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Krein's Spectral Shift Function and Fredholm Determinants as Efficient Methods to Study Supersymmetric Quantum Mechanics

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Abstract. A new method is proposed to study supersymmetric quantum mechanics. A basic relation is derived between Krein's spectral shift function and the Witten index as a powerful tool for explicit model investigations. The topological invariance of relevant quantities like the index, the anomaly, the spectral asymmetry and the spectral shift function is proved. As an illustration, some model calculations are presented, in particular the two-dimensional magnetic field problem, without assuming the magnetic flux to be quantized.

There has been considerable interest in fractionally charged stated during the last ten years [1]. It all started in field theory with the observation that soliton-monopole systems in the presence of Fermi fields show fractionalization of the soliton fermion number [2]. Now one knows that these fractionally charged states are phenomenologically realized in the physics of linearly conjugated polymers such as polyacetylene [3].

One of the approaches to study these phenomena is to build models starting from a Dirac operator with some external potential with nontrivial spatial asymptotics and to look at its zero modes [1]. This method is intimately connected with supersymmetry, an object of current interest in different fields of physics [4]. In this respect the investigation of supersymmetric quantum mechanical models is important. Such models serve as a laboratory for testing and understanding supersymmetry breakdown in realistic field theories [4, 5]. Furthermore, they provide a simple recipe for generating partner

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potentials, which can be used successfully in many physical problems, e.g., to substantially improve the convergence of large-N expansions [6].

In this Letter we propose a new method to study supersymmetric quantum mechanics.

We consider a general supersymmetric quantum mechanical system with Hamiltonian H and supercharge Q where

$$Q = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}, \qquad H = Q^2 = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, \quad H_1 = A^*A, H_2 = AA^*. \tag{1}$$

Two relevant quantities to be investigated are Witten's (resolvent) regularized index [7, 8], Δ, and the axial anomaly [8, 9], $\mathscr{A}$, given by

$$\Delta = \lim_{z \to 0} \Delta(z), \qquad \mathscr{A} = -\lim_{z \to \infty} \Delta(z), \tag{2}$$

$$\Delta(z) = -z \operatorname{Tr}[(H_1 - z)^{-1} - (H_2 - z)^{-1}], \quad \operatorname{Im} z \neq 0 \tag{3}$$

assuming that the trace on the r.h.s. of (3) exists. When A is Fredholm [10] we can immediately show, using the Laurent series around $z = 0$ for the resolvents appearing in (3), that the Witten index Δ equals the Fredholm index $i(A)$ [11]. They both precisely describe the difference in the number of bosonic and fermionic zero-energy states counting multiplicity (cf. also [12]).

An interesting question is then what happens if A is not Fredholm, e.g., in models were zero-energy resonances occur, in two-dimensional magnetic field problems, etc. (see [1, 9] and references therein). To answer this question in general, we introduce Krein's spectral shift function associated with the pair (H_1, H_2) [13]. In this context one establishes the existence of a real valued function ξ_{12} on $\mathbb{R}$, unique up to a constant, such that

$$\operatorname{Tr}[(H_1 - z)^{-1} - (H_2 - z)^{-1}] = -\int_{\mathbb{R}} d\lambda\, \xi_{12}(\lambda)(\lambda - z)^{-2}. \tag{4}$$

In the supersymmetric systems we consider, H_1 and H_2 are nonnegative. Furthermore, they have the same nonzero energy bound states with the same multiplicities [14]. Finally, the bottoms of the essential spectra of H_1 and H_2 coincide. Denoting the latter by Σ we infer that the spectral shift function may be chosen uniquely as

$$\xi_{12}(\lambda) = \begin{cases} 0, & \lambda > 0, \\ \xi_{12}(0_+), & \lambda > \Sigma, \\ -(2\pi i)^{-1} \ln \det S_{12}(\lambda), & \lambda > \Sigma, \end{cases} \tag{5}$$

where $S_{12}(\lambda)$ is the on-energy-shell S-matrix for the scattering system (H_1, H_2). (Under suitable conditions on the interaction V_{12}, defined by $V_{12} \equiv H_1 - H_2$, $\xi_{12}(\lambda)$ is continuous for $\lambda > \Sigma$.)

If the potential V_{12} can be factorized as $V_{12} = v_{12}u_{12}$ whereby $u_{12}(H_2 - z)^{-1}v_{12}$ has certain analyticity properties in z, then also Fredholm determinants [15–17] can be used to calculate $\xi_{12}(\lambda)$. In particular, on the basis of (4), especially the connection between its l.h.s. and Fredholm determinants, and standard properties of the Poisson kernel one can show that [12]

$$[\xi_{12}(\lambda_+) + \xi_{12}(\lambda_-)]/2 = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \ln \left\{ \frac{\det[1 + u_{12}(H_2 - \lambda - i\varepsilon)^{-1} v_{12}]}{\det[1 + u_{12}(H_2 - \lambda + i\varepsilon)^{-1} v_{12}]} \right\}. \tag{6}$$

We now use these methods to establish some new, general results on supersymmetric quantum mechanical systems, even when the associated operator A is non-Fredholm. First, let $\xi_{12}(\lambda)$ be bounded and piecewise continuous in $\mathbb{R}$ (see (5)), then one can prove [12], using (2)–(4), the following basic relations

$$\Delta = -\xi_{12}(0_+), \qquad \mathscr{A} = \xi_{12}(\infty). \tag{7}$$

Not only are these relations of some theoretical importance, they also turn out to be effective tools in explicit model calculations, as we will demonstrate later on. If $\Sigma > 0$ then $(-\xi_{12}(0_+))$ describes precisely the difference of zero-energy bound states of H_1 and H_2 counting multiplicity. It is thus equal to the Fredholm index $i(A)$, in agreement with what we have said before. If $\Sigma = 0$, such that the Fredholm property breaks down, then $\xi_{12}(0_+)$, and consequently Δ, can be fractional or even arbitrary real as we will see in the final examples.

Secondly, one can prove an important invariance property for $\Delta(z)$[11]. Let B be a relatively compact perturbation of A and define $A(\beta) \equiv A + \beta B$, β real. If in addition B fulfills a relative trace-class condition, technically speaking if, e.g., $[B \exp(-|A|)]$ is assumed to be trace-class, then we have (with obvious notation)

$$\Delta(z, \beta) = \Delta(z). \tag{8}$$

We emphasize the importance of the additional trace-class condition. For example, in the case of two-dimensional magnetic fields discussed at the end, perturbations B of A which destroy the magnetic flux of the system involved will, in general, be relatively compact but not relatively trace-class. Hence, the invariance result (8) will fail in general. It is the relative trace-class condition that distinguishes the distinct physical situations.

The idea of the proof [11] of (8) is based upon the fact that the trace appearing in (3), but with H_1, H_2 replaced by $H_1(\beta)$, $H_2(\beta)$ is differentiable with respect to β and that, because of certain commutation formulas [14], this derivative can be shown to be zero. The result (8) yields the topological invariance of the (resolvent) regularized $\Delta(z)$. Moreover, it proves the topological invariance of Δ and $\mathscr{A}$. When A is Fredholm, the invariance of the index $i(A)$ and thus of Δ under relatively compact perturbations is a standard result [18]. But the above result also works for A being not Fredholm. Furthermore, it implies the invariance of Krein's spectral shift function, viz.

$$\xi_{12}(\lambda, \beta) = \xi_{12}(\lambda). \tag{9}$$

Indeed, using (8), (3) and the connection between the r.h.s. of (3) and Fredholm determinants, one shows [12] that the Fredholm determinant itself is invariant such that (6) leads to the result (9). These results can also be used to substantially simplify supersymmetric model calculations. We further remark that also models of the type

$$Q_m = \begin{pmatrix} m & A^* \\ A & -m \end{pmatrix}, \qquad H_m = Q_m^2 = \begin{pmatrix} H_1 + m^2 & 0 \\ 0 & H_2 + m^2 \end{pmatrix}, \quad m \in \mathbb{R}\backslash\{0\} \tag{10}$$

can be treated analogously. For example, one can prove that the corresponding spectral asymmetry η_m (see, e.g., [1, 19]) is a topological invariant, and that it can be expressed directly in terms of ξ_{12} [11, 12]

$$\begin{aligned} \eta_m &= \lim_{t \to 0_+} \eta_m(t) = -(m/2) \int_0^\infty d\lambda \xi_{12}(\lambda)(\lambda + m^2)^{-3/2}, \\ \eta_m(t) &= \mathrm{Tr}[Q_m H_m^{-1/2} e^{-tH_m}] \\ &= m \int_0^\infty d\lambda \xi_{12}(\lambda) \frac{d}{d\lambda} [(\lambda + m^2)^{-1/2} e^{-t(\lambda + m^2)}], \quad m \in \mathbb{R}\backslash\{0\}. \end{aligned} \tag{11}$$

Next, we present a short discussion of some examples illustrating our method. We start with a one-dimensional model first discussed by Callias [8] and since then reconsidered by many authors

$$\begin{aligned} &A = d/dx + \phi, \lim_{x \to \pm\infty} \phi(x) = \phi_\pm, \quad \phi_-^2 \leqslant \phi_+^2, \\ &H_j = -d^2/dx^2 + \phi^2 + (-1)^j \phi', \quad j = 1, 2, x \in \mathbb{R}. \end{aligned} \tag{12}$$

Then it is known that

$$\Delta(z) = [\phi_+(\phi_+^2 - z)^{-1/2} - \phi_-(\phi_-^2 + z)^{-1/2}] \tag{13}$$

and, e.g., in the case that $\phi_- = 0$, $\phi_+ \neq 0$

$$\Delta = [\mathrm{sgn}(\phi_+)]/2.$$

Here the fractionization of Δ is due to a zero-energy (threshold) resonance of H_1. In addition one can calculate that [12]

$$\begin{aligned} \xi_{12}(\lambda) = \pi^{-1}\{&\theta(\lambda - \phi_+^2)\arctan[(\lambda - \phi_+^2)^{1/2}/\phi_+] - \\ &- \theta(\lambda - \phi_-^2)\arctan[(\lambda - \phi_-^2)^{1/2}/\phi_-]\} + \theta(\lambda)[\mathrm{sgn}(\phi_-) - \mathrm{sgn}(\phi_+)]/2 \end{aligned} \tag{14}$$

implying (13). (In (14) $\theta(x) = 1$ for $x \geqslant 0$ and $\theta(x) = 0$ for $x < 0$ and $\mathrm{sgn}(x) = \pm 1$ for $x \gtrless 0$ and $\mathrm{sgn}(0) = 0$.) Our calculation is efficient in the following respects. It is based on the use of Fredholm determinants (see (6)) which become simple Wronski determinants in one dimension, viz.

$$\begin{aligned} &\det[1 - 2|\phi'|^{1/2}\mathrm{sgn}(\phi')(H_2 - z)^{-1}|\phi'|^{1/2}] \\ &\quad = [W(f_{1-}(z), f_{1+}(z))][W(f_{2-}(z), f_{2+}(z))]^{-1}, \end{aligned} \tag{15}$$

where the $f_{j\pm}$, $j = 1, 2$ are Jost solutions associated with H_j and W denote the Wronskians [16, 20–22]. It exploits explicitly supersymmetry as expressed by the following relation [20]

$$W((Af_{1-})(z), (Af_{1+})(z)) = zW(f_{1-}(z), f_{1+}(z)). \tag{16}$$

This trick avoids, e.g., the use of additional comparison Hamiltonians. (Compare, e.g., [22].) We finally remark that, in agreement with our general results, we see that the quantities given by (13), (14) are topologically invariant since they only depend on the asymptotic values $\phi_\pm$ of $\phi(x)$ and not on its local properties.

This treatment can be extended to N-dimensional spherically symmetric systems, even with long-range interactions [12]. In that case, the calculation of the spectral shift function can be substantially simplified by using its topological invariance, as described in detail in [12]. This we also see in the last example we study here, i.e., the following two-dimensional magnetic field model:

$$\begin{aligned} &A = -(i\partial_1 + a_1) + i(i\partial_2 + a_2), \quad a = (\partial_2\phi, -\partial_1\phi), \partial_j = \partial/\partial x_j, j = 1, 2, \\ &\phi(\mathbf{x}) = -F\ln|\mathbf{x}| + C + O(|\mathbf{x}|^{-\varepsilon}) \quad \text{for } |\mathbf{x}| \to \infty, F \in \mathbb{R}. \end{aligned} \tag{17}$$

Then

$$\begin{aligned} &H_j = [(-i\nabla - a)^2 - (-1)^j b], \quad j = 1, 2, \\ &b(\mathbf{x}) = (\partial_1 a_2 - \partial_2 a_1)(\mathbf{x}) = -(\Delta\phi)(\mathbf{x}), \quad x \in \mathbb{R}^2 \end{aligned} \tag{18}$$

and the magnetic flux, F, is given by

$$F = (2\pi)^{-1} \int_{\mathbb{R}^2} d^2x\, b(\mathbf{x}). \tag{19}$$

These type of models are frequently used in the literature, e.g., in connection with gauge theories to study the nature of the Dirac spectrum in the presence of localized gauge vortices. (See, e.g., [1, 9, 23–27].) Here we prove, using only scaling arguments and the topological invariance of the spectral shift function, that

$$\xi_{12}(\lambda) = F\theta(\lambda), \qquad \Delta(z) = \Delta = -F, \qquad \mathscr{A} = F, \tag{20}$$

even when the flux is not quantized. To see this, we introduce a specific rotationally symmetric model for the magnetic field by putting [24]

$$\phi(\mathbf{x}) = \phi(r, R) = \begin{cases} -Fr^2/2R^2, & |\mathbf{x}| = r \leqslant R, R > 0, \\ -F/2[1 + \ln(r^2/R^2)], & r \geqslant R. \end{cases} \tag{21}$$

Then it is easy to check the following scaling property for H_j (now depending on R)

$$U_\varepsilon H_j(R) U_\varepsilon^{-1} = \varepsilon^2 H_j(\varepsilon R), \qquad (U_\varepsilon g)(x) = \varepsilon^{-1} g(x/\varepsilon), \quad \varepsilon > 0 \tag{22}$$

(U_ε is the unitary group of dilations in two dimensions, g is a square integrable function). This immediately implies that

$$S_{12}(\lambda, R) = S_{12}(\varepsilon^2\lambda, R/\varepsilon), \qquad \xi_{12}(\lambda, R) = \xi_{12}(\varepsilon^2\lambda, R/\varepsilon), \quad \lambda > 0. \tag{23}$$

Recalling now the topological invariance of ξ_{12} (see (9)) we infer that ξ_{12} cannot depend on R as long as F is kept fixed in (21). Therefore, (23) implies that ξ_{12} and, consequently, also $\Delta(z)$ are energy independent. So it suffices to calculate these quantities, e.g., at high energies, where this calculation is straightforward and can be done in either of two ways: one way using resolvent equations and trace estimates, the other way using the heat kernel index theorem type method [12]. This gives the results (20). We remark that the result in (20) for the Witten index Δ has been obtained in [27] in a path-integral approach by using certain approximations. Our treatment is the first nonperturbative and rigorous one and it works for all values of the flux F. For the situation described in Equation (10), the corresponding regularized spectral asymmetry (cf. Equations (11)) reads

$$\eta_m(t) = \operatorname{sgn}(m)F\,e^{-tm^2}, \quad m \in \mathbb{R}\backslash\{0\}, t > 0 \tag{24}$$

and, hence,

$$\eta_m = \operatorname{sgn}(m)F \tag{25}$$

in agreement with the result given in [1]. For more details and the discussion of other models we refer to [12].

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