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Internal Lifschitz Tails

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We consider an Anderson model in ν dimensions with a potential distribution supported in $(a, b) \cup (c, d)$, where $c-b>4\nu$. We prove the existence of Lifschitz tails at the edges of the internal gap at $b+2\nu$ and $c-2\nu$. This reproves results of Mezincescu.

KEY WORDS: Lifschitz tails; random Hamiltonians; Anderson model; gaps; density of states; periodic potentials.

1. INTRODUCTION

Many years ago, E. M. Lifschitz argued that the integrated density of states at the edge of the spectrum of a random system in ν dimensions should vanish roughly as $\exp[-(E-E_0)^{-\nu/2}]$. The rigorous proof of Lifschitz tails is a problem that interested Mark Kac over a period of time (see, e.g., Refs. 2 and 3). During the period I knew Mark, I invariably enjoyed my interactions with him, and I am pleased to dedicate this paper to his memory.

I will consider the simplest case of a random quantum Hamiltonian, the ν-dimensional Anderson model: The underlying Hilbert space is $l^2(\mathbb{Z}^\nu)$. The Hamiltonian is

$$(h_\omega u)(n) = \sum_{|j|=1} u(n+j) + V_\omega(n)\, u(n) \equiv (H_0 u)(n) + V_\omega(n)\, u(n)$$

where the $V_\omega(n)$ are i.i.d.'s with a distribution $d\kappa$. Any such model has an integrated density of states $k(E)$ defined, for example, in Ref. 1.

Suppose $a = \inf \operatorname{supp}(d\kappa)$, $d = \sup \operatorname{supp}(d\kappa)$, and that

$$\kappa[a, a+\varepsilon) \geqslant C\varepsilon^N; \qquad \kappa(d-\varepsilon, d] \geqslant C\varepsilon^N \tag{1}$$

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It follows that $\inf \operatorname{spec}(H) = a - 2v$; $\operatorname{supp} \operatorname{spec}(H) = d + 2v$. Then, the basic phenomenon of Lifschitz tails asserts that

$$\lim_{e \downarrow a-2v} \ln[-\ln k(e)]/\ln(e-a-2v) = -v/2 \tag{2a}$$

$$\lim_{e \uparrow d+2v} \ln\{-\ln[1-k(e)]\}/\ln(d-e+2v) = -v/2 \tag{2b}$$

Various approaches to this problem have been found by various authors (see Ref. 7 for references). I feel that the most elementary approach to this problem is the Dirichlet–Neumann bracketing method of Kirsch and Martinelli,[4] developed in the discrete case in Simon.[7]

In this paper, I want to consider the case of "internal" tails, i.e., suppose that

$$\operatorname{supp}(d\kappa) = [a, b] \cup [c, d]$$

where

$$\kappa(b-\varepsilon] \geqslant C\varepsilon^N, \qquad \kappa[c, a+\varepsilon) \geqslant C\varepsilon^N \tag{3}$$

and

$$c - d > 4v \tag{4}$$

Then

$$\operatorname{spec}(H_\omega) = [a-2v, b+2v] \cup [c-2v, d+2v]$$

My goal is to prove the following result.

Theorem 1.1. Let (1), (3), and (4) hold. Then

$$\lim_{e \uparrow b+2v} \ln\{-\ln[k(b+2v)-k(e)]\}/\ln(b+2v-e) = -v/2 \tag{5a}$$

$$\lim_{e \downarrow c-2v} \ln\{-\ln[k(c)-k(c-2v)]\}/\ln(e-c-2v) = -v/2 \tag{5b}$$

This result has already been proven by Mezincescu.[6] The present proof may shed additional light on this theorem.

It is not hard to show that $k(b+2v) = k(c-2v) = \kappa[a, b]$. While the result has been stated for two "bands" and one gap, it is not hard to extend it to several gaps; each gap in $\operatorname{supp} d\kappa$ must be at least $4v$ in size to produce a gap in $\operatorname{spec}(H)$. In addition, one could handle the case where $\operatorname{supp} \kappa = (-\infty, b] \cup [c, \infty)$ with some weak moment conditions on $d\kappa$. Essentially, the arguments in Section 3 require no change, while those in Section 4 require one to note that replacing $V(n)$ by $\max(a, V(n))$ with

$a<b$ can only increase $k(b+2v)-k(b+2v-\varepsilon)$, and that once the two classes of sites have been decoupled, one can replace $V(n)$ by $\min(V(n), d)$.

In order to understand the strategy of proof I will use, recall the ideas behind the Dirichlet–Neumann bracketing proof of (2). One gets a lower bound on $k(e)$ by using Dirichlet bracketing, essentially by using the variational principle to find enough trial functions to bound $k(e)$. In place of a linear trial function argument, which works at the edge of the spectrum, I use a quadratic energy estimate sometimes associated with the work of Temple[8] (this is related to, but distinct from, his lower bound inequalities for ground states). Since this lower bound result depends on finding the right trial functions, it is completely elementary.

The upper bound is more subtle. The proof makes sense of the following intuition: "levels repel" in quantum theory. Thus, the upper band should push levels away from the internal edge and should push them toward an external edge. Thus, one should be able to bound the number of states in an internal tail by the number on an external tail where we already have (2).

In Section 2, I present the necessary linear algebra results needed in the later sections. I prove the lower bound on k in Section 3, and the upper bound in Section 4. I only prove (5a); the proof of (5b) is similar.

Unfortunately, the present proof, as well as that of Mezincescu, depends essentially on independence at distinct points. Another "random" system with internal edges is the sum of a periodic and a random potential. If the randomness is smaller than a gap in the periodic potential, the sum has a gap and these should be Lifschitz tails at such internal edges. Kirsch and Simon[5] have derived external Lifschitz tails in such a situation, but the case of internal tails in this situation remains an important open question.

2. SOME LINEAR ALGEBRA

We require two elementary results from linear algebra: a Temple-type inequality yielding lower bounds on the number of eigenvalues and an expression of the repulsion of levels.

Theorem 2.1. Let A be a finite self-adjoint matrix, and suppose N, E, δ are given, and that there are N vectors $\varphi_1,\dots, \varphi_N$ obeying

(a) $\langle \varphi_i, \varphi_j\rangle = \delta_{ij}$

(b) $\langle A\varphi_i, \varphi_j\rangle = \langle A\varphi_i, A\varphi_j\rangle = 0$ if $i \neq j$

(c) $\|(A-E)\varphi_i\| \leqslant \delta$

Then A has at least N eigenvalues in $[E-\delta, E+\delta]$.

Proof. Let V be the span of the $\{\varphi_i\}$. Then (a)–(c) imply that if $\eta \in V$, then $\|(A-E)\eta\| \leqslant \delta\|\eta\|$. If $[E-\delta, E+\delta]$ had fewer than N eigenvalues, we could find η in V orthogonal to the span of those eigenvectors and for that η, we would have that $\|(A-E)\eta\| > \delta\,\|\eta\|$. Thus the interval has at least N eigenvalues. ▮

Our second result is an expression of the fact that coupling makes levels repel. This is well known to be true in second-order perturbation theory. The following nonperturbative result does not appear to be terribly well appreciated in the linear algebra literature, although, as Friedland has remarked, it follows from Cauchy's result on the interlacing of eigenvalues when one row and the corresponding column of a self-adjoint matrix are removed.

Theorem 2.2. Let A be a self-adjoint matrix of dimension $(n+m)$ of the block form

$$A = \begin{array}{c} \\ n \\ m \end{array} \begin{pmatrix} n & m \\ B & D \\ \hline D^* & C \end{pmatrix}$$

Let $e_1(A) \geqslant \cdots \geqslant e_{n+m}(A)$, $e_1(B) \geqslant \cdots \geqslant e_n(B)$, and $e_1(C) \geqslant \cdots \geqslant e_m(C)$ be the eigenvalues of A, B, and C, respectively. Suppose that $e_n(B) > e_1(C)$. Then

(i) $e_j(A) \subset \geqslant e_j(B)$, $j = 1,\dots, n$

(ii) $e_{j+n}(A) \leqslant e_j(C)$, $j = 1,\dots, m$.

Proof. I will prove (i); the proof of (ii) is similar. Given j, let V be the span of the eigenvectors of B corresponding to the eigenvalues $e_1(B),\dots, e_j(B)$. Then, for $\eta \in V$,

$$(\eta, B\eta) \geqslant e_j(B)(\eta, \eta)$$

But, for $\eta \in V$,

$$(\eta, A\eta) = (\eta, B\eta) \geqslant e_j(B)(\eta, \eta)$$

The min–max principle implies that

$$e_j(A) \geqslant e_j(B) \quad ▮$$

3. LOWER BOUND ON THE DENSITY OF STATES

In this section, I prove that

$$k(b+2v)-k(b+2v-\varepsilon) \geqslant C\exp[-D\varepsilon^{-v/2}\ln(\varepsilon^{-1})] \tag{6}$$

I start with the case $v=1$ and discuss the changes for general v later. I begin by analyzing a special situation where

$$|V(j)-b| < \varepsilon, \qquad j=1,..., L-1 \tag{7}$$

Define

$$\begin{aligned} \varphi_l(y) &= \sin(\pi jl/L), \qquad & j=1,..., L-1 \\ &= 0, & j \neq 1,..., L-1 \end{aligned}$$

the Dirichlet eigenfunctions, and let

$$\psi = \alpha\varphi_1 - \beta\varphi_3; \qquad \alpha = \sin(3\pi/L), \qquad \beta = \sin(\pi/L)$$

ψ is chosen to vanish at both $j=1$ and $j=L-1$. Since it vanishes there, $H_0\psi$ will vanish at 0, L. Thus

$$(H_0\psi) = 2\cos(\pi/L)\,\alpha\varphi_1 - 2\cos(3\pi/L)\,\beta\varphi_3 \tag{8}$$

For this to be correct with the infinite-volume H_0, it is essential that we arrange for ψ to vanish at $1, L-1$. [If φ_1 were chosen instead, $H_0\varphi_1$ would have a term of order $\varphi_1(1)$ and then $\|(H_0+V-b-2)\varphi_1\|/\|\varphi_1\| \sim L^{-3/2}$ rather than the L^{-2} we will get below for ψ.] Note that

$$\|(H_0+V-b-2)\psi\| \leqslant \|(V-b)\psi\| + \|(H_0-2)\psi\|$$

By the hypothesis (7), $\|(V-b)\psi\| \leqslant \frac{1}{2}\varepsilon\|\psi\|$. By (8) and $|2\cos(x)-2| \leqslant cx^2$

$$\|(H_0-2)\psi\| \leqslant dL^{-2}\|\psi\|$$

for a constant d.

Thus, given ε small, we pick L_0 so $dL_0^{-2} \sim \varepsilon/2$, i.e., $L_0(\varepsilon)$ so $dL_0^{-2} \leqslant \varepsilon/2$ while $L_0(\varepsilon) \leqslant \tilde{d}\varepsilon^{-1/2}$. For this, given ε, look at a volume $L=m(L_0+1)$ with m large. Break L into m disjoint blocks of size L_0+1. For each block in which (7) holds, look at the translate of ψ. If there are N blocks where (7) holds, we get N functions for which Theorem 2.1 holds with $E=b+2$,

$\delta = \varepsilon$, and $A = H_L$ and so a lower bound on the number of eigenvalues of H_L. Since $k(b+2+\varepsilon) = k(b+2)$ for b small, by taking $L \to \infty$, we see that

$$\begin{aligned} k(b+2) - k(b+2-\varepsilon) & \\ &\geqslant \lim[1/L \text{ number of blocks on which (7) holds}] \\ &= \text{Probability a fixed block obeys (7)} \end{aligned}$$

by the law of large numbers. Thus,

$$\begin{aligned} k(b+2) - k(b+2-\varepsilon) &\geqslant [\kappa(b, b-\varepsilon)]^{L_0(\varepsilon)} \\ &\geqslant \exp\{\tilde{d}\varepsilon^{-1/2}[\ln C - N \ln(\varepsilon^{-1})]\} \end{aligned}$$

which implies (6) in this case.

As for the case of general ν, we look for $L_0(\varepsilon) \times \cdots \times L_0(\varepsilon)$ blocks and take a product of the form $\prod_{i=1}^{\nu} \psi(\kappa_i)$. The argument is essentially the same with $L_0(\varepsilon) \sim d\varepsilon^{-1/2}$ and one gets

$$\begin{aligned} k(b+2\nu) - k(b+2\nu-\varepsilon) &\geqslant [\kappa(b, b-\varepsilon)]^{L_0(\varepsilon)^\nu} \\ &\geqslant \exp\{\tilde{d}^\nu \varepsilon^{-\nu/2}[\ln C - N \ln(\varepsilon^{-1})]\} \quad \blacksquare \end{aligned}$$

4. UPPER BOUNDS ON THE DENSITY OF STATES

In this section, I will bound from above the number of states near an internal spectral edge by the corresponding number of states near an external edge in a related model. By appealing to the results on the external edges,[7] one can then prove the necessary upper bound on k. Let $d\kappa$ have support in $[a, b] \cup [c, d]$ and let $\kappa_1 = \kappa \restriction [a, b]$, $\kappa_2 = \kappa \restriction [c, d]$. The $d\tilde{\kappa}$ will be a measure supported on $[e, f] \cup [a, b]$ with $f - e = d - c$ and $a - f = 4\nu + 1$. The $d\tilde{\kappa}$ is defined by requiring $\tilde{\kappa} \restriction [a, b] = \kappa \restriction [a, b]$, while $\tilde{\kappa} \restriction [e, f]$ is just the translation of $\kappa \restriction [c, d]$. Let $\tilde{k}$ be the integrated density of states for $\tilde{\kappa}$.

Theorem 4.1. $k(b+2\nu) - k(b+2\nu-\varepsilon) \leqslant \tilde{k}(b+2\nu) - \tilde{k}(b+2\nu-\varepsilon)$.

Proof. Fix a volume Λ and a random potential V distributed according to κ, and let H_Λ be the corresponding Hamiltonian. Define

$$\begin{aligned} \tilde{V}(n) &= V(n) && \text{if} \quad a \leqslant V(n) \leqslant b \\ &= V(n) - (c - e) && \text{if} \quad c \leqslant V(n) \leqslant d \end{aligned}$$

so that $\tilde{V}$ is ditributed according to $\tilde{k}$. I will prove a deterministic result

$$\begin{aligned}&\text{number of e.v. of } H_L \text{ in } (b+2v-\varepsilon, b+2v)\\ &\qquad \leqslant \text{number of e.v. of } \tilde{H}_L \text{ in } (b+2v-\varepsilon, b+2v)\end{aligned} \tag{9}$$

from which the theorem follows.

Break up the sites into two sets, $S_1 = \{n \mid a \leqslant V(n) \leqslant b\}$ and $S_2 = \{n \mid c \leqslant V(n) \leqslant d\}$. Let C be the matrix of couplings between sites in S_1 and potentials in S_1, and B the matrix of couplings between sites in S_2 and potentials in S_2, and let D be the coupling between S_1 and S_2. Then

$$H_L = \left(\begin{array}{c|c} B & D \\ \hline D^* & C \end{array}\right)$$

The eigenvalues of B are all larger than $c-2v$, while those in B are smaller than $b+2v$, so Theorem 2.2 applies. Thus,

$$\begin{aligned}&\text{number of e.v. of } H_L \text{ in } (b+2v-\varepsilon, b+2v)\\ &\qquad \leqslant \text{number of e.v. of } H_L^{\#} \text{ in } (b+2v-\varepsilon, b+2v)\end{aligned}$$

where

$$H_L^{\#} = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right)$$

Let

$$H'_L = \left(\begin{array}{c|c} B-(a-c)\mathbb{1} & 0 \\ \hline 0 & C \end{array}\right)$$

Then

$$\begin{aligned}&\text{number of e.v. of } H_L^{\#} \text{ in } (b+2v-\varepsilon, b+2v)\\ &\qquad = \text{number of e.v. of } H'_L \text{ in } (b+2v-\varepsilon, b+2v)\end{aligned}$$

since both are just the number of eigenvalues of C in $(b+2v-\varepsilon, b+2v)$.

Now reintroduce the couplings D. Since now the eigenvalues of C lie above those of $B-(c-e)\mathbb{1}$, the inequality goes in the opposite direction, i.e., by Theorem 2.2,

$$\begin{aligned}&\text{number of e.v. of } H'_L \text{ in } (b+2v-\varepsilon, b+2v)\\ &\qquad \leqslant \text{number of e.v. of } \tilde{H}_L \text{ in } (B+2v-\varepsilon, b+2v)\end{aligned}$$

as was to be proven. ∎

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