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Topological Invariance of the Witten Index

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We discuss the Witten index in terms of Krein's spectral shift function, and prove invariance of the Witten index under suitable relative trace class hypotheses.

1. INTRODUCTION

Let A be a bounded operator from one Hilbert space, $\mathscr{H}$, to another, $\mathscr{H}'$. A is Fredholm if and only if there are finite-dimensional subspaces, K in $\mathscr{H}$ and K' in $\mathscr{H}'$, so that $A=0$ on K, $A^*=0$ on K', and $A \restriction K^\perp$ is an invertible map from $K^\perp$ to $(K')^\perp$. The index of such an operator is defined by

$$\operatorname{ind}(A)=\dim(K)-\dim(K')=\dim(\operatorname{Ker}(A))-\dim(\operatorname{Ker}(A^*)).$$

It is classical that $\operatorname{ind}(A)$ is invariant under compact perturbations, i.e.,

$$\operatorname{ind}(A+C)=\operatorname{ind}(A) \tag{1}$$

if C is compact. Equation (1) is often called "topological invariance of the index."

It is often useful to extend this notion to unbounded A's which are closed operators, in which case we need only add $K\subset D(A)$, $K'\subset D(A^*)$. One can

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then prove (1) under the weaker hypothesis that C is relatively A-compact, i.e., $C(A-z)^{-1}$ is compact for all z in the resolvent set of A.

A rich class of such unbounded A's includes those with $\exp(-tA^*A)$ and $\exp(-tAA^*)$ both trace class, in which case one can prove [7] that, for any $t>0$,

$$\operatorname{ind}(A)=\operatorname{tr}(e^{-tA^*A})-\operatorname{tr}(e^{-tAA^*}). \tag{2}$$

In the Fredholm case, (2) is independent of t.

If A is not Fredholm and $\mathscr{H}\neq\mathscr{H}'$, there does not appear to be any reasonable definition of $\operatorname{ind}(A)$, but if $\mathscr{H}=\mathscr{H}'$, it might happen that while neither e^{-tAA^*} nor e^{-tA^*A} are trace class, their difference (which one can only consider if $\mathscr{H}=\mathscr{H}'$) might be trace class, in which case one defines the regularized Witten index [13] by

$$\operatorname{ind}_t(A)=\operatorname{tr}(e^{-tA^*A}-e^{-tAA^*}). \tag{3}$$

Our main goal in this note is to prove that

$$\operatorname{ind}_t(A+C)=\operatorname{ind}_t(A) \tag{4}$$

for a large (but not too large) class of C. That this is a subtle problem can be seen by noting first that $\operatorname{ind}_t(A)$ is now t dependent and non-integral, and by looking at an interesting example:

EXAMPLE 1.1. Let $B(x)$ be a magnetic field in two dimensions, and let

$$\mathbf{a}(x)=(a_1(x), a_2(x))$$

be a gauge potential for B, i.e., $B=\operatorname{curl}\mathbf{a}=\partial_1 a_2-\partial_2 a_1$. Define X on $L^2(\mathbb{R}^2, d^2x; \mathbb{C}^2)$ by

$$X=\begin{pmatrix}0 & A^*\\ A & 0\end{pmatrix},$$

where

$$A=(p_1-a_1)-i(p_2-a_2)$$

on $\mathscr{H}=L^2(\mathbb{R}^2, d^2x)$. Then

$$A^*A=(\mathbf{p}-\mathbf{a})^2+B$$
$$AA^*=(\mathbf{p}-\mathbf{a})^2-B.$$

If B decays suitably at infinity, e.g., if B has compact support, then $e^{-tA^*A}-e^{-tAA^*}$ is trace class. The proper "topological" invariance for

$\mathrm{ind}_t(A)$ is that $\mathrm{ind}_t(A)$ only depends on $(2\pi)^{-1} \int B\, d^2x$, the total flux, which is precisely what we will prove. (To be more precise, $\mathrm{ind}_t(A) = -(2\pi)^{-1} \int_{\mathbb{R}^2} B\, d^2x$ [5, 6].) However, if B, $\tilde{B}$ are *any* two B's (of compact support), one can choose $\mathbf{a}$, $\tilde{\mathbf{a}}$ so that $(A - \tilde{A})(A^*A + 1)^{-1/2}$ is compact; thus one cannot hope to prove invariance under relatively compact perturbations. The key is to prove invariance under suitable relative trace class conditions. For mathematical background on trace class theory, see [12].

The above example shows the inadequacy of previous proofs of the topological invariance of ind_t in the physics literature (e.g., [3]) which are formal and don't make precise the conditions on C for (4) to hold.

In Section 2, we discuss the connection between the regularized index and Krein's spectral shift function. In Section 3, we prove the topological invariance. In Section 4, we discuss some open questions. Examples are discussed in [6, 8].

2. Krein's Spectral Shift Function and the Regularized Witten Index

There is an intimate connection between the regularized Witten index and Krein's spectral shift function, developed by Birman and Krein [4, 11] over 25 years ago. To describe Krein's spectral shift function, we first treat the case of bounded self-adjoint operators. We must single out a special class of functions.

Definition. $\mathscr{G} = \{f\colon \mathbb{R} \to \mathbb{R} \mid \int_{-\infty}^{\infty} (|p| + 1)\, |\hat{f}(p)|\, dp < \infty\}$.
Any $f \in \mathscr{G}$ is C^1. If $\alpha > 1$ and R is finite, there is a $g \in \mathscr{G}$ with

$$g(x) = x^\alpha, \qquad 0 \leqslant x \leqslant R.$$

Theorem 2.1 [2, 10]. *Let X, Y be bounded self-adjoint operators on a Hilbert space so that $X - Y$ is trace class. Then there is a real valued, measurable function $\xi(\lambda)$ on $\mathbb{R}$ so that*

(i) $\int_{\mathbb{R}} |\xi(\lambda)|\, d\lambda \leqslant \|X - Y\|_1$, $\mathrm{tr}(X - Y) = \int_{\mathbb{R}} \xi(\lambda)\, d\lambda$;

(ii) *if $f \in \mathscr{G}$, then $f(A) - f(B)$ is trace class, and*

$$\mathrm{tr}[f(X) - f(Y)] = \int_{\mathbb{R}} d\lambda\, \xi(\lambda)\, f^1(\lambda); \tag{5}$$

(iii) *$\xi(\lambda) = 0$ on the complement of the largest interval containing $\sigma(X) \cup \sigma(Y)$.*

Moreover, ξ is uniquely determined (a.e.) by (ii) *and* (iii).

Formally, if ξ is continuous from above at λ_0,

$$\xi(\lambda_0+0)=\mathrm{tr}[E_{(-\infty,\lambda_0]}(X)-E_{(-\infty,\lambda_0]}(Y)] \qquad \text{(formal)},$$

and if it is continuous from below,

$$\xi(\lambda_0-0)=\mathrm{tr}(E_{(-\infty,\lambda_0)}(X)-E_{(-\infty,\lambda_0)}(Y)] \qquad \text{(formal)}$$

in the sense that there are trace class operators $Z_\varepsilon \to E_{(-\infty,\lambda_0]}(X)-E_{(-\infty,\lambda_0]}(Y)$ strongly with $\mathrm{tr}(Z_\varepsilon)\to\xi(\lambda_0+0)$, but these formulae are only formal; the examples in [6] yield situations where both sides are well defined but equality fails.

$\mathscr{G}$ arise naturally since

$$f(X)-f(Y)=(2\pi)^{-1/2}\int_{\mathbb{R}} dp\hat{f}(p)[e^{ipX}-e^{ipY}]$$

and

$$e^{ipX}-e^{ipY}=ip\int_0^1 d\alpha\, e^{i\alpha pX}(X-Y)\,e^{i(1-\alpha)pY}$$

so

$$\|f(X)-f(Y)\|_1 \leqslant (2\pi)^{-1/2}\,\|X-Y\|_1\int_{\mathbb{R}} dp\,|p|\,|\hat{f}(p)|.$$

It is easy to now extend this to the unbounded case.

THEOREM 2.2. *Let H, H_0 be positive, self-adjoint operators. Suppose that $e^{-tH}-e^{-tH_0}$ is trace class for some $t>0$. Then there exists a real-valued, measurable function $\xi(\lambda)$ on $(-\infty,\infty)$ so that $\xi(\lambda)=0$ for $\lambda<0$ and*

(i) $t\int_0^\infty d\lambda\,|\xi(\lambda)|\,e^{-t\lambda}\leqslant \|e^{-tH}-e^{-tH_0}\|_1$, $\quad \mathrm{Tr}(e^{-tH}-e^{-tH_0}) = -t\int_0^\infty d\lambda\,\xi(\lambda)\,e^{-t\lambda}$.

(ii) *if f is a function on $(-\infty,\infty)$ so that f vanishes for λ very negative and so that $f(-\ln\lambda)=g(\lambda)$ lies in $\mathscr{G}$, then* (5) *holds.*

Moreover, ξ is uniquely determined by (5) *and the property that $\xi(\lambda)=0$ for $\lambda<0$. In particular, $e^{-sH}-e^{-sH_0}$ is trace class if $s>t$.*

THEOREM 2.3. *Let H, H_0 be positive, self-adjoint operators. Suppose that $(H+1)^{-\alpha}-(H_0+1)^{-\alpha}$ is trace class for some $\alpha>0$. Then there exists a real-valued, measurable function $\xi(\lambda)$ on $(-\infty,\infty)$ so that $\xi(\lambda)=0$ for $\lambda<0$ and*

(i) $\alpha\int_0^\infty d\lambda\,|\xi(\lambda)|\,(\lambda+1)^{-\alpha-1}\leqslant \|(H+1)^{-\alpha}-(H_0+1)^{-\alpha}\|_1$,

(ii) *if f is a function on $(-\infty,\infty)$ so that f' vanishes for λ very negative, and so that $g(\lambda)=f(\lambda^{-1/\alpha}-1)$ lies in $\mathscr{G}$, then* (5) *holds.*

Moreover, ξ is uniquely determined by (5) *and the property that* $\xi(\lambda)=0$ *for* $\lambda<0$.

Sketch of Theorems 2.2 *and* 2.3. These are essentially translations of Theorem 2.1 using a change of variables. For example, for Theorem 2.2 with $t=1$, take $X=e^{-H}$, $Y=e^{-H_0}$. By Theorem 2.1, there is a function η supported on [0, 1] with $\int_0^1 d\lambda\, |\eta(\lambda)|<\infty$ and

$$\operatorname{Tr}[g(X)-g(Y)]=\int_0^1 d\lambda\, g'(\lambda)\,\eta(\lambda) \tag{6}$$

as long as g lies in $\mathscr{G}$. Now, define

$$\xi(\lambda)=-\eta(e^{-\lambda})$$

and, given f, define

$$g(\lambda)=f(-\ln \lambda)$$

and one finds that (6) yields (5) for the pair (H, H_0). ∎

The uniqueness shows that if the hypotheses of Theorem 2.3 hold (and thus those of Theorem 2.2), both ξ's are equal.

In particuar, if A is a closed map from $\mathscr{H}$ to $\mathscr{H}$ so that $e^{-tA^*A}-e^{-tAA^*}$ is trace class for all $t>0$, we can write the regularized Witten index in terms of Krein's spectral shift function by

$$\operatorname{ind}_t(A)=\operatorname{tr}(e^{-tA^*A}-e^{-tAA^*})=-t\int_0^\infty e^{-t\lambda}\xi(\lambda)\,d\lambda.$$

The invariance statement we will prove is an invariance statement for the full Krein's function $\xi(\lambda)$.

The Witten index is defined as [13]

$$\lim_{t\to\infty} \operatorname{ind}_t(A)=W(A).$$

If it exists, the Krein formalism lets us relate different regularizations if ξ has a regularity property.

THEOREM 2.4. *If* $\xi(\lambda)$ *is continuous from above at* $\lambda=0$, *then* $W(A)$ *exists and*

$$W(A)=-\xi(0+). \tag{7}$$

If $(A^*A+1)^{-1}-(AA^*-1)^{-1}$ *is also trace class, then*

$$W(A)=\lim_{c\downarrow 0} c\operatorname{tr}[(A^*A+c)^{-1}-(AA^*+c)^{-1}]. \tag{8}$$

Remark. Instead of taking c to zero, one can replace c by $-z$ and take $z \to 0$ in the complex plane as long as $\arg z$ stays bounded away from 0 (mod 2π).

Proof. By hypothesis and Theorem 2.2,

$$\int_0^\infty d\lambda e^{-t\lambda} |\xi(\lambda)| < \infty$$

so that for any $\delta > 0$

$$\lim_{t \to \infty} t \int_\delta^\infty d\lambda e^{-t\lambda} |\xi(\lambda)| = 0.$$

Since $\int_0^\infty d\lambda t e^{-t\lambda} = 1$, continuity implies (7). The same argument using

$$\operatorname{tr}[(A^*A + c)^{-1} - (AA^* + c)^{-1}] = -\int_0^\infty d\lambda (\lambda + c)^{-2} \xi(\lambda)$$

proves (8). ∎

Of course, *formally*

$$-\xi(0+) = \operatorname{tr}[E_{\{0\}}(A^*A) - E_{\{0\}}(AA^*)],$$

which explains why $-\xi(0+)$ is a reasonable notion of "index," but we emphasize that there are examples where both sides of this last relation are defined but unequal. However, in the case where A is a Fredholm operator, we have

THEOREM 2.5. *Let $e^{-tA^*A} - e^{-tAA^*}$ be trace class for some $t = t_0 > 0$ (and hence for all $t > t_0$) and assume A be Fredholm. Then*

$$\operatorname{ind}(A) = W(A) = -\xi(0_+).$$

Proof. Since A Fredholm is equivalent to A^*A Fredholm, we know that

$$\inf \sigma_{\text{ess}}(A^*A) = \delta > 0.$$

Our foregoing results imply

$$\operatorname{tr}(e^{-tA^*A} - e^{-tAA^*}) = \int_0^1 d\lambda \xi_t(\lambda) = -t \int_0^\infty d\lambda e^{-t\lambda} \xi(\lambda). \tag{9}$$

Moreover, we have, for a.e. $\lambda \in \mathbb{R}$ [10],

$$\xi_t(\lambda) = \pi^{-1} \lim_{\varepsilon \downarrow 0} \operatorname{Im} \ln \det[1 + (e^{-tA^*A} - e^{-tAA^*})(e^{-tAA^*} - \lambda - i\varepsilon)^{-1}]$$

and

$$\operatorname{Tr}[(e^{-tA^*A}-z)^{-1}-(e^{-tAA^*}-z)^{-1}]$$
$$= -\frac{d}{dz} \ln \det[1+(e^{-tA^*A}-e^{-tAA^*})(e^{-tAA^*}-z)^{-1}].$$

The last two equalities, together with the fact that A^*A and AA^* have the same eigenvalues including multiplicity, imply that

$$\xi_t(\lambda)=\xi_t(1_-)=\operatorname{ind}(A) \qquad \text{for} \quad \lambda \in (e^{-\delta t}, 1).$$

Hence $\xi(\lambda) = -\xi_t(e^{-\lambda t})$ stays constant near zero and

$$\xi(\lambda)=\xi(0_+)=\operatorname{ind}(A) \qquad \text{for} \quad \lambda \in (0, \delta).$$

The rest now trivially follows by splitting the integral on the right-hand side of (9) into $(0, \delta/2)$ and $(\delta/2, \infty)$. ∎

3. Topological Invariance of the Regularized Witten Index

Our main goal in this section is to prove the following:

THEOREM 3.1. *Let A be a closed operator from $\mathscr{H}$ to $\mathscr{H}'$ and let C be a closed operator from $\mathscr{H}$ to $\mathscr{H}'$ with $D(A) \subset D(C)$, $D(A^*) \subset D(C^*)$. Suppose that for some $T \geqslant 0$:*

(i) *C is A-bounded and C^* is A^*-bounded, each with relative bound zero (i.e., $C(A^*A+\mu)^{-1/2}$ and $C^*(AA^*+\mu)^{-1/2}$ are bounded with norms going to zero as $\mu \to \infty$).*

(ii) *Ce^{-tA^*A} and $C^*e^{-tAA^*}$ are trace class for all $t > T$.*

(iii) *For some $\alpha < \frac{1}{2}$ and $0 < t < 1$,*

$$\|Ce^{-tA^*A}\| \leqslant ct^{-\alpha}; \qquad \|C^*e^{-tAA^*}\| \leqslant ct^{-\alpha},$$

and for some $1 \leqslant p < \infty$, $\beta > 0$, and $0 < t < 1$,

$$\|Ce^{-tA^*A}\|_p \leqslant ct^{-\beta}; \qquad \|C^*e^{-tAA^*}\|_p \leqslant ct^{-\beta},$$

where $\|\cdot\|_p$ is the $\mathscr{I}_p$ – norm [12].

Let $\tilde{A} = A + C$. Then, for all t sufficiently large (and for all $t > 0$ if $T = 0$),

(a) $\exp(-t\tilde{A}^*\tilde{A}) - \exp(-tA^*A)$ *and* $\exp(-t\tilde{A}\tilde{A}^*) - \exp(-tAA^*)$ *are trace class,*

(b) $\operatorname{tr}[\exp(-t\tilde{A}^*\tilde{A}) - \exp(-tA^*A)]$
$= \operatorname{tr}[\exp(-t\tilde{A}\tilde{A}^*) - \exp(-tAA^*)]$.

The point of this theorem is that if $\mathscr{H} = \mathscr{H}'$ and $e^{-tA^*A} - e^{-tAA^*}$ is trace class, the index is invariant, i.e., the following is immediate from Theorem 3.1:

COROLLARY 3.2. *If A, C obey hypotheses of Theorem* 3.1 *with $\mathscr{H} = \mathscr{H}'$ and*

(iv) $e^{-tA^*A} - e^{-tAA^*}$ *is trace class for all $t > 0$, then*

(c) $\exp(-t\tilde{A}^*\tilde{A}) - \exp(-t\tilde{A}\tilde{A}^*)$ *is trace class,*

(d) $\operatorname{ind}_t(A + C) = \operatorname{ind}_t(A)$.

For simplicity, we only describe the proof when $T = 0$.

Before beginning the proof of Theorem 3.1, we want to reformulate it in a supersymmetric way. Introduce the Hilbert space $\mathscr{H} \oplus \mathscr{H}'$ and operators

$$Q = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}, \qquad S = \begin{bmatrix} 0 & C^* \\ C & 0 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

By von Neumann's theorem, Q is self-adjoint and S is symmetric on $\mathscr{D}(Q)$. Let $Q(\lambda) = Q + \lambda S$ so $\tilde{Q} = Q(1)$. One can reformulate hypotheses (i)–(iii), which say

(i′) S is Q-bounded with relative bound zero,

(ii′) $S \exp(-tQ^2)$ is trace class,

(iii′) $\|Se^{-tQ^2}\| \leqslant ct^{-\alpha}$; $\|Se^{-tQ^2}\|_p \leqslant ct^{-\beta}$.

The conclusions now read

(a′) $\exp(-t\tilde{Q}^2) - \exp(-tQ^2)$ is trace class,

(b′) $\operatorname{tr}(P[\exp(-t\tilde{Q}^2) - \exp(-tQ^2)]) = 0$.

Here is the formal calculation for the key equality (b′):

$$\frac{d}{d\lambda} \operatorname{tr}(P[e^{-tQ(\lambda)^2} - e^{-tQ^2}])$$

$$= -\int_0^t \operatorname{tr}(Pe^{-sQ(\lambda)^2}[Q(\lambda)S + SQ(\lambda)]\, e^{-(t-s)Q(\lambda)^2})\, ds. \qquad (10)$$

But Q commutes with Q^2 and anti-commutes with P, so

$$\operatorname{tr}[Pe^{-sQ^2}QSe^{-(t-s)Q^2}] = -\operatorname{tr}[Pe^{-sQ^2}SQe^{-(t-s)Q^2}] \qquad (11)$$

by using the commutation properties and the cyclicity of the trace to bring the Q "around." This argument is not a proof because one must use some care in invoking cyclicity of the trace with unbounded operators, or even if the operators are merely on an infinite-dimensional space. For example, if A is the operator on l^2 with $Ae_n = e_{n+1}$, then $\operatorname{tr}(A^*A - AA^*) = 1$. Thus, we will need to take care that certain operators are trace class. A key fact about cyclicity is the following (see, e.g., Simon [12]).

LEMMA 3.3. *If X and Y are bounded operators with both XY and YX trace class, then*

$$\operatorname{tr}(XY) = \operatorname{tr}(YX).$$

First, we need a technical result to prove that $e^{-t\tilde{Q}^2} - e^{tQ^2}$ is trace class:

PROPOSITION 3.4. *Under the hypotheses of* (i′)–(iii′),

(a) $e^{-t\tilde{Q}^2} - e^{-tQ^2}$ *is trace class for all* $t > 0$,

(b) $Se^{-tQ(\lambda)^2}$ *is trace class for all* $t > 0$.

Proof. By complex interpolation between $p = \infty$ and the initial p, we can increase p and suppose that $\beta < \frac{1}{2}$. Let

$$\delta_\lambda Q = \lambda SQ + \lambda QS + \lambda^2 S^2 = Q(\lambda)^2 - Q^2.$$

Formally

$$e^{-tQ(\lambda)^2} - e^{-tQ^2} = \sum_{n=1}^{\infty} A_n(\lambda) \tag{12a}$$

$$A_n(\lambda) = \int_R e^{-t_0 Q^2} (\delta_\lambda Q)\, e^{-t_1 Q^2} \cdots (\delta_\lambda Q)\, e^{-t_n Q^2}\, dt_1 \cdots dt_n, \tag{12b}$$

where R is the region $\{t_i \geqslant 0,\ i = 1, ..., n;\ 0 \leqslant t_1 + \cdots + t_n \leqslant t\}$ and $t_0 = t - \sum_{i=1}^{n} t_i$. It is very easy to see that if

$$\int_0^t [\lambda\, \|Se^{-uQ^2}Q\| + \lambda\, \|Qe^{-uQ^2}S\| + \lambda^2\, \|Se^{-uQ^2}\|]\, du < 1, \tag{13}$$

the series converges and gives $e^{-tQ(\lambda)^2} - e^{-tQ^2}$. By the hypotheses on $\|Se^{-uQ^2}\|$ each integrand in (13) bounded by $Cu^{-\alpha - 1/2}$ is integrable, and thus (13) holds for t small. Thus, for any λ, (12) holds for t sufficiently small.

We are reduced to showing that each $A_n(\lambda)$ and each $SA_n(\lambda)$ is trace class with $\sum \|A_n(\lambda)\|_1 < \infty$, $\sum \|SA_n(\lambda)\|_1 < \infty$. We need only do this for small t since the results then automatically hold for all $t > 0$ (for $Se^{-(t+\delta t)Q^2} = (Se^{-tQ^2})e^{-\delta t Q^2}$ and Theorem 2.1 implies that $\{t \mid e^{-t\tilde{Q}^2} - e^{-tQ^2}$ trace class$\}$ is a half-infinite interval).

By hypothesis

$$\lambda \|Se^{-uQ^2}Q\|_p + \lambda \|Qe^{-uQ^2}S\|_p + \lambda^2 \|Se^{-Q^2}S\|_p$$

is $O(u^{-1/2-\beta})$ at $u=0$, and so is integrable if p is picked with $\beta < \frac{1}{2}$. Pick n_0 so $n_0 p^{-1} > 1$. Then, for $n \geqslant n_0$, one can estimate $\|A_n(\lambda)\|_1$ and $\|SA_n(\lambda)\|_1$. Using Hölder's inequality and using (13) proves that

$$\sum_{n \geqslant n_0} \|A_n(\lambda)\|_1 < \infty; \qquad \sum_{n \geqslant n_0} \|SA_n(\lambda)\|_1 < \infty.$$

For $n < n_0$, we can break up the region of integration into $n+1$ pieces, so that on R_j, $t_j \geqslant t/n+1$ (take $R_j = \{t_i \mid \max_{i=0,\ldots,n}(t_i) = t_j\}$). By hypothesis, $\lambda \|Se^{-t_jQ^2}Q\|_1 + \lambda \|Qe^{-t_jQ^2}S\|_1 + \lambda^2 \|Se^{-t_jQ^2}S\|_1$ is bounded uniformly in $t_j \geqslant t/n+1$, and by (13), the integral of the operator norms over the other t's is finite. Thus, the required result is proven. ∎

Proof of Theorem 3.1. We can write

$$P[e^{-tQ(\lambda)^2} - e^{-tQ(\lambda_0)^2}] = -\int_0^t Pe^{-sQ(\lambda)^2}[(\lambda-\lambda_0)\, Q(\lambda_0) S + (\lambda-\lambda_0)\, SQ(\lambda_0) + (\lambda-\lambda_0)^2\, S^2]\, e^{-(t-s)Q(\lambda_0)^2}\, ds.$$

By the last proposition, the right side, and so the left, is trace class. From the equation it is easy to see that the left side is differentiable at $\lambda = \lambda_0$, and so we conclude that (10) holds. Thus, we need only prove (11).

Define $Q_\varepsilon = Q(1+\varepsilon Q^2)^{-1}$, $S_\varepsilon = (1+\varepsilon Q^2)^{-1} S$. Then we claim that

$$\operatorname{tr}[Pe^{-sQ^2} Q_\varepsilon S_\varepsilon e^{-(t-s)Q^2}] = -\operatorname{tr}[Pe^{-sQ^2} S_\varepsilon Q_\varepsilon e^{-(t-s)Q^2}]. \tag{14}$$

For $PQ_\varepsilon = -Q_\varepsilon P$, all operators are bounded, the products are trace class, and so we can use cyclicity by Lemma 3.3. By taking ε to 0, one obtains (11) and so completes the proof of the theorem. ∎

When to the basic conditions (i)–(iii) hold? Here is a sufficient condition:

PROPOSITION 3.5. *If, for some* $\gamma < \frac{1}{2}$,

$$C(A^*A+1)^{-\gamma} \textit{ and } C^*(AA^*+1)^{-\gamma}$$

are bounded and for some $\eta > 0$

$$C(A^*A+1)^{-\eta} \textit{ and } C^*(AA^*+1)^{-\eta}$$

are trace class, then the hypotheses of Theorem 3.1 *hold.*

Proof. For any positive operator B, and $0<t<1$,

$$\|e^{-tB}(B+1)^{\mu}\| \leqslant ct^{-\mu}.$$

From this, one immediately sees that

(a) $\|Ce^{-tA^*A}\| \leqslant ct^{-\gamma}$,

(b) $\|Ce^{-tA^*A}\|_1 \leqslant ct^{-\eta}$,

which implies (i)–(iii). ∎

With a little more effort, one needs only suppose $(AA^*+1)^{-\gamma_1}C(A^*A+1)^{-\gamma_2}$ is bounded for some $\gamma_1+\gamma_2<\frac{1}{2}$ and $(AA^*+1)^{-\eta_1}C(A^*A+1)^{-\eta_2}$ is trace class to obtain the results of (but not the hypotheses of) Theorem 3.1.

4. Open Problems

We have shown in [6] that Krein's spectral shift function is independent of the energy in the example of a two-dimensional magnetic field discussed in Section 1. The proof relies on the topological invariance, and the fact that scaling transformations yield equivalent Hamiltonians in the sense of the operators before and after scaling related by a relatively trace class perturbation. Is there any general condition that implies that Krein's function is energy independent?

It is a result of Aharanov and Casher [1] that in the magnetic field case the integral part of $\xi(0_+)$ (suitably defined for negative numbers on strict integers) is precisely $\dim \operatorname{Ker}(A^*) - \dim \operatorname{Ker}(A)$. Is there any general situation (different from the one considered in [9]) in which this is the case? Can one find a proof of the Aharanov–Casher theorem that "explains" why the topological index doesn't differ by more than it does from the analytic index?

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