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Corrections to the Classical Behavior of the Number of Bound States of Schrödinger Operators

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Let us denote by N_E the number of bound states of the Schrödinger operator $H = -\Delta - c/(1+|x|^2) + V_0$ below $-E$. V_0 is a potential decaying at infinity sufficiently fast. We prove that, for dimension $d=1$,

$$\lim_{E \downarrow 0} \frac{N_E}{|\ln E|} = \frac{1}{\pi}\sqrt{c - \frac{1}{4}}$$

and for $d=3$,

$$\lim_{E \downarrow 0} \frac{N_E}{|\ln E|} = \sum_{l=0}^{[\sqrt{c}-1/2]} (2l+1)\sqrt{c - \left(l + \frac{1}{2}\right)^2}.$$

1. INTRODUCTION

Suppose V is a potential decaying near infinity. Let us denote by $N_E(V)$ the number of eigenvalues of $H = -\Delta + V$ below $-E$. The finiteness (resp. infiniteness) of $N_0(V)$ is determined by the rate of decay of V at infinity (see Reed–Simon [4, XIII.3]) In fact, $N_0(V) < \infty$ if $V(x) \geqslant -c/|x|^{2+\varepsilon}$, while $N_0(V) = \infty$ for $V(x) \leqslant -c/|x|^{2-\varepsilon}$. For the borderline case $V(x) \sim c/|x|^2$ one even has that $N_0(V) < \infty$ (resp. $=\infty$) if $c < \frac{1}{4}(d-2)^2$ (resp. $> \frac{1}{4}(d-2)^2$) where d is the spatial dimension.

For potentials V behaving like $-c\,|x|^{-\beta}$ near infinity with $\beta < 2$, the behavior of

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$N_E(V)$ as $E \downarrow 0$ is exactly known (see Reed–Simon [4, Theorem XIII.82]). If we define

$$g(E) = \frac{\tau_d}{(2\pi)^d} \int_{\{|V(x)| < -E\}} (-V(X) - E)^{d/2}\, dx,$$

which is the classical phase space volume associated with V, then

$$\lim_{E \downarrow 0} \frac{N_E(V)}{g(E)} = 1 \qquad (V \sim c\,|x|^{-\beta} \text{ near infinity, } \beta < 2).$$

In this paper, we are concerned with the behavior of $N_E(V)$ in the borderline case $V(x) \sim c\,|x|^{-2}$ near infinity. In this case, the phase space volume $g(E)$ diverges logarithmically as E goes to zero. While one might expect a logarithmic divergence of $N_E(V)$ for $c > \frac{1}{4}$ (for $d = 1$ or 3), this is certainly not correct for $c < \frac{1}{4}$, in which case $N_E(V)$ is bounded as $E \downarrow 0$. To state our result, let us define $c_d = \frac{1}{4}(d-2)^2$. We will prove below:

THEOREM 1. *Suppose V_0 is a potential such that $C_0^\infty(\mathbb{R}^d)$ is a form core for the operators $H_\lambda = -\Delta + \lambda V_0$ for all $\lambda \in \mathbb{R}$, and such that H_λ has finitely many bound states below* 0 *for all $\lambda \in \mathbb{R}$. Then*

$$\lim_{E \downarrow 0} \frac{N_E(-c/(1+|x|^2) + V_0)}{|\ln E|} = f_d(c) < \infty.$$

$f_d(c)$ is zero iff $c \leqslant c_d$. Moreover, we have

$$f_1(c) = \frac{1}{\pi}\sqrt{c - \frac{1}{4}} \qquad \text{for} \quad c > \frac{1}{4}$$

$$f_2(c) = \frac{1}{2\pi}\sqrt{c} + \frac{1}{\pi}\sum_{l=1}^{[\sqrt{c}]} \sqrt{c - l^2} \qquad c > 0,$$

where $[x]$ denotes the integer part of x,

$$f_3(c) = \frac{1}{2\pi} \sum_{l=0}^{[\sqrt{c} - 1/2]} (2l+1)\sqrt{c - \left(l + \frac{1}{2}\right)^2} \qquad c > \frac{1}{4}.$$

We note that similar formulae for $f_d(c)$ $(d \geqslant 4)$ can be derived by the method employed below.

There is a systematic philosophy for analyzing corrections to the quasiclassical limit associated with the work of Fefferman–Phong. This would suggest that, rather than look at the volume in phase space of $S_0 = \{(x,p) \mid p^2 + V(x) < -E\}$, one looks at

$$S_1 = \{(x,p) \mid (x,p) \text{ is the center of box of volume 1 inside } S_0\}.$$

This actually predicts that $-d^2/dx^2 - c(|x|+1)^{-2}$ has infinitely many eigenstates only when $c > \frac{1}{4}$. We believe that it may correctly give $f_1(c)$, but doubt that it will correctly give $f_n(c)$ for $n \geqslant 2$.

In a forthcoming paper [1], we will give criteria involving local L^p-norms of V which imply that $-\Delta + \lambda V$ has finitely many bound states. For example, in dimension $d = 3$, we only require

$$|x|^2 \left(\int_{|y-x| \leqslant 1} |V(x)|^{3/2} \, dx \right)^{2/3} \to 0 \qquad \text{as} \quad |x| \to \infty.$$

Throughout this paper, we will keep the following assumptions: $C_0^\infty(\mathbb{R}^d)$ is a form core for $-\Delta + \lambda V + \mu V_0$ for all $\lambda, \mu \in \mathrm{R}$ and $\inf \sigma_{\mathrm{ess}}(-\Delta + \lambda V + \mu V_0) = 0$. This is satisfied, for example, if V and V_0 are relatively compact with respect to $-\Delta$.

The paper is organized as follows: In Section 2, we investigate a one-dimensional potential, $\tilde{V}$, which is derived from $c/|x|^2$ by first scaling and then perturbing it in a suitable way. This procedure is done in such a way that the Schrödinger equation with $\tilde{V}$ can be solved explicitly in a region of interest. We show, in Section 3, how to prove Theorem 1 from the results of Section 2. In Section 5, we prove Theorem 1 for $d > 1$.

2. A Scaled Eigenvalue Problem

We consider the potential

$$V^{(c)}(x) = \begin{cases} -c/|x|^2 & \text{for} \quad |x| \geqslant 1 \\ 0 & |x| < 1 \end{cases}$$

and the operator $H = -d^2/dx^2 + V^{(c)}(x)$ on $L^2(\mathbb{R})$. For any operator, A, we define $N_E(A)$ to be the number of eigenvalues of A (counting multiplicity) below $-E$. We are interested in the behavior of $N_E(H)$ as $E \downarrow 0$.

Define the scaling operator $U_\rho \varphi(x) = \rho^{+1/2} \varphi(\rho x)$. U_ρ is a unitary operator. Setting

$$V_\rho(x) = V_\rho^{(c)}(x) = \begin{cases} -c/|x|^2 & |x| \geqslant \rho \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_\rho = -\frac{d^2}{dx^2} + V_\rho,$$

we get

$$U_\rho^{-1} H U_\rho = \rho^2 H_\rho.$$

Consequently,

$$N_E(H_1^{(c)}) = N_{\rho^{-2}E}(H_\rho^{(c)}).$$

Taking $\rho = E^{+1/2}$, we finally obtain

PROPOSITION 1. $N_E(H) = N_1(H_{E^{1/2}})$.

Instead of investigating the potential $V_{E^{1/2}}$ directly, we consider a perturbed version of $V_{E^{1/2}}$, namely

$$\tilde{V}_{E^{1/2}}(x) = \begin{cases} -c/|x|^2 & |x| \geq \sqrt{c} \\ -c/|x|^2 - 1 & E^{1/2} \leq |x| < \sqrt{c} \\ -1 & \text{otherwise.} \end{cases}$$

$\tilde{V}_{E^{1/2}}$ is chosen in such a way that we find an explicit solution of $\tilde{H}_{E^{1/2}} u = -u$ (in $(E^{1/2}, \sqrt{c})$) very easily. We will eventually show that the perturbation does not change the behavior of $N_1(H_{E^{1/2}})$ for $E \downarrow 0$.

Let us define $\tilde{H}_{E^{1/2}} = -d^2/dx^2 + \tilde{V}_{E^{1/2}}$, and let $\tilde{\tilde{H}}_{E^{1/2}}$ be the operator obtained from $\tilde{H}_{E^{1/2}}$ by imposing Dirichlet boundary conditions at the points $\pm\sqrt{c}$ and $\pm\sqrt{E}$. Since the resolvents of $\tilde{\tilde{H}}_{E^{1/2}}$ and $\tilde{H}_{E^{1/2}}$ differ by a finite rank operator (in fact, an operator of rank 4), $|N_1(\tilde{\tilde{H}}_{E^{1/2}}) - N_1(\tilde{H}_{E^{1/2}})|$ is bounded (≤ 4) uniformly in E. $\tilde{\tilde{H}}_{E^{1/2}}$ is a direct sum of five pieces, namely

$$\tilde{\tilde{H}}_{E^{1/2}} = \tilde{\tilde{H}}_{E^{1/2}|L^2(-\infty, -\sqrt{c})} \oplus \tilde{\tilde{H}}_{E^{1/2}|L^2(-\sqrt{c}, -\sqrt{E})} \oplus \tilde{\tilde{H}}_{E^{1/2}|L^2(-\sqrt{E}, \sqrt{E})} \oplus \tilde{\tilde{H}}_{E^{1/2}|L^2(\sqrt{E}, \sqrt{c})} \oplus \tilde{\tilde{H}}_{E^{1/2}|L^2(\sqrt{c}, \infty)}.$$

Consequently (see, e.g., RS [4, XIII.15]),

$$N_1(\tilde{\tilde{H}}_{E^{1/2}}) = \sum_{j=1}^{5} N_1(\tilde{\tilde{H}})_{E^{1/2}}^{(j)}),$$

where $\tilde{\tilde{H}}_{E^{1/2}}^{(j)}$ denotes the jth term in the above direct sum. However, only the second and the fourth operators have eigenvalues below -1; moreover, by symmetry, $N_1(\tilde{\tilde{H}}_{E^{1/2}}^{(2)}) = N_1(\tilde{\tilde{H}}_{E^{1/2}}^{(4)})$. Hence, it suffices to consider $\tilde{\tilde{H}}_{E^{1/2}}^{(4)}$.

For $c > \frac{1}{4}$ and $0 < E < 1$,

PROPOSITION 2.

$$\left| N_1(\tilde{\tilde{H}}_{E^{1/2}}^{(4)}) + \frac{\sqrt{c - 1/4}}{2\pi} \ln E \right| \leq B < \infty$$

for an E-independent constant B.

Proof. We seek a solution, u, of

$$\tilde{\tilde{H}}^{(4)}_{E^{1/2}} u = -u,$$

i.e.,

$$\left(-\frac{d^2}{dx^2} - \frac{c}{|x|^2} \right) u = 0$$

for $\sqrt{E} \leqslant x \leqslant \sqrt{c}$.

Trying the ansats $u(x) = x^\alpha$ ($\alpha \in \mathbb{C}$), we obtain the solution $u(x) = x^{1/2} x^{i\sqrt{c-1/4}}$; hence $u_1(x) = x^{1/2} \sin(\sqrt{c - \frac{1}{4}} \ln x)$ is a real valued solution. The number of zeros of u_1 between $\sqrt{E}$ and $\sqrt{c}$ equals the number of (positive or negative) integers n between $(\sqrt{c-1/4}/2\pi) \ln E$ and $(\sqrt{c-1/4}/2\pi) \ln c$ which roughly equals the distance between these numbers. More precisely,

$$\left| \# \{ \sqrt{E} \leqslant x \leqslant \sqrt{c} \mid u_1(x) = 0 \} + \frac{\sqrt{c-1/4}}{2\pi} \ln E \right| \leqslant \frac{\sqrt{c-1/4}}{2\pi} |\ln c| + 2.$$

Furthermore, the number of zeros of any solution of $\tilde{\tilde{H}}^{(4)}_{E^{1/2}} u = -u$ equals the number of eigenvalues of $N_1(\tilde{\tilde{H}}^{(4)}_{E^{1/2}})$ up to an E- (and c-) independent constant, by Sturm's oscillation theorem. ∎

Summarizing, we get

PROPOSITION 3. *$N_1(\tilde{H}_{E^{1/2}}) + (\sqrt{c-1/4}/\pi) \ln E$ is bounded by an E-independent constant, M.*

Remark. The constant M can be chosen as $M = (\sqrt{c-1/4}/\pi) \ln |c| + M'$ where M' is independent of c. So M can be chosen uniformly in c on bounded sets.

Proposition 3 leads to the following corollary:

COROLLARY 1. *Let $c_0 > \frac{1}{4}$. Then*

$$\lim_{c \to c_0} \lim_{E \downarrow 0} \frac{N_1(\tilde{H}^{(c)}_{E^{1/2}})}{N_1(\tilde{H}^{(c_0)}_{E^{1/2}})} = 1.$$

3. PROOF OF THEOREM 1

We prove in this section that the behavior of $N_1(\tilde{H}_{E^{1/2}} + V_0)$ as $E \downarrow 0$ for V_0 bounded, and of compact support, is the same as the behavior of $N_1(\tilde{H}_{E^{1/2}})$. This gives us the behavior of $N_1(H_{E^{1/2}})$ since $\tilde{H}_{E^{1/2}} - H_{E^{1/2}}$ is a bounded function of compact support. Hence, by scaling, we know the behavior $N_E(-d^2/dx^2 + V^{(c)})$. Finally, we show that perturbing $V^{(c)}$ by a function which decays fas enough at infinity does not change the behavior of $N_E(H)$.

We need some preparation:

PROPOSITION 4. *Suppose that* A *and* B *are self-ajoint operators bounded below with* $\inf \sigma_{\text{ess}}(A) = \inf \sigma_{\text{ess}}(B) = 0$, *and assume that* A, B *and* $A+B$ *have a common core,* D_0. *Then, for any* $E > 0$ *and any* $0 < \varepsilon < 1$,

$$N_E(A+B) \leqslant N_{(1-\varepsilon)E}(A) + N_{\varepsilon E}(B).$$

Proof. By the min–max theorem [4, XIII.1]

$$\mu_n(A) = \sup_{\psi_1, \ldots, \psi_{n-1}} \inf_{\substack{\varphi \in D_0, \|\varphi\| = 1 \\ \varphi \perp \psi_1, \ldots, \psi_{n-1}}} \langle \varphi, A\varphi \rangle$$

is the nth eigenvalue of A, or is the bottom of the essential spectrum, in which case A has less than n eigenvalues below its essential spectrum.

We conclude that

$$\begin{aligned}
\mu_{m+k-1}(A+B) &= \sup_{\psi_1, \ldots, \psi_{m-1}, \rho_1, \ldots, \rho_{k-1}} \inf_{\substack{\varphi \perp \psi_i, \rho_j \\ \|\varphi\| = 1, \varphi \in D_0}} \langle \varphi, A\varphi \rangle + \langle \varphi, B\varphi \rangle \\
&\geqslant \sup_{\substack{\psi_1, \ldots, \psi_{m-1} \\ \rho_1, \ldots, \rho_{k-1}}} \{ (\inf_{\varphi \perp \psi_1, \ldots, \psi_{m-1}} \langle \varphi, A\varphi \rangle) + (\inf_{\varphi \perp \rho_1, \ldots, \rho_{k-1}} \langle \varphi, B\varphi \rangle) \} \\
&= \mu_m(A) + \mu_k(B).
\end{aligned}$$

Hence, if $N_{(1-\varepsilon)E}(A) = m$ and $N_{\varepsilon E}(B) = k$, then

$$\mu_{m+k+1}(A+B) \geqslant \mu_{m+1}(A) + \mu_{k+1}(B) > (1-\varepsilon)\, E + \varepsilon E = E$$

so $N_E(A+B) \leqslant m+k$. ∎

PROPOSITION 5. *Let* V, W *be potentials on* $\mathbb{R}^d$, *s.t. the operators* $H_{\lambda,\mu} = -\Delta + \lambda V + \mu W$ *have a common form domain for all* λ, μ *and such that* $\inf \sigma_{\text{ess}}(H_{\lambda,\mu}) = 0$. *Then, for any* $E > 0$ *and* $0 < \varepsilon < 1$:

(i) $N_E(-\Delta + V + W) \leqslant N_E(-\Delta + (1/(1-\varepsilon))\, V) + N_E(-\Delta + (1/\varepsilon)\, W)$

(ii) $N_E(-\Delta + V + W) \geqslant N_E(-\Delta + (1-\varepsilon)\, V) - N_E(-\Delta - ((1-\varepsilon)/\varepsilon)\, W)$.

Proof. (i) is an immediate consequence of Proposition 4, if we note that $N_{\alpha E}(A) = N_E(\alpha^{-1} A)$. (ii) follows from (i) by

$$\begin{aligned}
N_E(-\Delta + (1-\varepsilon)\, V) &= N_E(-\Delta + (1-\varepsilon)(V+W) - (1-\varepsilon)\, W) \\
&\leqslant N_E(-\Delta + V + W) + N_E\left(-\Delta - \frac{(1-\varepsilon)}{\varepsilon}\, W\right). \quad ∎
\end{aligned}$$

We no come back to the potentials we are concerned with here:

PROPOSITION 6. *If V_0 is a bounded function of compact support and then if $c > \frac{1}{4}$*

$$\lim_{E \downarrow 0} \frac{N_1(\tilde{H}^{(c)}_{E^{1/2}} + V_0)}{N_1(\tilde{H}^{(c)}_{E^{1/2}})} = 1.$$

Proof. Set

$$\chi_c(x) = \chi_{\{x \mid |x| \leqslant \sqrt{c}\}}(x)$$

and

$$W_\varepsilon = (1-\varepsilon)\, \chi_{(1/(1-\varepsilon))c} - \chi_c.$$

By Proposition 5(i), we know

$$\begin{aligned} N_1(\tilde{H}^{(c)}_{E^{1/2}} + V_0) &= N_1\left(-\frac{d^2}{dx^2} + V^{(c)}_{E^{1/2}} - (1-\varepsilon)\, \chi_{(1/(1-\varepsilon))c} + V_0 + W_\varepsilon\right) \\ &\leqslant N_1\left(-\frac{d^2}{dx^2} + \frac{1}{1-\varepsilon)} V^{(c)}_{E^{1/2}} - \chi_{(1/(1-\varepsilon))c}\right) + N_1\left(-\frac{d^2}{dx^2} + \frac{1}{\varepsilon}(V_0 + W_\varepsilon)\right) \\ &= N_1\left(-\frac{d^2}{dx^2} + \tilde{V}^{((1/(1-\varepsilon))c)}_{E^{1/2}}\right) + N_1\left(-\frac{d^2}{dx^2} + \frac{1}{\varepsilon}(V_0 + W_\varepsilon)\right). \end{aligned}$$

Dividing by $N_1(-d^2/dx^2 + \tilde{V}^{(c)}_{E^{1/2}})$ and taking $E \downarrow 0$ and then $\varepsilon \to 0$, we get that the limit in question is $\leqslant 1$.

The lower bound is similar and uses (ii) of Proposition 5 instead of (i). ∎

As an immediate consequence, we remark:

COROLLARY 2.

$$\lim \frac{N_E(H^{(c)})}{|\ln E|} = \frac{1}{\pi}\sqrt{c - \frac{1}{4}} \qquad \textit{if} \quad c > \frac{1}{4};$$

in particular

$$\lim_{c \to c_0} \lim_{E \downarrow 0} \frac{N_E(H^{(c)})}{N_E(H^{(c_0)})} = 1 \qquad \left(c_0 > \frac{1}{4}\right).$$

Corollary 2 and Proposition 5 allow us to prove Theorem 1 for the one-dimensional case:

Proof (Theorem 1, $d = 1$). The proof is closely related to that of Proposition 6. We may write the potential

$$-\frac{c}{(1+|x|^2)} + V_0 \qquad \text{as} \quad V^{(c)} + \tilde{V}_0;$$

then

$$N_E\left(-\frac{d^2}{dx^2}+V^{(c)}+\tilde{V}_0\right)\leqslant N_E\left(-\frac{d^2}{dx^2}+\frac{1}{1-\varepsilon}V^{(c)}\right)+N_E\left(-\frac{d^2}{dx^2}+\frac{1}{\varepsilon}\tilde{V}_0\right).$$

The second term in the above sum is bounded as $E\downarrow 0$ for fixed $\varepsilon>0$. Thus, dividing by $N_{-E}(-d^2/dx^2+V^{(c)})$ and taking the limit $E\downarrow 0$, we get

$$\lim_{E\downarrow 0}\frac{N_E(-d^2/dx^2+V^{(c)}+\tilde{V}_0)}{N_E(-d^2/dx^2+V^{(c)})}\leqslant\lim_{E\downarrow 0}\frac{N_E(H^{(1/(1-\varepsilon))c})}{N_E(H^{(c)})}.$$

This limit converges to one as $\varepsilon\to 0$ by Corollary 2. Again, the lower bound goes along the same lines. ▮

4. The Higher-Dimensional Case

Now, we consider a potential

$$V(x)=\begin{cases}-c/|x|^2 & \text{for } |x|>1\\ 0 & |x|\leqslant 1\end{cases}$$

in arbitrary dimension $x\in\mathbb{R}^d$, $d>1$. Since V is rotation invariant, we may separate the Schrödinger equation in spherical coordinates and obtain (see Reed–Simon [2, X.1]) the radial equation

$$H_l:=-\frac{d^2}{dr^2}+\left(\frac{(d-1)(d-3)}{4}+l(l+d-2)\right)\frac{1}{|x|^2+V(x)},$$

$l=0, 1, \ldots$.

We will restrict ourselves to the cases $d=2$, $d=3$. For the higher-dimensional cases we refer to Reed–Simon [3, X.1 (Appendix)] and Müller [2]. All the computations below are easily done for $d>3$ as well, and nothing changes qualitatively in those dimensions.

Define by $N_l(E)$ the number of eigenvalues below $-E$ of the operator H_l on $L^2(0,\infty)$ with Dirichlet boundary condition at the origin. Then $N(H_0+V, E)$ is given by

$$N_E(H_0+V)=\begin{cases}N_0(E)+2\sum\limits_{l=1}^{\infty}N_l(E) & \text{for } d=2\\ \sum\limits_{l=1}^{\infty}(2l+1)\,N_l(E) & \text{for } d=3.\end{cases}$$

We may now employ the techniques and results of Section 3 to obtain the small E behavior of $N_l(E)$.

Suppose first that $d=2$. In this case, we have that $N_0(E)=\infty$ for arbitrary $c>0$, because of the "angular momentum" term $-1/4r^2$. For $l=0$, the effective potential is given by $-(c+\frac{1}{4})/|x|^2$; hence $\lim_{E\downarrow 0}(N_0(E)/|\ln E|)=(1/2\pi)\sqrt{c}$. (The $\frac{1}{2}$-term appears since we consider an operator on the *half* line.)

For $0<c\leqslant 1$, $N_l(E)<\infty$ for all $l>0$. If $c>1$, then $N_1(E)=\infty$; in fact, the effective potential is $-(c-\frac{3}{4})/|x|^2$; hence $\lim_{E\downarrow 0}(N_1(E)/|\ln E|)=(1/2\pi)\sqrt{c-1}$ and so on. By this procedure, we obtain the result of Theorem 1 for the potential $V=V^{(c)}$ in the case $d=2$. The case $d=3$ (as, in fact, $d>3$) goes along the same line.

We now use Proposition 5 and the fact that

$$\lim_{c\to c_0}\lim_{E\downarrow 0}\frac{N_E(-\Delta+V^{(c)})}{N_E(-\Delta+V^{(c_0)})}=1 \quad \text{if} \quad c>0 \quad \text{for} \quad d=2, \qquad \text{resp. } c>\frac{1}{4} \quad \text{for} \quad d=3$$

to prove Theorem 1 in full strength.

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References

1. W. Kirsch and B. Simon, *J. Funct. Anal.*, to appear.
2. C. Müller, "Lecture Notes in Mathematics," Vol. 17, Springer-Verlag, Berlin/New York, 1966.
3. M. Reed and B. Simon, "Methods of Modern Mathematical Physics II," Academic Press, New York, 1975.
4. M. Reed and B. Simon, "Methods of Modern Mathematical Physics IV," Academic Press, New York, 1978.