SOME PICTORIAL COMPACTIFICATIONS OF THE REAL LINE

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1. Introduction. The general theory of compactifications of a completely regular space, X, either from a classical Tychonoff cube or from the modern Gelfand point of view is well known (see, e.g., [1] pp. 223–227). It turns out that in case the original space is not compact, there are many different compactifications; in fact, there is one for every algebra of bounded continuous real-valued functions on X which is closed in the uniform norm and which contains enough functions to separate points from closed sets. Even in the case where X is the real line, R, one rarely talks about anything but the one-point, the two-point and the Stone-Čech compactifications. The first two are quite tame while the last is impossible to picture. The purpose of this paper is to present a certain class of compactifications of R which are quite easy to picture.

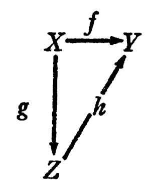
2. The main result. First, we state the basic definition:

If X is a topological space, a compactification *of X is a compact Hausdorff space, Y, together with a map $f\colon X\to Y$ such that:*

(i) *f is a homeomorphism of X and* im f *(the image of f) where* im f *has the relative topology which it inherits as a subset of Y.*

(ii) Im f *is dense in Y.*

Two compactifications $f\colon X\to Y$ and $g\colon X\to Z$ are said to be equivalent if there is a homeomorphism $h\colon Z\to Y$ such that



commutes, i.e., $f=h\circ g$.

One normally associates X with its image in the compactification, in which case the commutative diagram is replaced with the statement that h leaves X pointwise fixed.

We will be concerned with compactifications of $[0, \infty)$. If $f\colon [0, \infty)\to Y$ is a compactification, we will say that Y-im f has been added to make the compactification. The main result is the theorem:

Let X be a compact Hausdorff space and let $g\colon [0, \infty)\to X$ be a continuous map with the property that for each $a>0$, $g([a, \infty))$ is dense in X. Then $[0, \infty)$ has a compactification in which X has been added to make the compactification.

We note that g was not required to be either injective, or if injective, a homeomorphism onto im g. We also emphasize that X is not itself the compact extension.

Proof. Let $I=[0, 1]$ and define f: $[0, \infty)\rightarrow X\times I$ by $f(a)=(g(a), h(a))$ where $h(a)=a/(1+a)$ is a homeomorphism of $[0, \infty)$ and $[0, 1)$. For convenience set $G=g\circ h^{-1}$.

We first show that f is a homeomorphism of $[0, \infty)$ and $\operatorname{im} f$. It is obvious that f is continuous since its coordinates are continuous and 1-1 since h is 1-1. Moreover, f is open, for if $A\subset[0, \infty)$ is open, $h[A]\subset[0, 1)$ is open and thus $f[A]=(X\times h[A])\cap\operatorname{im} f$ is relatively open in $\operatorname{im} f$.

Next, we show that $\overline{\operatorname{im} f}=(X\times\{1\})\cup\operatorname{im} f$. For let $(x, r)\in X\times I$ with $r\neq 1$ and $x\neq G(r)$. We show that $(x, r)\notin\overline{\operatorname{im} f}$; for let B and C be disjoint open sets in X about x and $G(r)$ respectively. Then $(B\times G^{-1}[C])$ is a neighborhood of (x, r) which does not intersect $\operatorname{im} f$. On the other hand any $(x, 1)\in\overline{\operatorname{im} f}$; for let $U\times(b, 1]$ be a rectangular neighborhood of $(x, 1)$, and let $a=h^{-1}(b)$. Then, by the density assumption, $U\cap g((a, \infty))\neq\emptyset$; say $g(c)\in U$. Then $f(c)=(g(c), h(c))\in U\times(b, 1]$. Thus $(x, 1)\in\overline{\operatorname{im} f}$.

Thus our result is proven; for f: $[0, \infty)\rightarrow\overline{\operatorname{im} f}$ is a compactification and $\overline{\operatorname{im} f}-\operatorname{im} f=X$.

3. Some examples. Since $(0, \infty)$ is homeomorphic to the real line, given g as in the main theorem (and given a *specific* homeomorphism of $(-\infty, \infty)$ and $(0, \infty)$), we can regard f: $(0, \infty)\rightarrow\overline{\operatorname{im} f}$ as a compactification of R. We get this compactification "by putting a point at one end of R and X at the other end"; thus we will call it the point-X compactification (actually a point-X compactification since the way R lies in $\overline{\operatorname{im} f}$ depends not only on X but on the exact map g and on the homeomorphism of $(-\infty, \infty)$ and $(0, \infty)$). Given two compactifications of $[0, \infty)$ following the theorem, say by adding X and Y respectively, we can view one as a compactification of $(-\infty, 0)$ and join the two together at 0 and so get an $X-Y$ compactification or if $X=Y$ a two$-X$ compactification. Finally given a map g satisfying the hypothesis of the theorem, one can consider the two$-X$ compactification and "glue" the two copies of X together; a "one$-X$" compactification results. This terminology agrees with the usual one-point, two-point terminology in the case that X is a single point.

The prime example of an X and a g obeying the conditions of the theorem, in fact, the example that motivated the theorem is the winding line on the torus $S^1\times S^1$. If S^1 is represented by real coordinates mod 2π, and g: $[0, \infty)\rightarrow S^1\times S^1$ is defined by $g(a)=(a, ta)$ with t a fixed irrational number, then g meets the hypothesis of the theorem. In this way, one can construct one- and two-torus compactifications.

To obtain a geometric picture of a torus-point compactification, we imbed $(S^1\times S^1)\times I$ in R^3 as a toroidal shell. In fact, without changing the construction of the main result, we can shrink $(S^1\times S^1)\times\{0\}$ into a circle and so view our compactification as being embedded in a solid torus. Then we take a copy of the real line, start it at the center of a cross-section and let it spiral out towards the surface, winding around longitudinally as we spiral outward; only the surface need be added to give us a compact set.

Of course, we need not stop with two dimensions or with the torus. We can get a winding line on an n-dimensional torus or we can go to a countable number of dimensions or even an uncountable number of dimensions since the reals have uncountable dimension over the rationals. Or one can wind about a two-dimensional sphere as if one were winding a ball of yarn and thereby find sphere-torus, point-sphere and assorted other compactifications. Again, one is not restricted to two-dimensional spheres. More exotic spaces (like $S^n \times S^m$ or a nest of circles tangent at one point) can be used.

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Reference

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LEFT ARTINIAN RINGS THAT ARE DIVISION RINGS

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Zariski and Samuel point out that if R is a commutative ring with identity and no proper divisors of 0, and R satisfies the descending chain condition, then it is a field [1, p. 203]. Surprisingly, we can omit the assumptions of commutativity and an identity and prove the following theorem:

If R is a left Artinian ring with no proper divisors of 0, *then R is a division ring.*

Proof. Recall that a left Artinian ring is one in which every properly descending chain of left ideals is finite. A semisimple ring is a left Artinian ring with zero radical [2, Chapter 2]. A semisimple ring has a multiplicative identity [2, p. 29]. Now if R is left Artinian with no proper divisors of 0, then it is semisimple and hence has an identity 1. Consider R as a left R-module. All the submodules are left ideals, so R is an Artinian left R-module. Let $b \in R$, $b \neq 0$, and define a function f on R:

$$f(x) = xb.$$

Then f is an endomorphism of R as a left R-module and f is one-to-one. Now a one-to-one endomorphism of an Artinian module is an automorphism [3, p. 23]. Hence for all b in R different from 0, there exists x in R such that $xb = 1$. Thus R is a division ring.

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