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Localization in General One-Dimensional Random Systems

II. Continuum Schrödinger Operators

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Dedicated to Walter Thirring on his 60th birthday

Abstract. We discuss two ways of extending the recent ideas of localization from discrete Schrödinger operators (Jacobi matrices) to the continuum case. One case allows us to prove localization in the Goldshade, Molchanov, Pastur model for a larger class of functions than previously. The other method studies the model $-\Delta + V$, where V is a random constant in each (hyper-) cube. We extend Wegner's result on the Lipschitz nature of the ids to this model.

1. Introduction

Localization for continuum and discrete random Schrödinger operators has been heavily studied. This note contributes to this literature. Our main goal is to extend to the study of operators on $L^2(R^\nu)$ [especially $L^2(R)$] a set of ideas recently developed to discuss localization for operators on $\ell^2(\mathbb{Z}^\nu)$. These ideas, which have their roots in work of Carmona [2, 3], were developed by Kotani [14, 15] and brought to fruition in Delyon-Levy-Souillard [5–7] and Simon-Wolff [22, 23, 21]. As a by-product, we will extend Wegner's result on the Lipschitz nature of the integrated density of states to certain continuum models.

The models that we will study can be described as follows: Let (Ω, μ) be a probability measure space and let $\{T_x(\omega)\}$ be a one-parameter group of μ-preserving transformations on Ω which is ergodic. Let F be a measurable function from Ω to $\mathbb{R}$. We want to study the family of Schrödinger operators on $L^2(\mathbb{R}^\nu)$:

$$-\Delta + q_\omega(x),$$

where $q_\omega(x) = F(T_x(\omega))$. We always suppose that, for a.e. ω: $q_\omega(x)$ is continuous in x and

$$|q_\omega(x)| \leqq C_\omega(1 + |x|^2)$$

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so there is no problem with selfadjointness (see e.g. [19]). For virtually all our arguments in one dimension, one can easily deal with $q_\omega(x)$ which are in L^2_{loc} with $\left(\int_{|y-x|\leqq 1} |q_\omega(y)|^2 dy\right)^{1/2} \leqq C(1+|x|^2)$ and suitable L^p requirements in higher dimensions.

In Sects. 2–4 we will deal with subsets $\Omega_0 \subset \Omega$ of positive measure and want to take a conditional expectation onto a Σ-algebra Σ_0 of the form

$$E(F\chi_{\Omega_0} \mid \Sigma_0).$$

We will call this the Ω_0-restricted conditional expectations.

A basic object in the study of one-dimensional Schrödinger operators is the transfer matrix, $U(a,b;E)$ defined by looking at

$$-\varphi'' + (q(x)-E)\varphi = 0$$

with initial conditions $\varphi(a)$, $\varphi'(a)$ and solving for $\varphi(b)$, $\varphi(b')$. U is defined by

$$U(a,b;E)\begin{pmatrix}\varphi(a)\\ \varphi'(a)\end{pmatrix} = \begin{pmatrix}\varphi(b)\\ \varphi'(b)\end{pmatrix}.$$

Constancy of the Wronskian implies that U has a determinant 1.

We will also often want to "projectivize" such a U; i.e. given a 2×2 invertible matrix and $\theta \in [0,\pi)$ define $\varphi \in [0,\pi)$ by

$$U\begin{pmatrix}\sin\theta\\ \cos\theta\end{pmatrix} = \pm r\begin{pmatrix}\sin\varphi\\ \cos\varphi\end{pmatrix}.$$

We will write

$$\varphi = \tilde{U}(\theta).$$

The key idea in the papers quoted above is that averaging the spectral measure over a finite region of space (more properly, conditioning on q outside a finite region) should produce a measure absolutely continuous with respect to Lebesgue measure. In Sect. 2, we show this is implied by an absolutely continuous density for $\tilde{U}(\varphi)$ when conditioned properly. In many ways, this result is behind the work of Carmona [2, 3], so Sect. 2 should be viewed as Carmona's theory with certain irrelevancies eliminated and the useful extension to Ω_0-restricted expectations. We then apply this theory in Sects. 3 and 4. In particular, we are able to extend the celebrated result of Goldshade, Molchanov, Pastur [10] to arbitrary smooth non-constant functions on their underlying manifold rather than being restricted to isolated critical points.

In Sects. 5 and 6, we shift models and discuss the random constant model introduced by Holden-Martinelli [11]. Section 5 discusses localization while Sect. 6 discusses a Wegner-type result. These sections show that the approach of Wolff-Simon can be extended to handle cases which are not rank 1 or even finite rank (indeed, one could replace $-\Delta$ by a bounded function on $-\Delta$ and make a partial proof without even local compactness, although there would be a problem with the continuum eigenfunction expansion). What is critical is the positivity of the perturbation (although see [8] for a discussion of some non-positive perturbations by the Delyon-Levy-Souillard method).

For background in probability theory, see [25].

2. An Abstraction of Carmona's Theory

Here we will develop an extension of some ideas of Carmona [2, 3]. Fix $a>0$ and let $q_\omega(x)$ be a stochastic potential of the type described in Sect. 1. Fix $\theta \subset [0, \pi)$ and let $\varphi(\theta, a, q_\omega; E)$ denote the value of

$$\varphi = \tilde{U}_{q_\omega}(0, a; E)\theta .$$

Theorem 2.1. *Suppose there exists a measurable* $\Omega_0 \subset \Omega$ *with* $\mu_0(\Omega_0)>0$ *and* $a, c>0$ *so that for a measurable set* $\Omega_1 \subset \{q \upharpoonright R\backslash[0,a]\}$ *with* $\mu_0\{q\in\Omega_0 \mid q\upharpoonright \mathbb{R}\backslash[0,a]\in\Omega_1\}>0$, *we have that for each fixed* θ, *the* Ω_0*-restricted conditional distribution, conditioned on fixing* $q\in\Omega_1$, *of* φ *has the form*

$$G(\varphi)d\varphi$$

with $|G(\varphi)|<c$ *if* $|E|<B$ *(c is independent of* θ *but may depend on B and* $q\upharpoonright\mathbb{R}\backslash[0,a]$*). Suppose, moreover, that* $\sup\{|q(x)| \mid q\in\Omega_0, x\in[0,a]\}<\infty$. *Then, with probability* 1, $-\frac{d^2}{dx^2}+q_\omega(x)$ *has only pure point spectrum with exponentially decaying eigenfunctions.*

In fact, the decay will be at the Lyaponov exponent rate.

We will prove this result by reducing it to a lemma concerning the averaged spectral measure. Let $W(x)$ be a continuous potential obeying $W(x)\geqq -C(x^2+1)$, and let $H=-\frac{d^2}{dx^2}+W(x)$. The spectral measure $d\varrho(E)$ is defined by

$$(\delta_0, e^{itH}\delta_0)+(\delta_0', e^{-itH}\delta_0')=\int d\varrho(E)e^{-itE}, \tag{2.1}$$

where δ_0, δ_0' are the delta function and its derivative at the origin. Using ODE techniques [4, 18] one can uniquely define $d\varrho$ and show that it is formally given by (2.1). If $w=q_\omega$, we denote the spectral measure as ϱ_q. The key lemma is:

Lemma 2.2. *Under the hypotheses of Theorem 2.1, fix g, a function on* $\mathbb{R}\backslash[0,a]$, *lying in* Ω_1. *Then the integral of* $d\varrho_q(E)$ *over the* Ω_0*-conditional distribution of q's with* $q\upharpoonright\mathbb{R}\backslash[0,a]=g$ *is absolutely continuous with respect to Lebesgue measure, dE.*

Proof of Theorem 2.1, Assuming Lemma 2.2. We follow the strategy of [14, 5, 22]. The hypotheses of Theorem 2.1 imply that the process $q_\omega(x)$ is nondeterministic, so by the method of [13] (extended as in [12] to handle unbounded q's) shows that the Lyaponov exponent, $\gamma(E)$, is strictly positive for a.e. E, say, for all $E\notin S_1$, where S_1 has zero Lebesgue measure. Fix $g\in\Omega_1$. By the definition of γ and Fubini's theorem, for a typical g (i.e. for a.e. g) there is a set $S_2(g)\subset\mathbb{R}$ of zero Lebesgue measure, so that if $E\notin S_1\cup S_2(g)$ and if $q\upharpoonright\mathbb{R}\backslash[0,a]=g$, then the transfer matrix for $-\frac{d^2}{dx^2}+q(x)$ at energy E has Lyaponov behavior at both $-\infty$ and $+\infty$. Thus, by the Osceledec-Ruelle theorem, for such E, every solution of

$$-u''+qu=Eu \tag{2.2}$$

decays exponentially or grows exponentially at both $+\infty$ and $-\infty$. By Lemma 2.2, for a.e. q with respect to the conditional measure, $d\varrho_q(S_1\cup S_2)=0$, so for such

q's, $d\varrho_q$ is supported on $\{E \mid \text{every solution of (2.2) decays or grows exponentially}\}$. By the BGK eigenfunction theory [1, 20], $d\varrho_q$ is supported on $\{E \mid (2.2) \text{ has polynomially bounded solutions}\}$. Thus, since an exponentially growing solution is not polynomially bounded, $d\varrho_q$ is supported on the set of E where (2.2) has a solution exponentially decaying at both $+\infty$ and $-\infty$, i.e. on the set of point eigenvalues. This set is countable, so $d\varrho_q$ is supported on a countable set and so it is pure point.

We have therefore shown that for a.e. $q \in \Omega_0$ with $q \upharpoonright \mathbb{R}\backslash[0,a] \in \Omega_1$, $-\frac{d^2}{dx^2}+q$ has only pure point spectrum. Thus, for the original process, we have only point spectrum on a set of potentials of positive measure. Since the spectral type is non-random [16], we have only point spectrum for a.e. q. In the appendix we will show that having only Lyaponov-exponentially decaying eigenfunctions is a non-random event, so showing it with positive probability proves it for a.e. ω. $\square$

In addition to the spectral measure, $d\varrho$, for the whole line problem, we will need the spectral measure σ_-^θ for the half-line problem on $(-\infty, 0]$ with boundary condition θ at $x=0$. Explicitly, let $H_\omega^{-,\theta} = -\frac{d^2}{dx^2}+q_\omega$ on $L^2(-\infty,0]$ with the boundary condition $u(0)/u'(0)=\tan\theta$ [with the understanding that $\theta=\pi/2$ corresponds to Neumann boundary conditions $u'(0)=0$]. Then σ_-^θ is given by (2.1) with ϱ replaced by (2.1) and H by $H_\omega^{-,\theta}$. As a preliminary, we recall the following standard result (see Carmona [2, 3]):

Proposition 2.3. (a) $\sigma_-^\theta(dE) = \underset{y\to-\infty}{w\text{-lim}}\, \delta(\tilde{U}_{q_\omega}(y,0;E)\beta-\theta)dE$ *for any fixed* β.

(b) $\varrho(dE) = \underset{x\to\infty,\, y\to-\infty}{w\text{-lim}}\, \delta(\tilde{U}_{q_\omega}(x,0;E)\alpha - \tilde{U}_{q_\omega}(y,0;E)\beta)dE$ *for any fixed* α *and* β.

Sketch. The key fact that one uses is that if η solves

$$-\eta''+(q-E)\eta=0 \tag{2.3}$$

with $\eta(a)/\eta'(a)=\tan\alpha$ and if $\Phi(E)=\tilde{U}(a,0;E)\alpha$, then

$$\frac{d\Phi}{dE} = \int_0^a |\eta(x)|^2 dx/[\eta(0)^2+\eta'(0)^2]\,. \tag{2.4}$$

Accepting (2.4) for the moment, we see that the spectral measure for the operator on $[0,a]$ with α-boundary conditions at a and θ boundary conditions at 0 is just

$$\sum_{\substack{\text{eigenvalues}\\ E_n}} \left(\frac{d\Phi}{dE}\right)^{-1} \delta(E-E_n)dE = \delta(\Phi-\theta)\,,$$

since eigenvalues correspond precisely to solutions of $\Phi=\theta$. Similarly, we see that the right side of (b) is just the spectral measure for the operator on $[y,x]$, α boundary conditions at y and β at x. Thus, the proposition follows from the weak convergence of the spectral measures for finite volume to the infinite volume spectral measure which is not hard.

Equation (2.4) follows by writing out Prüfer variables for η, i.e. $\eta(x) = r(x)\sin\theta(x)$, $\eta'(x)=r(x)\cos\theta(x)$, deducing the differential equations in x for θ and

r and then differentiating the equation for θ with respect to E. The net result is that if $d\theta/dE=\gamma$, then

$$\frac{d\gamma}{dx}=\sin^2\theta(x)-\left[\frac{d}{dx}(\ell nr(x))\right]\gamma(x),$$

and the method of integrating factors yields (2.4). □

For the next element, we need the following lemma which is well known (see e.g. [2, 3]):

Lemma 2.4. *Let $U\in SL(2,R)$. Then*

$$\int_0^\pi f(\tilde{U}\theta)d\theta=\int_0^\pi f(\theta)\|U\hat{\theta}\|^{-2}d\theta, \tag{2.5}$$

where $\hat{\theta}=\begin{pmatrix}\sin\theta\\ \cos\theta\end{pmatrix}$.

Sketch. Any U can be written R_1DR_2 with R_1,R_2 rotations and D diagonal. Moreover, it is easy to see that (2.5) for matrices A,B implies it for AB. Since it is trivial for rotations, the result follows from the calculation for diagonal matrices:

$$\frac{d}{d\theta}\arctan[\alpha^2\tan\theta]=(\alpha^2\sin^2\theta+\alpha^{-2}\cos^2\theta)^{-1}. \quad \square$$

The last two results imply the formula of Carmona:

Proposition 2.5 (Carmona [2, 3]). *With the notation of Proposition 2.3:*

$$\varrho(dE)=\frac{1}{\pi}\operatorname*{w-lim}_{x\to\infty}\int_0^\pi d\theta\|U_q(0,x)\hat{\theta}\|^{-2}\sigma_-^\theta(dE).$$

Proof. Since Proposition 2.3(b) holds for all α,β and the measures can be shown to be uniformly bounded, the result also holds for the average, i.e.

$$\begin{aligned}\varrho(dE)&=\frac{1}{\pi^2}\operatorname*{w-lim}_{\substack{x\to+\infty\\ y\to-\infty}}\int_0^\pi d\alpha\int_0^\pi d\beta[\delta(\tilde{U}(x,0;E)\alpha-\tilde{U}(y,0;E)\beta)dE]\\ &=\frac{1}{\pi^2}\operatorname*{w-lim}_{\substack{x\to+\infty\\ y\to-\infty}}\int_0^\pi d\alpha\int_0^\pi d\beta\int d\theta\delta(\tilde{U}(x,0;E)\alpha-\theta)\delta(\tilde{U}(y,0;E)\beta-\theta)dE)\\ &=\frac{1}{\pi^2}\operatorname*{w-lim}_{\substack{x\to+\infty\\ y\to-\infty}}\int_0^\pi d\theta[\|U(0,x;E\hat{\theta})\|^{-2}\|U(0,y;E)\hat{\theta}\|^{-2}dE].\end{aligned}$$

Similarly, averaging Proposition 2.3(a) over β:

$$\begin{aligned}\sigma_-^\theta(dE)&=\frac{1}{\pi}\operatorname*{w-lim}_{y\to-\infty}\int_0^\pi d\beta[\delta(U(y,0;E)\beta-\theta)dE]\\ &=\frac{1}{\pi}\operatorname*{w-lim}_{y\to-\infty}\|U(0,y;E)\hat{\theta}\|^{-2}dE. \quad \square\end{aligned}$$

We are now ready for

Proof of Lemma 2.2. With $\varphi = \tilde{U}(0,a)\theta$, we have for $x > a$ that

$$\|U(0,x)\hat{\theta}\| = \|U(a,x)[U(0,a)\hat{\theta}]\| = \|U(a,x)\hat{\varphi}\| \, \|U(0,a)\hat{\theta}\| .$$

By hypothesis, $q \in \Omega_0$ is uniformly bounded on $[0,a]$, so by an elementary argument, for each such q, $\|U(0,a)\hat{\theta}\| \geqq C^{-1}$ with C independent of q and of E on compact sets. Thus, by Proposition 2.5:

$$\varrho(dE) \leqq \frac{C^2}{\pi} \, w\text{-lim} \int_0^\pi d\theta \|U(x,a)\hat{\varphi}\|^{-2} \sigma_-^\theta(dE) . \tag{2.6}$$

Fix now $q_0 \equiv q \upharpoonright R \backslash [0,a] \in \Omega_1$ and take the Ω_0-restricted conditional distribution over q with $q \upharpoonright R \backslash [0,a] = q_0$, call it $E_{q_0,\Omega_0}(\,\cdot\,)$. Then, by hypothesis and (2.6),

$$E_{q_0,\Omega_0}(\varrho(dE)) \leqq \frac{C^2}{\pi} w\text{-lim} \int_0^\pi \sigma_-^\theta(dE) d\theta \int \|U(x,a)\hat{\varphi}\|^{-2} G(\varphi) d\varphi ,$$

where G is dependent on E, θ, and q_0. By hypothesis, G is bounded, so

$$E_{q_0,\Omega_0}(\chi_{[-B,B]}(E)\varrho(dE)) \leqq \frac{C_2}{\pi} \, \underset{x\to\infty}{w\text{-lim}} \left[\int_0^\pi \sigma_-^\theta(dE) d\theta \int d\varphi \|U(x,a)\hat{\varphi}\|^{-2}) \right].$$

But by (2.5) with $f=1$, $\int d\varphi \|U(x,a)\hat{\varphi}\|^{-2} = \pi$, so

$$E_{q_0,\Omega_0}(\chi_{[-B,B]}(E)\varrho(dE)) \leqq \tilde{c} \, w\text{-lim} \int_0^\pi \sigma_-^\theta(dE) d\theta .$$

But by Proposition 2.3,

$$\int \sigma_-^\theta(dE) d\theta = \underset{y\to\infty}{w\text{-lim}} \int d\theta \delta(\tilde{U}(y,0;E)\beta - \theta) dE = dE ,$$

so

$$E_{q_0,\Omega_0}(\chi_{[-B,B]}(E)\varrho(dE)) \leqq \tilde{c} dE ,$$

as was to be proven. □

3. Wiggling One Parameter

The most direct way of trying to prove that the conditional distribution of φ is absolutely continuous is to have q depend on one real parameter t which has an a.c. distribution, in which case the map from t to φ needs only have a non-vanishing Jacobian for φ to be absolutely continuous:

Theorem 3.1. *Let $q(x,t)$ be a jointly continuous function on $[0,a] \times [-\delta,\delta]$, so that*

$$\frac{\partial q}{\partial t}(x,t) \geqq 0 ,$$

and for each $t \in [-\delta,\delta]$, $\frac{\partial q}{\partial t}(x_t,t) > 0$ for some $x_t \in [0,a]$. Let $\varphi(\theta_0,t,E)$ be the angle

$\tilde{U}_{q_t}(0,a;E)\theta_0$. *Then, there exists a constant C depending only on B with*

$$C^{-1} < \left|\frac{\partial\varphi}{\partial t}\right| < C$$

for all θ_0, t, E *with* $|E| < B$.

Proof. Let

$$\theta(x,t,E,\theta_0) = \tilde{U}_{q_t}(0,x;E)\theta_0 .$$

Let r be the Prüfer variable discussed in the proof of Proposition 2.3. Then the differential equation for r can easily be integrated to yield

$$r(x) = r(a)\exp\left[\int_x^a (-1+E-q(y,t))\sin 2\theta(y)dy\right]. \tag{3.1}$$

The analog of (2.3) gives $\partial\varphi/\partial t$ explicitly as

$$\frac{\partial\varphi}{\partial t} = -\int_0^a dx\sin^2\theta(x)\frac{r(x)}{r(a)}\frac{\partial q(x,t)}{\partial t}. \tag{3.2}$$

The upper bound is trivial from (3.1), (3.2). To prove the lower bound, we note by (3.1) that $\frac{r(x)}{r(a)} \geqq \exp[-a(\|q\|_\infty + |E| + 1)]$ so that, since the integrand of (3.2) is positive, there is only a problem with the zeros of $\sin\theta(x)$. But since θ obeys

$$\frac{\partial\theta}{\partial x} = \cos^2\theta(x) + (E-q)\sin^2\theta,$$

$\frac{\partial\theta}{\partial x} = 1$ if $\sin\theta(x) = 0$. Thus the zeros of $\sin\theta$ are isolated, so $\frac{\partial\varphi}{\partial t} < 0$. By continuity and compactness, $\partial\varphi/\partial t$ is bounded away from zero. □

Example 1. Let $\{\zeta_n;\ n\in\mathbb{Z}\}$ be i.i.d. random variables with a density $d\kappa(\lambda)$ which has an absolutely continuous component $d\kappa_{ac}(\lambda) = g(\lambda)d\lambda$. Let f be a non-negative continuous function on $\mathbb{R}$ of compact support. Let

$$H_\omega = -\frac{d^2}{dx^2} + \sum_{n\in\mathbb{Z}} \zeta_n(\omega)f(x-n).$$

(This model is discussed in [12] and references quoted there.) Then, for a.e. ω, H_ω has only pure point spectrum with exponential decaying eigenfunctions.

Proof. By translating f, renumbering n, and scaling, we can suppose $\operatorname{supp} f \subset [0,a]$. Fix $\{\zeta_n(\omega_0)\}_{n\neq 0}$ and let

$$q(x,t) = (t+\alpha)f(x) + \sum_{n\neq 0} \zeta_n(\omega_0)f(x-n),$$

where $\alpha\in\operatorname{supp}(d\kappa_{ac})$. By the arguments above, if we restrict ζ_0 to a set S on which g is bounded and $|t| < d$, the required distribution on φ is a.e. with bounded density. Thus, the arguments of Sect. 2 imply the claim. □

Example 2. Let $\{X_x(\omega)\,|\,x\in\mathbb{R}\}$ be a stationary Gaussian process with mean zero and covariance spectral measure $C(d\eta)$ obeying

$$C(d\eta) = \zeta(\eta)d\eta; \qquad \zeta(\eta) \geqq c(1+|\eta|)^{-\varrho}$$

for some $c, \varrho > 0$. Let F be a bounded C^1 function on R which is not constant. Then, for a.e. ω, the random Schrödinger operator

$$H(\omega) = -\frac{d^2}{dx^2} + F(X_x(\omega))$$

has a pure point spectrum with exponentially decaying eigenfunctions.

Proof. Choose an open interval $(\alpha, \beta) \subset \mathbb{R}$ with $F'(x) > 0$ on (α, β) (if F' is only negative, a similar argument will work). Let $\delta = \frac{1}{4}(\beta - \alpha)$ and let

$$\Omega_0 = \{\omega \in \Omega \mid X_x(\omega) \in (\alpha + \delta, \beta - \delta) \text{ for all } x \in [0, a]\}.$$

Then $P(\Omega_0) >$;. Pick a C^∞ function $q_0(x)$ on $\mathbb{R}$ so that $q_0 \equiv 0$ on $\mathbb{R} \backslash [0, a]$ and $0 < q_0(x) < 1$ on $(0, a)$. Let

$$q(x, t) = F(X_x(\omega) + t q_0(x)).$$

Since $q(x, t)$ may not be non-deterministic (though X is), one needs additional arguments to show that the Lyaponov exponent is positive; see [26]. By Theorem 3.1,

$$H(\omega, t) = -\frac{d^2}{dx^2} + F(X_x(\omega) + t q_0(x))$$

has pure point spectrum with exponentially decaying eigenfunctions for a.e. t and a.e. $\omega \in \Omega_0$. The hypotheses on the covariance show that $\{X_x(\omega)\}$ and $\{X_x(\omega) + t q_0(\omega)\}$ are mutually absolutely continuous, so that the desired result is proven. □

Example 3. Let $\{B_x \mid x \in \mathbb{R}\}$ be two-sided Brownian motion on the n-torus T^n and let f be a C^1-nonconstant function on T^n. Then

$$-\frac{d^2}{dx^2} + F(B_x)$$

has pure point spectrum with exponentially decaying eigenfunctions for a.e. B.

Proof. By the Cameron-Martin formula, if $\varphi(x) \in C_0^\infty$, $\{B_x\}$ and $\{B_x + \varphi(x)\}$ are mutually absolutely continuous. Given this, the argument in Example 2 extends. □

By using rotations, one can prove a similar result for Brownian motion on a sphere. Unfortunately, unless the manifold supports many isometries, one cannot use this argument for Brownian motion on a general Riemannian manifold.

4. The GMP Argument Revisited

The first rigorous result on localization was the following theorem proven by Goldshade et al. [10]:

Theorem 4.1. *Let $\{B_x \mid x \in R\}$ be two-sided Brownian motion on a compact Riemannian manifold, M. Let $F: M \to R$ be a C^∞ function with only isolated critical* *points. Then, for a.e. B,*

$$-\frac{d^2}{dx^2}+F(B_x)=H_B \tag{4.1}$$

has only pure point spectrum with exponentially decaying eigenfunctions.

A key element of their proof was an appeal to Hörmander's hypoellipticity theorem to conclude that a certain heat kernel is smooth (we will describe this kernel soon). We had hoped to find a direct argument that would eliminate the appeal to hypoellipticity but, in fact, with one important proviso, the applicability of Theorem 2.1 is essentially equivalent to the boundedness of this integral kernel. Let us begin by sketching a proof of the GMP result:

Sketch of a Proof of Theorem 4.1. Instead of conditioning on the potential $\{F(B_x)\,|\,x\notin[0,a]\}$, we can condition on the actual path $\{B_x\,|\,x\notin[0,a]\}$ and carry all the arguments through. By the properties of Brownian motion, conditioning on $\{B_x\,|\,x\notin[0,a]\}$ is the same as conditioning on $\{B_0\}$ and $\{B_a\}$. Thus, fix $B_0=m_0\in M$, $B_a=m_1\in M$ and $\theta\in[0,\pi)$. For each Brownian path and energy, E, $[B_t\,|\,0\leqq t\leqq a]$ with $B_0=m_0$, $B_a=m_1$, we can solve the Schrödinger equation for (4.1) at energy E with initial condition θ at $x=0$ and obtain an angle ϕ at $x=a$. Averaging over all Brownian paths with the relevant end points gives one precisely

$$P_a(m_0,\theta;m_1,\phi)d\phi\,,$$

where P_a is the heat kernel studied by GMP. The hypoellipticity implies the boundedness needed to apply Theorem 2.1. In fact, the boundedness of the integral kernel is equivalent to the boundedness of this conditional distribution. □

In addition to one major possible extension which we will discuss below, we want to note several ways in which this argument uses less than the full hypoellipticity:

(a) Only boundedness of P_a and not smoothness is required.

(b) One only needs boundedness for some a which is sufficiently large.

We believe that by combining these ideas with the one we will discuss next, one can handle function F with much less regularity than being C^∞.

The major weakening in the hypothesis involves the fact that one need not control all paths but only a subset of paths. This allows one to extend the GMP theorem to treat the case where F is constant on a part of the manifold:

Theorem 4.2. *Theorem 4.1 extends to the case of any nonconstant C^∞ function $F:M\to\mathbb{R}$.*

Proof. Let $M_0\subset M$ be an open set on which $\nabla F\neq 0$ and let $G:M\to\mathbb{R}$ be a C^∞ function with isolated nondegenerate critical points, so that $G\restriction M_0=F\restriction M_0$. Let

$$\Omega_0=\{B_x\,|\,B_x\in M_0 \text{ for } 0\leqq x\leqq a\}\,.$$

Then the restricted conditional expectation of ϕ, conditioned over Ω_0 and multiplied by $P(\Omega_0)$ is given by a path integral which is the same for both G and F. It is therefore dominated by the unrestricted conditional expectation for G, i.e. by $P_a^{(G)}(m_0,\theta;m_1,\phi)d\phi$ which is bounded by hypoellipticity. □

5. An N-Dimensional Result

In this section, we will discuss a different approach to continuum models which works in higher dimensions, but which applies only to the somewhat artificial model which most closely resembles the discrete case. It will illustrate the fact that the method of Simon and Wolff [23] depends primarily on positivity of the perturbation and not very much on its finite rank or even compactness. We are heading towards a proof of the following:

Theorem 5.1. *Fix* ν, *and for* $n\in\mathbb{Z}^\nu$, *let* χ_n *be the characteristic function of those* $x\in\mathbb{R}^\nu$ *with* $n_j-\frac{1}{2}\leqq x_j<n_j+\frac{1}{2}$. *For* $\{\lambda_n\,|\,n\in\mathbb{Z}^\nu\}$, *a set of uniformly bounded reals, define*

$$H(\lambda)=-\Delta+\sum_{n\in\mathbb{Z}}\lambda_n\chi_n. \tag{5.1}$$

Fix $\varphi\in L^2(\operatorname{supp}\chi_0)$ *and let* $\{\lambda_n\}_{n\neq 0}$ *be fixed. Let* $d\mu_{\lambda_0}$ *be the spectral measure for* $H(\lambda)$ *and the vector* φ. *Then*

$$\int\frac{d\lambda}{1+\lambda^2}\,d\mu_\lambda(E)\leqq c\,dE. \tag{5.2}$$

Lemma 5.2. *Let* $C=A+iB$ *with* A,B *bounded and selfadjoint. Suppose that* $B\geqq d1$. *Then* C *is invertible and*

$$\|C^{-1}\|\leqq d^{-1}.$$

Proof. Write $C=B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}+i)B^{\frac{1}{2}}$. B is invertible, and by the spectral theorem applied to $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$, $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}+i$ is invertible with

$$\|(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}+i)^{-1}\|\leqq 1.$$

Thus

$$\|C^{-1}\|\leqq\|B^{-\frac{1}{2}}\|^2=d^{-1}.\quad\square$$

Proof of Theorem 5.1. Let $A(\lambda)=H(\{\lambda_n\}_{n\neq 0},\ \lambda_0=\lambda)$, and let $R(\lambda,z)=\chi_0(A(\lambda)-z)^{-1}\chi_0$ as an operator on $L^2(\operatorname{supp}\chi_0)\equiv\mathscr{H}$, where $\operatorname{Im}z>0$. Then, by the second resolvent equation

$$R(\lambda,z)=R(0,z)-\lambda R(\lambda,z)R(0,z)$$

or, formally,

$$R(\lambda,z)=R(0,z)(1+\lambda R(0,z))^{-1}. \tag{5.3}$$

Suppose $\operatorname{Im}\lambda<0$. Then $\operatorname{Im}(\lambda^{-1}+R(0,z))\geqq\operatorname{Im}(\lambda^{-1})$, so by the lemma, $(1+\lambda R(0,z))$ is invertible, and for $\tilde\lambda$ real and $\operatorname{Im}z>0$, it is not hard to show (by the lemma again) that $(H(\tilde\lambda)-z)^{-1}=\lim_{\varepsilon\downarrow 0}(H(\tilde\lambda-i\varepsilon)-z)^{-1}$. Thus (5.3) holds for $\operatorname{Im}\lambda<0$ and boundary values can be taken so

$$\int_{-\infty}^{\infty}\frac{d\lambda}{(1+\lambda^2)}R(\lambda,z)$$

can be evaluated by closing a contour in the lower half-plane. The result is

$$\int_{-\infty}^{\infty}(1+\lambda^2)^{-1}R(\lambda,z)d\lambda=-\pi(-R(0,z)^{-1}+i)^{-1}$$

$\left(\text{where } R(0,z)^{-1} \text{ is interpreted as } \lim_{\varepsilon\downarrow 0} R(-i\varepsilon,z)^{-1}\right)$. By the lemma, we see that

$$\left\| \int_{-\infty}^{\infty} (1+\lambda^2)^{-1} R(\lambda,z)d\lambda \right\| \leqq \pi\,,$$

and that

$$\left| \int_{-\infty}^{\infty} \frac{d\lambda}{1+\lambda^2} \int \frac{1}{E-z} d\mu_\lambda(E) \right| \leqq \pi\,,$$

which implies (5.2). □

With this result and previous work, one can obtain localization of states in certain multidimensional models, a result already proven by Fröhlich, Martinelli, Scoppola, and Spencer [9] using different methods. We will require the following result which combines ideas of Holden and Martinelli [11] and Martinelli and Scoppola [17]:

Theorem 5.3. *Consider the Hamiltonian* (5.1), *where* $\{\lambda_n\}$ *are i.i.d.'s with distribution* $g(\lambda)d\lambda$ *with* $g \in L^\infty$ *and* supp g *bounded. Then, for each sufficiently small energy, E:*
For a.e. λ, *every polynomially bounded solution of* $H(\lambda)u = Eu$ *is exponentially decaying.*

Given this result, we can prove:

Theorem 5.4. *Consider the Hamiltonian* (5.1), *where* $\{\lambda_n\}$ *are i.i.d.'s with distribution* $g(\lambda)d\lambda$ *with* $g \in L^\infty$ *and* supp g *bounded. Then for all sufficiently small energies, H has only pure point spectrum with exponentially decaying eigenfunctions.*

Proof. Follow the proof of Theorem 2.1 replacing the Furstenburg and Osceledec theorem by Theorem 5.3 and replacing Lemma 2.2 by Theorem 5.1. □

6. A Continuum Wegner-Type Theorem

In this section, we want to show how to obtain an estimate on the integrated density of states, $k(E)$, from Theorem 5.1 of the following form:

Theorem 6.1. *Let* $H_\omega = -\Delta + q_\omega$ *on* $L^2(\mathbb{R}^\nu)$, *where*

$$q_\omega(x) = \sum_n W_\omega(n)\chi_n(x)$$

and the $W_\omega(n)$ *are i.i.d.'s with distribution* $g(\lambda)d\lambda$. *Suppose that* $\|g\|_\infty < \infty$ *and that g has compact support. Then, for each R, there exists* C_R *with*

$$|k(E)-k(E')| \leqq C_R|E-E'| \quad \textit{if} \quad |E|, |E'| \leqq R\,.$$

Remark. C_R is only a function of R, $\|g\|_\infty$, and (supp$g$).

This result is an analog of a celebrated result of Wegner [24]; to explain where it comes from, it is perhaps worth explaining how to recover Wegner's precise theorem from the calculation of Simon-Wolff.

Theorem 6.2 (Wegner [24]). *Let $k(E)$ be the integrated density of states for a ν-dimensional Anderson model with potential density $g(\lambda)d\lambda$ with $\|g\|_\infty$ bounded. Then*

$$|k(E)-k(E')| \leqq \|g\|_\infty |E-E'|.$$

Proof. Suppose first that g has compact support. Let $d\mu_{\lambda,\{v_i\}_{i\neq 0}}$ be the spectral measure for H with potential $v(n)=v_n$ $(n\neq 0)$ and $v(0)=\lambda$. By the calculation of Simon-Wolff:

$$\int \frac{d\lambda}{1+\varepsilon\lambda^2} d\mu_{\lambda,\{v_i\}_{i\neq 0}}(E) \leqq dE$$

for any ε. It follows that

$$\int g(\lambda) d\lambda d\mu_{\lambda,\{v_i\}_{i\neq 0}}(E) \leqq \|g\|_\infty dE,$$

$(p_L^2+1)^{-\alpha}$ is trace class and uniformly in L:

$$\|\chi_n(p^2+1)^{-\alpha}\chi_m\|_1 \equiv \text{Trace class norm of } (\chi_n(p^2+1)^{-\alpha}\chi_m) \leqq ce^{-d|n-m|}.$$

To see this, one need only use the Payley-Weiner theorem to note that the integral kernel $K(x,y)$ of $(p^2+1)^{-\alpha/2}$ is a function of $(x-y)$ with

$$\int |e^{d|x|} f(x)|^2 dx < \infty.$$

Thus, the Hilbert-Schmidt norm obeys

$$\|\chi_n(p^2+1)^{-\alpha}\chi_l\|_2 \leqq c_1 e^{-d|n-m|},$$

from which the above estimate follows. Thus

$$\begin{aligned} &\text{Exp}(\text{Tr}[(p^2+1)^{-\alpha}\,\text{Im}(H_L-z)^{-1}]) \\ &= \sum_{n,m} \text{Exp}(\text{Tr}(\chi_m(p^2+1)^{-\alpha}\chi_n\,\text{Im}(H-z)^{-1}\chi_m) \\ &\leqq \sum_{n,m} \|\chi_m(p^2+1)^{-\alpha}\chi_n\|_1 \|\text{Exp}(\chi_n\,\text{Im}(H-z)^{-1}\chi_m)\| \\ &\leqq cL^\nu. \end{aligned}$$

Now, if $E_{(a,b)}$ is the spectral projection,

$$E_{(a,b)}(A) \leqq \pi^{-1} \lim_{\varepsilon\downarrow 0} \int_e^b (A-x-i\varepsilon)^{-1} dx,$$

so

$$\text{Exp}\,\text{Tr}((p^2+1)^{-\alpha}E_{(a,b)}(H_L)) \leqq cL^\nu(b-a).$$

Let ψ_n^L and e_n^L be the eigenfunctions and eigenvalues of H_L. Suppose $|e_n^L|<R$; then, by Hölder's inequality

$$\begin{aligned} 1 &= (\psi_n^L, \psi_n^L) \leqq (\psi_n^L, (p^2+1)^{-\alpha}\psi_n^L)(\psi_n^L, (p^2+1)^\alpha\psi_n^L) \\ &\leqq (R+1+\|V\|_\infty)^\alpha(\psi_n^L, (p^2+1)^{-\alpha}\psi_n^L) \\ &\equiv d(\psi_n^L, (p^2+1)^{-\alpha}\psi_n^L). \end{aligned}$$

Thus, if $|a|, |b| < R$:

$$\text{Tr}(E_{(a,b)}(H_L)) = \sum_{a \leqq e_n^L \leqq b} 1.$$

But $dk(E) = \mathrm{Exp}(d\mu_v(E))$, so taking expectations over $\{v_i\}_{i \neq 0}$ in this last expression, we obtain

$$dk \leqq \|g\|_\infty dE.$$

By a limiting argument, one obtains the case of general g. □

Proof of Theorem 6.1. Place the system in a box of side L with periodic boundary conditions. The proof of Theorem 5.1 shows that, for any values of $\{W(n)\}_{n \neq 0}$,

$$\left\| \int \frac{d\lambda}{(1+\lambda^2)} \chi_0 (H_L + \lambda\chi_0 - z)^{-1} \chi_0 \right\| \leqq \pi,$$

$$\| \int g(\lambda) d\lambda \chi_0 (H_L + \lambda\chi_0 - z)^{-1} \chi_0 \| \leqq c.$$

Thus, averaging over $\{W(n)\}_{n \neq 0}$,

$$\| \mathrm{Exp}[\chi_0 (H_L - z)^{-1} \chi_0] \| \leqq c.$$

Since zero is not special

$$\| \mathrm{Exp}\, \chi_n (H_L - z)^{-1} \chi_n \| \leqq c$$

for each n. Since $\mathrm{Im}(H_L - z)^{-1} \equiv A(z, \omega)$ is positive, we have

$$(\varphi, \chi_n A(z,\omega) \chi_m \varphi) \leqq (\varphi, \chi_n A(z,\omega) \chi_n \varphi)^{\frac{1}{2}} (\varphi, \chi_m A(z,\omega) \chi_m)^{\frac{1}{2}},$$

so by the Schwarz inequality (on the probability space):

$$\| \mathrm{Exp}\, \chi_n \,\mathrm{Im}(H_L - z)^{-1} \chi_m \| \leqq c$$

for all n, m.

Now let p_L be the momentum in the box. Then, for $\alpha > \nu/2$, $(p_L^2 + 1)^{-\alpha}$ is trace class and uniformly in L:

$$\| \chi_n (p^2 + 1)^{-\alpha} \chi_m \|_1 \equiv \text{Trace class norm of } (\chi_n (p^2+1)^{-\alpha} \chi_m) \leqq c e^{-d|n-m|}.$$

To see this, one need only use the Payley-Weiner theorem to note that the integral kernel $K(x, y)$ of $(p^2+1)^{-\alpha/2}$ is a function of $(x-y)$ with

$$\int |e^{d|x|} f(x)|^2 dx < \infty.$$

Thus, the Hilbert-Schmidt norm obeys

$$\| \chi_n (p^2+1)^{-\alpha} \chi_l \|_2 \leqq c_1 e^{-d|n-m|},$$

from which the above estimate follows. Thus

$$\begin{aligned} &\mathrm{Exp}(\mathrm{Tr}[(p^2+1)^{-\alpha} \mathrm{Im}(H_L - z)^{-1}]) \\ &\quad = \sum_{n,m} \mathrm{Exp}(\mathrm{Tr}(\chi_m (p^2+1)^{-\alpha} \chi_n \mathrm{Im}(H-z)^{-1} \chi_m) \\ &\quad \leqq \sum_{n,m} \| \chi_m (p^2+1)^{-\alpha} \chi_n \|_1 \| \mathrm{Exp}(\chi_n \mathrm{Im}(H-z)^{-1} \chi_m) \| \\ &\quad \leqq c L^\nu . \end{aligned}$$

Now, if $E_{(a,b)}$ is the spectral projection,

$$E_{(a,b)}(A) \leqq \pi^{-1} \lim_{\varepsilon \downarrow 0} \int_e^b (A - x - i\varepsilon)^{-1} dx,$$

so

$$\operatorname{Exp}\operatorname{Tr}((p^2+1)^{-\alpha}E_{(a,b)}(H_L))\leqq cL^\nu(b-a).$$

Let ψ_n^L and e_n^L be the eigenfunctions and eigenvalues of H_L. Suppose $|e_n^L|<R$; then, by Hölder's inequality

$$\begin{aligned}1&=(\psi_n^L,\psi_n^L)\leqq(\psi_n^L,(p^2+1)^{-\alpha}\psi_n^L)(\psi_n^L,(p^2+1)^{\alpha}\psi_n^L)\\&\leqq(R+1+\|V\|_\infty)^\alpha(\psi_n^L,(p^2+1)^{-\alpha}\psi_n^L)\\&\equiv d(\psi_n^L,(p^2+1)^{-\alpha}\psi_n^L).\end{aligned}$$

Thus, if $|a|,|b|<R$:

$$\begin{aligned}\operatorname{Tr}(E_{(a,b)}(H_L))&=\sum_{a\leqq e_n^L\leqq b}1\\&\leqq d\sum_{c\leqq e_n^L\leqq b}(\psi_n^L,(p^2+1)^{-\alpha}\psi_n^L)\\&=d\operatorname{Tr}((p^2+1)^{-\alpha}E_{(a,b)}(H_L))\\&\leqq cdL^\nu|b-a|.\end{aligned}$$

Thus, dividing by L^ν and taking $L\to\infty$:

$$|k(b)-k(a)|\leqq cd|b-a|,$$

as was to be proven. □

Remarks. 1. The contortions involving putting in $(p^2+1)^{-\alpha}$ and then taking it out are needed because it is natural to average the resolvent, but the resolvent is not trace class due to high energy contributions.

2. In some ways, it might seem more natural to cut off the high energies using the interacting resolvents. But this makes the averaging harder, so we use the free Hamiltonian p^2.

3. One can extend the above argument to show that if $g\in L^p$ (with compact support) (rather than L^∞) and $p>1$, then $k(E)$ is Hölder continuous of order $q^{-1}=1-p^{-1}$. The basic idea is to note that $\operatorname{Im}[\int d\mu(E)/(E-z)]\leqq c(\operatorname{Im}z)^{-1+\theta}$ implies $\left|\int_a^b d\mu\right|\leqq c|b-a|^\theta$. When we control $\int\frac{d\lambda}{1+\lambda^2}R(\lambda,z)$ we deduce a bound:

$$\int g(\lambda)d\lambda(\varphi,\operatorname{Im}R(\lambda,z)\varphi)\leqq\|g\|_{L^p}\left(\int\frac{d\lambda}{(1+\lambda^2)}|(\varphi,\operatorname{Im}R(\lambda,z)\varphi)|^q\right)^{1/g},\qquad(6.4)$$

where the L^p norm is with respect to a measure $(1+\lambda^2)^{p/q}d\lambda$. Since $\|R(\lambda,z)\|\leqq(\operatorname{Im}z)^{-1}$ we find that

$$\text{RHS of }(6.1)\leqq c(\operatorname{Im}z)^{-(q-1)/q}.$$

Appendix

Lyaponov Decay is a Tail Event. In this appendix, we will prove the following result which we need in Sect. 2, which we feel is of wider interest:

Theorem A.1. *Let $q_\omega(x)$ be an ergodic stochastic process with $|q_\omega(x)| \leqq C_\omega(1+|x|^2)$ with probability* 1 *and let $H_\omega = -\frac{d^2}{dx^2} + q_\omega(x)$. Then the probability that the spectral measure for H_ω is supported on the set of energies, E, where the transfer matrix has Lyaponov decay is either* 0 *or* 1.

Remarks. 1. The hypothesis $|q_\omega| \leqq C_\omega(1+|x|^2)$ is not essential. A similar bound on $\left(\int_{x-1}^{x+1} |q_\omega(y)|^2 dy\right)^{1/2}$ would certainly suffice.

2. Actually, the theorem is only colloquially started. In order to talk of the probability of the set in question, one needs to show it is measurable, but by ergodicity, once one has the measurability, the probability 0 and 1 is immediate. A more proper statement would say that the set in question is measurable (see Lemma A.3 below).

Define $U_\omega(E,x)$ to be the transfer matrix for H_ω and $\gamma(E) = \lim_{|x|\to\infty} |x|^{-1} \ell n \| U_\omega(E,x)\|$ for a typical ω. Also define $\gamma_+(\omega,E) = \overline{\lim}_{|x|\to\infty} |x|^{-1} \ell n \| U_\omega(E,x)\|$ and $\gamma_-(\omega,E)$ with $\overline{\lim}$ replaced by $\underline{\lim}$. Since the limits can be taken through subsequences and $U_\omega(E,x)$ is jointly measurable in E, ω, x:

Lemma A.2. *$\gamma_\pm(\omega,E)$ and $\gamma(E)$ are measurable (jointly in ω, E and in E).*

Let $\varrho_\omega(dE)$ be the spectral measure of H_ω as defined in Sect. 2. Define

$$A_\omega = \{E \mid \gamma_+(\omega,E) = \gamma_-(\omega,E) = \gamma(E)\}.$$

A_ω is measurable in E by Lemma A.2.

The key fact we need is

Lemma A.3. *$\varrho_\omega(A_\omega^c)$ is a measurable function of ω.*

Proof of Theorem A.1 Given Lemma A.3. Let T_x be the family of translations. It is trivial that $\gamma_\pm(T_x\omega, E) = \gamma_\pm(\omega,E)$, so $A_{T_x\omega} = A_\omega$. Moreover, ϱ_ω and $\varrho_{T_x\omega}$ are mutually equivalent as spectral measures for H_ω, and thus

$$\{\omega \mid \varrho_{T_x\omega}(A_{T_x\omega}^c) = 0\} = \{\omega \mid \varrho_\omega(A_\omega^c) = 0\} \equiv \tilde{\Omega}.$$

By ergodicity, $\tilde{\Omega}$ either has measure 0 or 1. □

To prove Lemma A.3, we need a little bit of measure theory. We call a family of finite measures μ_ω on $\mathbb{R}$ indexed by $\omega \in \Omega$ measurable if and only if $\int e^{itx} d\mu_\omega(x)$ is jointly measurable in ω and t.

Proposition A.4. (a) *If μ_ω are measurable in ω and F is a bounded measurable function, then $\int F(x) d\mu_\omega(x)$ is measurable in ω.*

(b) *If μ_ω are measurable in ω and $F(x,\omega)$ is jointly measurable in (x,ω), then $\omega \mapsto \int F(x,\omega) d\mu_\omega(x)$ is measurable in ω.*

Proof. (a) Let $\mathscr{F}$ be the set of bounded measurable functions for which $\omega \mapsto \int F(x) d\mu_\omega(x)$ is measurable. By hypothesis, $e^{ixt} \in \mathscr{F}$ for all t and clearly if $F_n \in \mathscr{F}$, $\sup_n |F_n(x)| < \infty$, and $F_n(x) \to F(x)$, then $F \in \mathscr{F}$. It follows that all bounded measurable functions are in $\mathscr{F}$ since $\mathscr{F}$ is the smallest set closed under the indicated limits containing all e^{ixt}.

(b) Let $\mathscr{F}_0$ be the set of all $\mathscr{F}$ with $\omega \mapsto \int F(x,\omega)d\mu_\omega(x)$ bounded. $\mathscr{F}_0$ is closed under the same limits as in part (a) and, by part (a), contains any finite sum of product function $g(x)h(\omega)$ with g, h measurable. Thus, by the definition of the Borel class of functions, $\mathscr{F}_0$ is all Borel functions. □

Proof of Lemma A.3. We must begin by coping with the fact that $d\varrho_\omega$ is not a finite measure. However,

$$(1+\varepsilon E^2)^{-1} d\varrho_\omega(E) \equiv d\mu_\omega^{(\varepsilon)}(E)$$

is a finite measure and it is measurable in ω, since $e^{-itH_\omega}(\varepsilon H_\omega^2+1)^{-1}$ has a measurable integral kernel. Let $F(\omega, E)$ be the characteristic function of A_ω^c. F is measurable by Lemma A.2 so, by Proposition A.4, $\int F(\omega, E)d\mu_\omega^{(\varepsilon)}(E)$ is measurable, and thus

$$\varrho_\omega(A_\omega^c) = \lim_{\varepsilon \downarrow 0} \mu_\omega^{(\varepsilon)}(A_\omega^c)$$

is measurable. □

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