is the diagonal operator $V_\omega(n)$, with $V_\omega(n)$ independent identically distributed random variables with distribution $d\kappa(v)$.

Theorem 1 (Ref. 1).—In the one-dimensional case ($\nu=1$), suppose that $d\kappa$ has the form $d\kappa(v) = F(v)dv$, where F has compact support and $\hat{F}(k) \equiv \int e^{-ikv}F(v)dv$ obeys[4]

$$|\hat{F}(k)| \leq C(1+|k|)^{-\alpha}, \quad \alpha > \tfrac{1}{2}.$$

Then, the integrated density of states, $k(E)$, is an infinitely differentiable function.

This result, whose proof we discuss at the end of this Letter, says that $k(E)$ can be much smoother than the distribution of $V(n)$. Previous results either proved some form of continuity weaker than differentiability[5] or proved that k is only at least as smooth as $\int_{-\infty}^{E} d\kappa(v)$.[6] The result applies to the case where $V(n)$ is uniformly distributed in some interval $[a,b]$. Infinite smoothness of $k(E)$ is consistent with, and suggested by, the phenomenon of Lifshitz tails.

Halperin[7] has proven that when $d\kappa = \theta\delta(v-a) + (1-\theta)\delta(v-b)$ and either $|a-b|$ is large or $\theta(1-\theta)$ is small, then k is not differentiable; indeed, it is not Hölder continuous of any prescribed order. This shows that some hypothesis on $d\kappa$ is needed.

The second result is a criterion for localization; $G_\omega(n,m;z)$ is the Green's function $(\delta_m, (H_\omega - z)^{-1}\delta_n)$:

Theorem 2 (Ref. 2).—Suppose that for almost all $E \in (a,b)$ and almost all ω we have that

$$\sup_{0<\epsilon<1} \left[\sum_n |G_\omega(0,n;E+i\epsilon)|^2 \right] < \infty. \tag{1}$$

If $d\kappa$ has an absolutely continuous component[8] and $\nu=1$, or $d\kappa$ is absolutely continuous and ν is arbitrary, then for almost all ω, H_ω has only a point spectrum[9] in (a,b). If the essential support of the absolutely continuous component of $d\kappa$ is $(-\infty,\infty)$,[10] then (1) is not only sufficient for a pure point spectrum, it is also necessary.

The quantity on the lefthand side of (1) increases as ϵ decreases, and so we need only treat sufficiently small ϵ. One estimate which clearly implies (1) is

$$|G_\omega(0,n;E+i\epsilon)| \leq C_{\omega,E} \exp[-C(E)|n|] \tag{2}$$

for almost every (a.e.) ω and all sufficiently small ϵ. In this case, one can prove[2] that the eigenfunctions decay with a localization length[11] no larger than $C(E)^{-1}$ so long as $d\kappa$ is purely absolutely continuous.

Before discussing the proof of theorem 2, we note that there are two cases where one knows how to prove (1) [in fact, to prove (2)]: in the general one-dimensional case, and in the higher-dimensional case at strong coupling. In the one-dimensional case, Ishii and Deift-Simon[12,13] proved (2). This provides a new proof of localization in this case. The point is not so much that our hypothesis on $d\kappa$ is weaker than that in existing proofs,[14] but that the proof via theorem 2 is mathematically and conceptually quite simple, and more significantly, explains why (2) can hold in certain almost-periodic models[15] which only have singular continuous spectra.[16] Ishii's bounds, together with general lower bounds on eigenfunctions,[17] imply that the localization length is the inverse of the Lyaponov exponent.[18]

Fröhlich and Spencer[19] have proven (2) in the multidimensional Anderson model under two circumstances: (i) $d\kappa$ Gaussian and $|E|$ very large, and (ii) $d\kappa = g(E)\,dE$ with $\sup_E |g(E)|$ sufficiently small (large coupling or large randomness). While it was known that these estimates imply the absence of extended states,[20] it was not known until now that the estimates of Ref. 19 imply a point spectrum. Recently, Fröhlich *et al.*[21] and Goldsheid[22] have announced results on localization in the multidimensional situations. The Fröhlich-Spencer estimates[19] and our remarks on the localization length imply that the localization length goes to zero in the infinite-randomness or large-energy limit.

Theorem 2 comes from an analysis of the spectrum of self-adjoint operators under a random rank-one perturbation. The basic deterministic theory of such perturbations was developed by Aronszajn[23] and Donaghue,[24] and our own interest was kindled by the recent work of Kotani[25] on the special case of random boundary conditions in half-line problems. Indeed, the proof of theorem 3 below is essentially a synthesis of ideas of Aronszajn and Kotani.

Let A be a self-adjoint operator, let P be the projection onto a unit vector, ϕ, and let $A_\lambda = A + \lambda P$. Let $d\mu_\lambda$ be the spectral measure[26] defined by

$$(\phi, e^{-itA_\lambda}\phi) = \int e^{-itx} d\mu_\lambda(x).$$

We need two functions related to these measures:

$$F_\lambda(z) = \int (x-z)^{-1} d\mu_\lambda(x),$$
$$B(x) = \left[\int (x-y)^{-2} d\mu_0(x)\right]^{-1}. \tag{3}$$

The Steiltjes transform, $F_\lambda(z)$, is analytic in the upper half-plane, and the general theory of such functions[27] implies that boundary values $F_\lambda(x+i0)$ exist (for λ fixed) for almost all x. Since $\mathrm{Im}F_0(x+i\epsilon) \leq (\mathrm{Im}\epsilon)B(x)^{-1}$, at most one of $\mathrm{Im}F_0(x+i0)$ and $B(x)$ is nonzero at any point.

Theorem 3.—$d\mu_\lambda$ has a vanishing singular continuous part for almost all λ if and only if $B(x) + \mathrm{Im}F_0(x+i0) > 0$ for almost all x.

Before discussing the proof of this theorem, we explain how it implies theorem 2. If $d\mu_0$ is the spectral measure for H_ω associated to δ_0, then a simple calculation shows that the left-hand side of Eq. (1) is $B(E)^{-1}$, and so (1) says that for a.e. ω and a.e. $E \in (a,b)$, $B(E) > 0$. As noted above, this implies that $\mathrm{Im}F_0(E+i0) = 0$, and thus by Eq. (4) below, $\mathrm{Im}F_\lambda(E+i0) = 0$. The general theory of boundary values of Steiltjes transforms[27] implies that $d\mu_\lambda^{\mathrm{ac}} = 0$ on (a,b). Thus, theorem 3 says that for a.e. λ, $H_\omega + \lambda P_0$ has only a point spectrum for a.e. ω and a.e. λ. The λP_0 just shifts the value of $V(0)$. Since $V(0)$ is independent of the other $V(n)$'s and $d\kappa$ has an absolutely continuous component, we have a point spectrum in the original H_ω with nonzero probability, and so with probability 1 by general results (see the first reference in Ref. 15).

Here is a sketch of the proof of theorem 3: (i) By taking expectations in $(A_\lambda - z)^{-1} = (A_0 - z)^{-1} - \lambda(A_0 - z)^{-1}P(A_\lambda - z)^{-1}$, we obtain the basic equation of Aronszajn[23]:

$$F_\lambda(z) = F_0(z)/[1 + \lambda F_0(z)]. \tag{4}$$

(ii) Since $\mu_\lambda(\{E_0\}) = \lim_{\epsilon\downarrow 0} i\epsilon F_\lambda(E_0 + i\epsilon)$, one deduces from (4) that $\mu_\lambda(\{E_0\}) > 0$ if and only if $F_0(E_0 + i0) = -\lambda^{-1}$ and $B(E_0) > 0$; in fact, $\mu_\lambda(\{E_0\}) = \lambda^{-2}B(E_0)$ if $F_0(E_0+i0) = -\lambda^{-1}$. (iii) By using Eq. (4), one can study the measure $d\eta$ defined by

$$\int g(E)\,d\eta(E) = \int \left[\int g(E)\,d\mu_\lambda(E)\right](1+\lambda^2)^{-1} d\lambda.$$

From (4), one finds that

$$F^{(\eta)}(z) \equiv \int (x-z)^{-1} d\eta(x) = \pi/[F_0(z)^{-1} - i]. \tag{5}$$

From this, one can deduce that $d\eta(x) = H(x)\,dx$, where H is almost everywhere nonzero. For example, since $\mathrm{Im}F_0(z) > 0$, $\mathrm{Im}F^\eta(z) \leq \pi$, which implies that $H(x) = \pi^{-1}\mathrm{Im}F^\eta(x+i0) \leq 1$. (iv) Let $C = \{x | F_0(x+i0) = -\lambda^{-1};\ B(x) = 0\}$. The theorem of de Vallée Poussain[27] says that the singular continuous part of $d\mu_\lambda$, call it $d\mu_\lambda^{\mathrm{sc}}$, is supported on the set where $F_\lambda(x+i0) = \infty$, which, by (4), is the set of where $F_0(x+i0) = -\lambda^{-1}$. By (ii), the subset of this set where $B(x) > 0$ consists of point masses of $d\mu_\lambda$ so that it is countable. Thus $\mu_\lambda(C) = \mu_\lambda^{\mathrm{sc}}(R)$. (v) By (iii), $|C| = \int_C dx = 0$ if and only if $\int \mu_\lambda(C)(1+\lambda^2)^{-1} d\lambda = 0$ which, by (iv), is true if and only if $\mu_\lambda^{\mathrm{sc}}(C) = 0$ for almost all λ. This completes the proof of theorem 3.

In some ways, the key step is (iii) (related to Kotani's work[25]), which says that [if (1) holds] under random changes of $V(0)$, sets of measure zero do not matter. It is precisely pathological behavior on sets of measure zero which are responsible for singular continuous spectra in those almost-periodic models where they occur.[17] The difference between random and almost-periodic models is the decoupling of infinity [which is responsible for (1)] and $V(0)$.[28]

Finally, we describe some aspects of the proof of theorem 1. Like so much in the one-dimensional theory, it depends on an analysis of the transfer matrix, $\Phi(n)$, which takes data for solutions of the time-independent Schrödinger equation at 0 to data at n; i.e.,

$$\Phi(n) = A(n) \cdots A(1); \quad A(j) = \begin{pmatrix} E - V(j) & -1 \\ 1 & 0 \end{pmatrix}. \tag{6}$$

Since the $V(j)$ are random variables, $\Phi(n)$ is a random matrix lying in the group $SL(2,R)$ (for E real). The key technical input for the proof of theorem 1 is that for any k, one can find n so that $\Phi(n)$ has a distribution of the form $G_n(A,E)\,dA$, where G is C^k in A and E and dA is Haar measure on $SL(2,R)$. This is proven by showing that $G_3(A,E)$ [29] has a fractional derivative in A,[30] noting that G_{3n} is the n-fold $SL(2,R)$ convolution of G_3, and that repeated convolutions of functions with a fractional derivative are smoother and smoother.[31]

Next, one uses the basic fact noted already by Schmidt[32] that one should look at the distribution $d\nu_E(x)$ on $x = u(1)/u(0)$ left invariant by applying an independent random transfer matrix to $(u(1), u(0))$ for[32] $k(E) = \int_0^\infty d\nu_E(x)$; so smoothness of $d\nu$ in E implies smoothness of k. Moreover, for each n, ν_E is an eigenfunction of a compact operator built out of G_n. Since the corresponding eigenvalue is simple by results of Furstenberg,[33] the general theory of eigenvalue perturbation theory[34] implies that ν_E is at least C^k. Since k is arbitrary, ν_E is C^∞.

After this paper was submitted, we received two papers from Delyon, Levy, and Souillard,[35,36] who, also motivated by Kotani,[25] discuss localization via a procedure related to, but distinct from, ours in theorems 2 and 3.

Two of us (B.S. and M.T.) would like to thank T. Spencer for valuable discussions related to theorem 1, and one of us (B.S.) would like to thank S. Kotani and T. Spencer for useful discussions related to theorems 2 and 3. This research was partially supported by the National Science Foundation under Grants No. MCS-81-20833, No. MCS-82-01766A01, and No. DMS-84-07099.

[1]B. Simon and M. Taylor, "Harmonic analysis on SL(2,R) and smoothness of the density of states in the one-dimensional Anderson model" (to be published).

[2]B. Simon and T. Wolff, "Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians" (to be published); B. Simon, "Localization in general one-dimensional random systems" (to be published).

[3]P. Anderson, Phys. Rev. **109**, 1492 (1958).

[4]In Ref. 1, only a weaker condition on F is needed, namely, that F has compact support and lies in a Sobolev space L^1_α with $\alpha > 0$. The condition stated in the text is technically simpler to state, implies the L^1_α condition, and holds for any piecewise C^1 function such as $F(x) = 1$ (0) if $a < x < b$ ($x \le a$ or $x \ge b$).

[5]L. Pastur, Commun. Math. Phys. **75**, 179 (1980); W. Craig and B. Simon, Commun. Math. Phys. **90**, 207 (1983); F. Delyon and B. Souillard, Commun. Math. Phys. **89**, 415 (1983); *Probability Measures on Groups VII,* edited by E. LePage, Springer Lecture Notes in Mathematics, Vol. 1064 (Springer, Berlin, 1984), p. 309.

[6]F. Wegner, Z. Phys. B **44**, 9 (1981); S. Edwards and D. Thouless, J. Phys. C **4**, 453 (1971); F. Constantinescu, J. Fröhlich, and T. Spencer, to be published.

[7]B. Halperin, Adv. Chem. Phys. **31**, 123–177 (1967).

[8]Any measure $d\kappa$ on $(-\infty,\infty)$ has a unique decomposition $d\kappa = d\kappa^{pp} + d\kappa^{ac} + d\kappa^{sc}$, where the pure point piece is a countable sum of point masses, the absolutely continuous part has the form $g(E)\,dE$, and the singular continuous part is a measure like the Cantor measure which has no pure points, but lives on a set of Lebesgue measure zero. For spectral measures, the corresponding states are sometimes called "localized," "extended," and "exotic" in the physics literature.

[9]This implies the weak form of localization that for any ϵ there is an R with $\sup_t \sum_{|n| \ge R} |(e^{-itH}\delta_0)(n)|^2 < \epsilon$ but does not imply boundedness in time of moments like $\sum_n n^2 |(e^{-itH}\delta_0)(n)|^2$.

[10]That is, $d\kappa^{ac}(E) = g(E)\,dE$ with $g(E)$ almost everywhere nonzero.

[11]For definiteness, we define the localization length of an eigenfunction ψ to be

$$\lim_{N\to\infty} \left[(-2N)^{-1} \ln \left(\sum_{|n| \ge N} |\psi(n)|^2 \right) \right]^{-1}.$$

[12]K. Ishii, Prog. Theor. Phys. Suppl. **53**, 77 (1973).

[13]Ishii proved (2) for the half-line problem, but his argument works to control the full-line Green's function; see Ref. 2. We note that, as in Ishii's analysis, (1) implies that

$$\mathrm{Im}[G(0,0;E_0+i\epsilon)] = \epsilon \sum_n |G(0,n;E_0+i\epsilon)|^2$$

goes to zero as $\epsilon \downarrow 0$. This can be used to replace the arguments of F. Martinelli and E. Scoppola, to be published, in the higher-dimensional case.

[14]H. Kunz and B. Souillard, Commun. Math. Phys. **78**, 201 (1980); F. Delyon, H. Kunz, and B. Souillard, J. Phys. A **16**, 25 (1983).

[15]Ishii's arguments for (2) only depend on positive Lyapunov exponents, and so they imply that (2) holds in the case studied by J. Avron and B. Simon, Duke Math. J. **50**, 369 (1983).

[16]Avron and Simon, Ref. 15.

[17]W. Craig and B. Simon, Duke Math. J. **50**, 551 (1983).

[18]Such results in a related model were first found by R. Carmona, Duke Math. J. **49**, 191 (1982).

[19]J. Fröhlich and T. Spencer, Commun. Math. Phys. **88**, 151 (1983).

[20]Martinelli and Scoppola, Ref. 13.

[21]J. Fröhlich, F. Martinelli, E. Scoppola, and T. Spencer, to

be published; new estimates beyond those in Ref. 19 are required.

[22]I. Goldsheid, talk presented at the Conference on Information Theory, Tbilisi, U.S.S.R., September 1984 (unpublished).

[23]N. Aronszajn, Am. J. Math. **79**, 597 (1957).

[24]W. Donoghue, Commun. Pure Appl. Math. **18**, 559 (1965).

[25]S. Kotani, in Proceedings of the American Mathematics Society Conference on Random Matrices and Their Applications, Brunswick, Maine, 17–23 June 1984 (to be published).

[26]See, e.g., M. Reed and B. Simon, *Methods of Modern Mathematical Physics* (Academic, New York, 1972), Vol. I.

[27]S. Saks, *Theory of the Integral* (Dover, New York, 1964); Y. Katznelson, *An Introduction to Harmonic Analysis* (Dover, New York, 1976).

[28]This remark can be used to prove a point spectrum in many one-dimensional models with nonindependent $V(n)$; see Ref. 1.

[29]$\Phi(1)$ has a distribution concentrated on a curve in SL(2,R) so that $\Phi(3)$ is the first $\Phi(n)$ that can have a distribution of the form $G(A)dA$ since SL(2,R) is three dimensional. This is why G_3 rather than G_2 appears.

[30]Even if $d\kappa/dE$ is C^∞, $G_3(A)$ has an non-L^1 gradiant and so the theory of fractional derivatives is essential.

[31]This only explains smoothness in A; an additional argument (Ref. 1) is needed to get smoothness in E.

[32]H. Schmidt, Phys. Rev. **105**, 425 (1957).

[33]H. Furstenberg, Trans. Am. Math. Soc. **108**, 377 (1963).

[34]T. Kato, *Perturbation Theory for Linear Operators* (Springer, New York, 1966).

[35]F. Delyon, Y. Levy, and B. Souillard, "Anderson localization for multidimensional systems at large disorder or large energy" (to be published).

[36]F. Delyon, Y. Levy, and B. Souillard, "An approach 'a la Borland' to multidimensional localization" (to be published).