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Localization for Off-Diagonal Disorder and for Continuous Schrödinger Operators [1]

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Abstract. We extend the proof of localization by Delyon, Lévy, and Souillard to accommodate the Anderson model with off-diagonal disorder and the continuous Schrödinger equation with a random potential.

1. Introduction

New proofs of localization were recently found by Delyon et al. [1–3], Simon et al. [4], and Simon and Wolff [5]. These proofs arose in an effort to understand some very interesting work of Kotani [6] on the sensitivity of the nature of the spectrum with respect to boundary conditions for one-dimensional systems on a half-line, a work itself connected with a work of Carmona [7]. These proofs work for the Anderson model with diagonal disorder, which is the Hamiltonian H on $l^2(\mathbb{Z}^d)$ given by

$$(Hu)(x) = \sum_{|y-x|=1} u(y) + b_n u(x). \tag{1}$$

The results hold for one-, quasi-one, and multi-dimensional systems in appropriate domains of the parameters and for large classes of (possibly correlated) random processes for the b's.

In this note we will study the case of off-diagonal disorder for discrete equations and also the case of the continuous Schrödinger equation with a random potential. It is already known in one dimension in some situations (for example assuming independence of the random parameters and some regularity of the distribution of the random variables) that a discrete Schrödinger equation with off-diagonal disorder has only pure point spectrum and exponentially decaying eigenfunctions [8]. The same result is also known for the continuous Schrödinger operator, under some conditions of independence of the potential and regularity of its distributions [9–12]. For a large bibliography on the topic of random discrete

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and continuous Schrödinger equations, we refer to [13] and for a review of the mathematical results in this field we refer to [14]. In the present paper, we give a new proof of these results, based on the proof of [1–3] which is much simpler than the previous ones and in addition applies to many new situations. The preprint [15] presents a distinct but related proof of Theorem 1′ below.

The approach of the present paper is basically the same as the one of [1–3] for the Anderson model (1) and may be summarized as follows: we assume that the realization of the random parameters is such that for one-dimensional systems the Lyapunov exponent associated to the equation $Hu=Eu$ is strictly positive for almost every E (alternatively for higher dimensions, we assume that the realization of the random parameters is such that for almost every E the Green's function decays exponentially with the distance). These hypothesis imply that for a.e. E any solution of $Hu=Eu$ (H being possibly modified by a local perturbation) either decays or increases exponentially. Then we get this property also for spectrally almost every E, at least for almost every perturbation of H in some appropriate class. Since generalized eigenfunctions are known to increase at most polynomially, it follows that this set of operators H has only pure point spectrum and exponentially decaying eigenfunctions. It then remains to check if the previous set of H has full measure when the parameters are chosen according to some given random process.

In Sect. 2 below, we give the basic results for discrete and continuous one-dimensional systems. Then in Sect. 3 we give with some details the proof for the case of the one-dimensional discrete model with off-diagonal disorder and in Sect. 4 the proof for the continuous one-dimensional Schrödinger equation. Finally in Sect. 5 we apply the results of Sects. 2–4 to various specific random distributions of the parameters, state some extensions and present the results for multi-dimensional systems together with the applications.

2. Definitions and Results for the Discrete and Continuous One-Dimensional Cases

Let us first introduce the discrete one-dimensional model with off-diagonal disorder: It is the self-adjoint operator, acting on $L^2(\mathbb{Z})$, defined as

$$(Hu)(n)=a_{n+1}u(n+1)+a_n u(n-1), \tag{2}$$

and we suppose for simplicity the a_n to be strictly positive. If u is a solution of the eigenvalue equation $Hu=Eu$, then its components satisfy an equation of the form

$$(u_{n+1}, u_n)=M_n \cdot (u_n, u_{n-1}),$$

where M_n is a two by two matrix called the transfer matrix which depends on a_n, a_{n+1} and E.

Let B be a fixed interval of $\mathbb{R}$ and denote by L the normalised Lebesgue measure on B. We assume the following hypothesis:

Hypotheseis H1. The a_n's (for $n>1$) are such that for L-a.e. E, there exists a vector $v=(u(1), u(2))$ satisfying:

$$\limsup_{n\to\infty} n^{-1} \log\|M_n M_{n-1} \dots M_2 \cdot v\| < 0. \tag{3}$$

In Sect. 3 we will prove the following basic theorem and in Sect. 5 we will give examples of applications of it:

Theorem 1. *Let H1 be true: then for* a.e. (a_0, a_1, a_2) *and spectrally* a.e. E *on B any generalized eigenfunction of H decays exponentially at* $+\infty$.

The second model we consider in this section is the Schrödinger operator on $L^2(\mathbb{R})$,

$$H = -d^2/dx^2 + V(x). \tag{4}$$

Again any solution Ψ of $H\Psi = E\Psi$ satisfies an equation

$$(\Psi(x), \Psi'(x)) = M(x, y)(\Psi(y), \Psi'(y)),$$

where $M(x, y)$ is a 2×2 transfer matrix depending on the potential V in the interval $[x, y]$ and on the energy E. The corresponding hypothesis and basic theorem are:

Hypothesis H1'. The potential $V(x)$ is such that for L-a.e. E, there exists a vector $\Phi = (\Psi(1), \Psi'(1))$ satisfying:

$$\limsup_{x \to \infty} x^{-1} \log \|M(x, 1)\Phi\| < 0. \tag{5}$$

Theorem 1'. *Let hypothesis H'1 be true and set* $H_\lambda = H + \lambda V_0$, *where* $V_0(x)$ *has support* $[0, 1]$ *and is strictly positive on this interval; then for Lebesgue* a.e. λ, *for spectrally almost every E on B any generalized eigenfunction of* H_λ *decays exponentially at* $+\infty$.

Applications of Theorems 1 and 1' and extensions of them are given in Sect. 5 below. There the multi-dimensional case is also treated.

3. Proof of Theorem 1

Let $P(dE)$ be the spectral projection for H, and set:

$$\mu_{k,l}(dE) = (\delta_k, P(dE) \cdot \delta_l). \tag{6}$$

Since H is a real symmetric matrix, $\mu_{k,l}(dE)$ is a real measure and $|\mu_{k,l}(dE)|$ denotes its absolute value. We first show that Theorem 1 is a direct consequence of the following lemma:

Lemma 2. *Let H1 be true; then for* a.e. a_0 *and* $|\mu_{0,1}(dE)|$ a.e. E *on B any generalized eigenfunction of H decays exponentially at* $+\infty$.

Proof of Theorem 1 from Lemma 2. Since any non-identically zero solution of the eigenvalue problem cannot be zero on two neighboring sites (all the a_n's are non-zero by assumption), the spectral measure is absolutely continuous with respect to $\mu_{1,1}(dE) + \mu_{2,2}(dE)$. Also by definition of the spectral measure we have

$$E\mu_{1,1}(dE) = a_0\mu_{0,1}(dE) + a_1\mu_{2,1}(dE), \qquad E\mu_{2,2}(dE) = a_1\mu_{2,1}(dE) + a_2\mu_{2,3}(dE).$$

Assuming Lemma 2, we get that for a.e. (a_0, a_1, a_2), for $|\mu_{i,i+1}(dE)|$ a.e. E in B $(0 \leqq i \leqq 2)$ any generalized eigenfunction of H decays exponentially at $+\infty$.

Consequently this conclusion holds for $\mu_{1,1}(dE)+\mu_{2,2}(dE)$ a.e. E, and hence for spectrally a.e. E. This ends the proof of Theorem 1 assuming Lemma 2. □

We now turn to the

Proof of Lemma 2. The proof relies on the following lemma that we will prove later:

Lemma 3. *The average of $|\mu_{0,1}(dE)|$ with respect to a_0 is uniformly absolutely continuous with respect to the Lebesgue measure, that is*

$$\int da_0 |\mu_{0,1}(dE)| \leqq C \cdot dE. \tag{7}$$

Indeed, assuming $H1$ and Lemma 3 we have that for a.e. a_0 then for $|\mu_{0,1}(dE)|$-a.e. E in B, the inequality (3) of the hypothesis $H1$ is satisfied. Since $\det(M_n M_{n-1} \dots M_2) = a_1/a_n$ is bounded from above and below, for $|\mu_{0,1}(dE)|$-a.e. E on B any solution of $Hu=Eu$ decreases or increases exponentially at $+\infty$. Furthermore, any generalized eigenfunction is polynomially bounded and thus has to decrease exponentially at $+\infty$ for $|\mu_{0,1}(dE)|$-a.e. E on B. This yields Lemma 2 assuming Lemma 3. □

We now prove the Lemma 3:

Proof of Lemma 3. Let H^L be the restriction of H to $l^2([-L, L])$ and let $|\mu^L_{0,1}(dE)|$ denote the absolute value of its spectral measure; thus

$$|\mu^L_{0,1}(dE)| = \sum_{1 \leqq j \leqq 2L+1} \delta(E-E_j) \cdot |u_j(0) u_j(1)|, \tag{8}$$

where E_j and u_j are the eigenvalues and normalized eigenvectors of H^L. As L goes to infinity, the measures $\mu^L_{0,1}(dE)$ converge vaguely to $\mu_{0,1}(dE)$, so for any open interval A

$$|\mu_{0,1}(A)| \leqq \liminf_{L \to +\infty} |\mu^L_{0,1}(A)|. \tag{9}$$

Thus, by Fatou's lemma it remains to prove that there exists a constant C such that

$$\int da_0 \cdot |\mu^L_{0,1}(A)| \leqq C \cdot |A|. \tag{10}$$

By (8) we have:

$$\int da_0 \cdot |\mu^L_{0,1}(A)| = \int da_0 \sum_j \delta(E-E_j) \cdot |u_j(0) u_j(1)|, \tag{11}$$

where the sum in the right-hand side runs over all the eigenvalues in A. By Rayleigh perturbation theory, E_i is a smooth function of a_0 (in the one-dimensional case all the eigenvalues E_i are simple, hence globally smooth and in the higher dimensional case considered later they are piecewise smooth) and

$$dE_j/da_0 = 2u_j(0) u_j(1). \tag{12}$$

Thus:

$$\int da_0 \sum_{E_j \in A} \delta(E-E_j) \cdot |u_j(0) u_j(1)| = (1/2) \int \sum_{e_j \in A} |dE_j/da_0| \cdot da_0. \tag{13}$$

Let $n_L(E)$ denote the number of distinct values of a_0 for which E is an eigenvalue of the corresponding H_L. If some E_j does not depend on a_0, $dE_j/da_0=0$ and $n_L=\infty$ but such values of a_0 are in finite number so they do not affect either side of (13). For any nonconstant E_j we can break it up into branches, where E_j is monotonically increasing or decreasing with respect to a_0; thus we get

$$\int da_0 \cdot |\mu_{0,1}^L(A)| = (1/2) \int_A dE n_L(E). \tag{14}$$

Since $n_L(E)$ is the number of solutions of $\det[H^L(a_0)-E]=0$ which is a quadratic polynomial in a_0, $n_L(E)$ is at most two and Lemma 2 holds with $C=1$. This yields Lemma 3 and consequently completes the proof of Theorem 1.

4. Proof of Theorem 1′

We turn now to the proof of Theorem 1′. Since no solution of the stationary Schrödinger equation can be zero on an interval, it is equivalent to Theorem 1′ to prove that for any square integrable Ψ with support in [0, 1], for a.e. λ, then for $(\Psi, P(dE)\Psi)$ a.e. E any generalized eigenfunction of H_λ decays exponentially at $+\infty$. We denoted by $P(dE)$ the spectral projection of H_λ. We also denote $\mu_\Psi(dE)$ the positive measure $(\Psi, P(dE)\Psi)$. The proof relies on the following lemma corresponding to the previous Lemma 3.

Lemma 3′. *Let F be any bounded interval, and $m(d\lambda)$ the corresponding normalized Lebesgue measure on F. Let H_λ be defined as in Theorem 1′. Then the average of $\mu_\Psi(dE)$ with respect to λ is uniformly absolutely continuous with respect to the Lebesgue measure, that is:*

$$\int d\lambda \mu_\Psi(dE) \leqq C \cdot dE. \tag{15}$$

Proof of Theorem 1′ from Lemma 3′. The proof is very similar to the proof of Theorem 1: assuming Lemma 3′ and $H1'$, we get for a.e. λ in F, then for $\mu_\Psi(dE)$ a.e. E on B, (5) holds. That is, since $\det M(x,1)=1$, any solution of $H_\lambda\Psi=E\Psi$ increases or decreases exponentially at $+\infty$. Since generalized eigenfunctions are polynomially bounded, they are exponentially decreasing at $+\infty$ for μ_Ψ a.e. E on B. Since F is arbitrary, this proves Theorem 1′ from Lemma 3′. □

Proof of Lemma 3′. We introduce the operators H_λ^L (with Dirichlet boundary conditions at $+L$ and $-L$) and the corresponding (discrete and positive) spectral measures $\mu_\Psi^L(dE)$, and we still have (see (9), (10)) for any open interval A,

$$\int_F d\lambda \mu_\Psi(A) \leqq \int_F d\lambda \liminf_{L\to+\infty} \mu_\Psi^L(A) \leqq \liminf_{L\to+\infty} \int_F d\lambda \mu_\Psi^L(A). \tag{16}$$

Thus we have to prove that there exists a constant C such that

$$\int_F d\lambda \mu_\Psi^L(A) \leqq C \cdot |A|. \tag{17}$$

Now (8) becomes

$$\mu_\Psi^L(dE) = \sum_j \delta(E-E_j) \left[\int_{[0,1]} |\Psi_j(x)\Psi(x)| dx \right]^2, \tag{18}$$

where E_j and Ψ_j are the eigenvalues and normalized eigenfunctions of H^L_λ. Since V_0 is positive and bounded from below, we get by Schwarz inequality

$$\mu^L_\Psi(dE) \leqq C' \cdot \sum_j \delta(E-E_j) \int_{[0,1]} V_0(x)|\Psi_j(x)|^2 dx, \tag{19}$$

where $C' = \int_{[0,1]} [|\Psi(x)|^2/V_0(x)]dx$. And again by perturbation theory the integral in (19) is equal to $|dE_j/d\lambda|$; thus we get

$$\int_F d\lambda \mu^L_\Psi(A) \leqq C' \int_F d\lambda \sum_j |dE_j/d\lambda|, \tag{20}$$

where the sum runs over the eigenvalues belonging to A. Denoting $n_L(E)$ the number of values of λ in F for which E is an eigenvalue of H^L_λ, we have

$$\int_F d\lambda \mu^L_\Psi(A) \leqq C' \int_A n_L(E) dE. \tag{21}$$

But $n_L(E)$ can also be seen as the number of λ for which there is a solution Ψ of the equation $H^L_\lambda \Psi = E\Psi$ on the interval $[0, 1]$ and with boundary conditions at 0 and 1 which are respectively the images of the Dirichlet boundary conditions at $-L$ and $+L$ by the equation $H^L_\lambda \Psi = E\Psi$. Because the potential V_0 is non-negative, such solutions on the interval $[0, 1]$ are necessarily in finite number, uniformly with respect to the boundary conditions at 0 and 1. Thus $n_L(E)$ is uniformly bounded by a constant depending on A and F. This in view of (21) yields Lemma 3′ and consequently concludes the proof of Theorem 1′. □

5. Applications – Extensions – The Multi-Dimensional Case

In this section we discuss the previous theorems and give some examples of random systems fulfilling our hypotheses. We also indicate some extensions and in particular we show how the results of the present paper do extend to multi-dimensional cases.

We start with one-dimensional cases. Our Theorems are specially useful in the case where hypotheses $H1$ or $H1'$ are fulfilled on both sides, that is at $+\infty$ and $-\infty$. Then since the intersection of two sets of full measure is still of full measure, we conclude (both in the discrete and the continuous case) that the generalized eigenfunctions have to decrease exponentially on both sides; thus they are square integrable and are eigenfunctions so that the operator has only pure point spectrum and exponentially decaying eigenfunctions. We give some specific applications below:

Applications of Theorem 1. It is known that for large classes of ergodic random sequences $\{a_n\}_{n\in\mathbb{Z}}$ the Lyapunov exponent of the product $\prod M_n$ of transfer matrices is positive, so that $H1$ is fulfilled on both sides with probability one for Lebesgue a.e. E. The simplest case consists of a_n to be independent and identically distributed random variables: the transfer matrices M_n are nevertheless not independent but from a generalization [16] of Furstenberg's Theorem, for any E, with probability one, a property such as (3) holds on both sides and thus by Fubini's Lemma it holds with probability one, for Lebesgue a.e. E. This ensures that with probability one the hypothesis $H1$ holds on both sides. In particular if the probability distribution of each a_n is absolutely continuous, Theorem 1 implies that (3) has almost surely only pure point spectrum and exponentially decaying eigenvectors. The result is in fact more general: if the probability distribution of the a_n has some absolutely continuous component the same conclusion holds with a non-zero probability, and because of ergodicity it thus holds with probability one. This may be summarized in the

Corollary 1. *Let H be defined as in* (3), *and let the a_n's be independent identically distributed strictly positive random variables whose probability distribution has an absolutely continuous component, then with probability one the spectrum of H is pure point with exponentially decaying eigenfunctions.*

In fact the positivity of the Lyapunov exponent and hence the almost sure realization of hypothesis $H1$ on both sides are known as soon as the sequence of the a_n's is non-deterministic [17–19]; on the other hand Theorem 1 only requires in addition that the conditional expectation of (a_0, a_1, a_2) given the remaining of the sequence is absolutely continuous with respect to the Lebesgue measure (or possesses an absolutely continuous component in the ergodic case). This allows us to apply Theorem 1 to a much wider class of situations than those of Corollary 1 and to extend correspondingly the proof os localization.

Finally let us note that even in the case where the distributions of the a_n's has no absolutely continuous component, one still gets something from Theorem 1: since the hypothesis $H1$ is still satisfied on both sides, Theorem 1 implies that for almost every value of a_0, a_1, and a_2 and in particular for almost any (w.r.t. Lebesgue measure) arbitrarily small perturbation of these three coefficients, H has almost surely pure point spectrum and exponentially decaying eigenfunctions.

Applications of Theorem 1′. i) We suppose now that the potential $V(x)$ in (4) is random and is given as a sum of potentials with compact support in $[n, n+1]$:

$$V(x)=\sum_n \lambda_n V_0(x-n),$$

where V_0 satisfies the hypothesis of Theorem 1′, and the λ_n's is given according to any non-deterministic ergodic process. Then by a result of Kotani [17] the Lyapunov exponent associated to the transfer matrices $M(x, y)$ is positive for a.e. E so that Hypothesis $H1'$ is satisfied at $+\infty$ and $-\infty$. If in addition the conditional distribution of λ_0 and λ_1 has an absolutely continuous part, then from Theorem 1′, we get that almost surely H has only pure point spectrum and exponentially decaying eigenvectors.

ii) In fact Theorem 1′ implies that as soon as the potential V is such that the corresponding H has a positive Lyapunov exponent for a.e. E (for instance by Kotani's result if it is not deterministic), then for a.e. λ the operator $H_\lambda = H + \lambda V_0$ gives rise to pure point spectrum.

iii) Some extension of Theorem 1′: In fact V_0 need not to be bounded away from zero as we assumed for simplicity in Sects. 2 and 4 because it is sufficient to consider functions $\Psi(x)$ such that $\int_{[0,1]} [|\Psi(x)|^2/V_0(x)]dx$ is bounded in order to prove Lemma 3′, and then to recover the entire spectral measure.

Applications of the Ideas of the Present Paper to Other Types of One-Dimensional Problems. In fact the ideas developed in the present paper have been presented in the specific case of models with off-diagonal disorder and the case of the continuous Schrödinger equation; but they allow one to solve many other different situations. As an example, let us note the following problem concerning a purely diagonal disorder which can be solved according to the same ideas: consider the Anderson diagonal model:

$$[Hu](n)=u(n+1)+u(n-1)+V(n)u(n), \qquad V(n)=a_n+b\cdot a_{n-1},$$

where b is a constant and $\{a_n\}$ is a sequence of independent and identically distributed random variables. If b is positive, it is not hard to extend the methods of [1, 3–5] to prove localization when the distribution of the a_n's have some absolutely continuous component. However when b is negative, one must use the methods of this paper and the above arguments carry out without any essential changes. The issue is how to go from the fact that $\mu_{j,j}(dE)+b\cdot\mu_{j+1,j+1}(dE)$ is pure point to the fact that $\mu_{j,j}(dE)$ is pure point. Without loss, we can suppose $|b|\leqq 1$. If $|b|<1$, we get that $\mu_{0,0}(dE)-|b|^n\mu_{n,n}(dE)$ is pure point; so taking $n\to+\infty$, we obtain the result that $\mu_{0,0}(dE)$ is pure point. If $|b|=1$, then $\mu_{j,j}(dE)-\mu_{j+1,j+1}(dE)$ is pure point so the continuous part of $\mu_{j,j}(dE)$ is independent of j and so, by ergodicity a.s. independent of the a_n's. But the positivity of the Lyapunov exponent implies through Pastur's argument [20] that the spectral measure is almost-surely orthogonal to any fixed measure. Thus the continuous part of $\mu_{j,j}(dE)$ vanishes. As a result, we see that if a_n has a purely absolutely continuous distribution, then H has pure point spectrum. □

Extensions to the Multi-Dimensional Cases. The model (3) can be extended to higher dimensions becoming

$$(Hu)(x)=\sum_{|y-x|=1} a_{x,y}u(y)$$

for x and y in $\mathbb{Z}^d$.

The ideas developed in the present paper allow us to extend the methods of [1–2] to various such cases with off-diagonal disorder for multi-dimensional systems.

As in [2], the analogue of Hypothesis $H1$ is the hypothesis that for almost every E in some interval the Green's function at energy E, $G_E(x,y)$ decays exponentially with distance. More precisely let us consider that the coefficients $a_{x,y}$ are given according to an ergodic process and that almost surely for almost every E in A,

$$|G_E(x,y)|<C\exp\{-K|x-y|\},$$

where C can depend on the realization of the set of coefficients and on x. If in addition the conditional distribution fo the $a_{x,y}$ has an absolutely continuous component, then H has almost surely only pure point spectrum and exponentially decaying eigenfunctions in A. The extension of the argument of [2] is straightforward in the light of the present Sect. 3 above.

Such properties of decay of the Green's function for off-diagonal disorder have been proven in [21] for independent random variables at sufficiently strong disorder or small energy; there, in addition, the pure point character of the spectrum was also derived by a method different from the one of the present paper.

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