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Constructing Solutions of the *mKdV*-Equation

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Using commutation methods (i.e., $N=1$ supersymmetry) underlying Miura's transformation, an explicit construction of solutions of the modified Korteweg–de Vries equation, given a solution of the (ordinary) Korteweg–de Vries equation, is provided.

Under the hypothesis

$$V, \phi \in C^\infty(\mathbb{R}^2) \text{ real-valued}, \ \partial_x^n V, \ \partial_x^n \phi \in L^\infty(\mathbb{R}^2), \qquad n=0, 1 \tag{H.1}$$

we are interested in constructing solutions ϕ of the modified Korteweg–de Vries (*mKdV*-) equation

$$mKdV(\phi) := \phi_t - 6\phi^2\phi_x + \phi_{xxx} = 0, \qquad (t, x) \in \mathbb{R}^2 \tag{1}$$

given a solution V of the Korteweg–de Vries (*KdV*-) equation

$$KdV(V) := V_t - 6VV_x + V_{xxx} = 0, \qquad (t, x) \in \mathbb{R}^2. \tag{2}$$

A key step in our analysis is Miura's transformation [14]

$$V_j(t, x) = \phi(t, x)^2 + (-1)^j \phi_x(t, x), \qquad (t, x) \in \mathbb{R}^2, j=1, 2 \tag{3}$$

and his identity

$$KdV(V_j) = [2\phi + (-1)^j \partial_x] \, mKdV(\phi), \qquad j=1, 2. \tag{4}$$

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As is obvious from (4), whenever ϕ satisfies the *mKdV*-equation (1), then $V_j = \phi^2 + (-1)^j \phi_x$, $j = 1, 2$, will satisfy the *KdV*-equation (2). Our main objective in this paper is to reverse this process, i.e., given a solution V_1 of (2), construct ϕ and V_2 that satisfy (1), (2), and (3), respectively. As will turn out later, Eq. (3), being a Riccati-type equation for ϕ given V_j, will have nonsingular solutions ϕ only if the associated Schrödinger operators $H_j(t) = -\partial_x^2 + V_j(t, \cdot)$, $j = 1, 2$, are nonnegative. We first recall [12, 17]

THEOREM 1. (i) *Assume V satisfies* (H.1), $V_t(t, \cdot) \in L^\infty(\mathbb{R})$, $t \in \mathbb{R}$ *and the KdV-equation* (2). *Then the Schrödinger operator $H(t)$ in $L^2(\mathbb{R})$*

$$H(t) = -\partial_x^2 + V(t, \cdot),\ \mathscr{D}(H(t)) = H^2(\mathbb{R}), \qquad t \in \mathbb{R} \tag{5}$$

is unitarily equivalent to $H(0)$ for all $t \in \mathbb{R}$; i.e., there exists a family of unitary operators $U(t)$, $t \in \mathbb{R}$, $U(0) = 1$ in $L^2(\mathbb{R})$ s.t.

$$U(t)^{-1} H(t) U(t) = H(0), \qquad t \in \mathbb{R}. \tag{6}$$

(ii) *Assume ϕ satisfies* (H.1), $\phi_{xx} \in L^\infty(\mathbb{R}^2)$, $\phi_t(t, \cdot)$, $\phi_{tx}(t, \cdot) \in L^\infty(\mathbb{R})$, $t \in \mathbb{R}$ *and the mKdV-equation* (1). *Then the Dirac operator $Q_m(t)$ in $L^2(\mathbb{R}) \otimes \mathbb{C}^2$*

$$Q_m(t) = \begin{bmatrix} m & A(t)^* \\ A(t) & -m \end{bmatrix},\ \mathscr{D}(Q_m(t)) = H^1(\mathbb{R}) \otimes \mathbb{C}^2, \qquad (m, t) \in \mathbb{R}^2, \tag{7}$$

$$A(t) = \partial_x + \phi(t, \cdot), \qquad \mathscr{D}(A(t)) = H^1(\mathbb{R}), \qquad t \in \mathbb{R} \tag{8}$$

is unitarily equivalent to $Q_m(0)$ for all $t \in \mathbb{R}$; i.e., there exists a family of unitary operators $W_m(t)$, $t \in \mathbb{R}$, $W_m(0) = 1$ in $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ s.t.

$$W_m(t)^{-1} Q_m(t) W_m(t) = Q_m(0), \qquad (m, t) \in \mathbb{R}^2. \tag{9}$$

Proof. It is well known, that the Lax pair

$$\begin{gathered} L(t) = H(t), \\ B_V(t) = -4\partial_x^3 + 6V(t, \cdot)\,\partial_x + 3V_x(t, \cdot),\ \mathscr{D}(B_V(t)) = H^3(\mathbb{R}), \qquad t \in \mathbb{R} \end{gathered} \tag{10}$$

together with

$$\frac{d}{dt} L(t) = [B_V(t), L(t)],\ \frac{d}{dt} U(t) = B_V(t) U(t),\ U(0) = 1, \qquad t \in \mathbb{R} \tag{11}$$

proves (6) (see, e.g., [7, 12, 13]). In the Dirac case, $B_V(t)$ in (10) goes into [17]

$$B_{(\phi^2 - \phi_x)}(t) \oplus B_{(\phi^2 + \phi_x)}(t), \qquad t \in \mathbb{R} \tag{12}$$

and $L(t)$ into $Q_m(t)$. ∎

Remark 2. The fact that

$$Q_m(t)^2 = \begin{bmatrix} H_1(t)+m^2 & 0 \\ 0 & H_2(t)+m^2 \end{bmatrix} \quad \text{on} \quad H^2(\mathbb{R}) \oplus \mathbb{C}^2, \ t \in \mathbb{R}, \tag{13}$$

where

$$\begin{aligned} H_1(t) &= A(t)^* A(t) = -\partial_x^2 + V_1(t,\cdot) \quad \text{on} \quad H^2(\mathbb{R}), \ t \in \mathbb{R}, \\ H_2(t) &= A(t) A(t)^* = -\partial_x^2 + V_2(t,\cdot) \quad \text{on} \quad H^2(\mathbb{R}), \ t \in \mathbb{R} \end{aligned} \tag{14}$$

and V_j, $j=1,2$, are defined in (3), together with Theorem 1(ii) explains why commutation (or $N=1$ supersymmetry) underlies Miura's transformation (3). This has first been exploited in [1, 6]. Commutation for one-dimensional systems; i.e., the factorization into $A(t)$, $A(t)^*$ in (14), has a long history (see, e.g., [6, 9] and the references therein) and has recently become popular again in connection with supersymmetric quantum mechanics (cf., e.g., [2–5, 8, 11, 18] and the references therein). Equation 13 also explains the direct sum in (12).

In order to start with our analysis we assume from now on that V and $H(t)$ defined in (5) satisfy the hypothesis

$$\begin{aligned} &\text{(i)} \quad V \text{ satisfies (H.1) and the } KdV\text{-equation (2).} \\ &\text{(ii)} \quad H(t) \geqslant 0 \text{ for some (and hence for all) } t \in \mathbb{R}. \end{aligned} \tag{H.2}$$

(H.2) (ii) is motivated by (3), (13), and (14) since obviously $H_j(t) \geqslant 0$, $j=1,2$, $t \in \mathbb{R}$. As discussed in [9], one can deal with semibounded Schrödinger operators $H(t) \geqslant c$, $c \in \mathbb{R}$, by changing (3) into $V_j = \phi^2 + (-1)^j \phi_x - c$, $j=1,2$, and using an additional Galilei transformation $(t,x) \to (t, x-6ct)$, $c \in \mathbb{R}$ in $V(t,x)$.

By Theorem 1(i) or Lemma 3 below

$$\inf[\sigma(H(t))] = \inf[\sigma(H(0))] \geqslant 0, \qquad t \in \mathbb{R}. \tag{15}$$

A prominent role in our construction of solutions ϕ of (1) is played by distributional, zero-energy solutions $\psi(t)$ of $H(t)$

$$(H(t)\psi(t))(x) = 0, \qquad (t,x) \in \mathbb{R}^2 \tag{16}$$

(i.e., $\psi(t,\cdot)$, $\psi_x(t,\cdot) \in AC_{\text{loc}}(\mathbb{R})$, $t \in \mathbb{R}$). We start with

LEMMA 3. *Suppose* (H.2) *and let* $\psi_0 \in C^\infty(\mathbb{R})$ *be a real-valued, distributional solution of* $H(0)\psi_0 = 0$. *Then* $H(t)\psi(t) = 0$, $t \in \mathbb{R}$, *has a unique, real-valued, distributional solution* $\psi \in C^\infty(\mathbb{R}^2)$ *that satisfies*

$$\psi_t(t,x) = -4\psi_{xxx}(t,x) + 6V(t,x)\,\psi_x(t,x) + 3V_x(t,x)\psi(t,x), \tag{17a}$$

or equivalently, by using $\psi_{xx} = V\psi$,

$$\psi_t(t, x) = 2V(t, x)\,\psi_x(t, x) - V_x(t, x)\psi(t, x), \qquad (t, x) \in \mathbb{R}^2, \quad (17b)$$

with

$$\psi(0, x) = \psi_0(x), \qquad x \in \mathbb{R}. \quad (17c)$$

In particular, if $\psi(t)$, $\tilde{\psi}(t)$, $t \in \mathbb{R}$, *are two such zero-energy solutions of* $H(t)$, *then their Wronskian*

$$W(\psi(t), \tilde{\psi}(t)) = W(\psi_0, \tilde{\psi}_0) \quad (18)$$

is independent of $(t, x) \in \mathbb{R}^2$.

Proof. Consider the following Volterra-type integral equation

$$\psi(t, x) = c(t) + d(t)x + \int_0^x dx'(x - x')\,V(t, x')\psi(t, x'), \qquad (t, x) \in \mathbb{R}^2,$$

$$c, d \in C^\infty(\mathbb{R}) \text{ real-valued.} \quad (19)$$

Iterating (19) one infers $\psi \in C^\infty(\mathbb{R}^2)$, $H(t)\,\psi(t) = 0$, and

$$\psi_{txx} - V\psi_t = V_t\psi. \quad (20)$$

Similarly, Ψ defined by

$$\Psi(t, x) := 2V(t, x)\,\psi_x(t, x) - V_x(t, x)\psi(t, x), \qquad (t, x) \in \mathbb{R}^2 \quad (21)$$

also satisfies

$$\Psi_{xx} - V\Psi = V_t\psi. \quad (22)$$

Thus $H(t)[\Psi - \psi_t] = 0$. From

$$\begin{aligned} &\psi_t(t, 0) = \dot{c}(t), \ \psi_{tx}(t, 0) = \dot{d}(t), \\ &\Psi(t, 0) = 2V(t, 0)\,d(t) - V_x(t, 0)\,c(t), \\ &\Psi_x(t, 0) = V_x(t, 0)\,d(t) + [2V(t, 0)^2 - V_{xx}(t, 0)]\,c(t); \qquad t \in \mathbb{R} \end{aligned} \quad (23)$$

we finally infer $\psi_t = \Psi$ (and hence (17)) iff

$$\begin{bmatrix} \dot{c}(t) \\ \dot{d}(t) \end{bmatrix} = \begin{bmatrix} -V_x(t, 0) & 2V(t, 0) \\ 2V(t, 0)^2 - V_{xx}(t, 0) & V_x(t, 0) \end{bmatrix} \begin{bmatrix} c(t) \\ d(t) \end{bmatrix}, \qquad t \in \mathbb{R}. \quad (24)$$

Equation (18) finally follows by a simple calculation using (17), $H(t)\psi(t) = 0$, $H(t)\tilde{\psi}(t) = 0$, $t \in \mathbb{R}$, and (2). ∎

Next we state

LEMMA 4. *Assume* V *satisfies* (H.1) *and suppose there exists a* $0<\psi\in C^2(\mathbb{R}^2)$ *with* $H(t)\psi(t)=0$, $t\in\mathbb{R}$. *Define*

$$\phi(t,x):=-\psi_x(t,x)/\psi(t,x), \qquad (t,x)\in\mathbb{R}^2. \tag{25}$$

Then ϕ *and* $\tilde{V}:=\phi^2+\phi_x$ *satisfy* (H.1).

Proof. As shown in Corollary XI.6.5 of [10], $V\in L^\infty(\mathbb{R}^2)$ implies

$$\phi\in L^\infty(\mathbb{R}^2). \tag{26}$$

Since $V=\phi^2-\phi_x$, we infer $\phi_x\in L^\infty(\mathbb{R}^2)$. The rest is trivial. ∎

Concerning positive, zero-energy solutions of $H(t)$ we recall

LEMMA 5 [13]. *Suppose* (H.2) *and assume* $\psi\in C^\infty(\mathbb{R}^2)$ *satisfies* $H(t)\psi(t)=0$, $t\in\mathbb{R}$, *and* (17). *If* $\psi(t,x(t))=0$, $t\in\mathbb{R}$, *then* x *solves*

$$\dot{x}=-2V(t,x), \qquad t\in\mathbb{R}. \tag{27}$$

Conversely, if $\psi(t_0,x_0)=0$ *for some* $(t_0,x_0)\in\mathbb{R}^2$, *solve*

$$\dot{x}=-2V(t,x), \qquad t\in\mathbb{R},\ x(t_0)=x_0 \tag{28}$$

to get $\psi(t,x(t))=0$, $t\in\mathbb{R}$. *In particular, if*

$$\psi(0,x)>0,\ x\in\mathbb{R} \qquad \text{then} \quad \psi(t,x)>0,\ (t,x)\in\mathbb{R}^2. \tag{29}$$

Proof. Equation (27) is obvious from $d\psi=0$. On the other hand, if $x(t)$, $t\in\mathbb{R}$, is the unique solution of (28), then

$$d\psi=\psi_t\,dt+\psi_x\,dx=-V_x\psi\,dt \tag{30}$$

yields $\psi(t,x(t))=0$, $t\in\mathbb{R}$, since $\psi(t_0,x_0)=0$. Finally, if $\psi(0,x)>0$, $x\in\mathbb{R}$, assume that $\psi(t_0,x_0)=0$ for some $(t_0,x_0)\in\mathbb{R}^2$. Then, propagating ψ from t_0 to 0 in time, $\psi(0,x(0))$ would be zero as discussed above. This contradiction proves the last statement. ∎

Before we state our main theorem we would like to mention a classification of Schrödinger operators originally introduced in [16] (see also [15]). In the special case of one-dimensional, nonnegative Schrödinger operators this classification reads as follows: $H(0)\geqslant 0$ is called subcritical iff there are two linearly independent, positive, distributional zero-energy solutions $\psi_{0,\pm}$ of $H(0)$, i.e.,

$$\psi_{0,\pm}(x)>0, \qquad x\in\mathbb{R},\ H(0)\,\psi_{0,\pm}=0,\ W(\psi_{0,-},\psi_{0,+})\neq 0. \tag{31}$$

$H(0)$ is called critical iff $H(0) \geqslant 0$ and there are not two linearly independent, positive zero-energy solutions of $H(0)$; i.e., $H(0)$ is critical iff $H(0)$ has a unique (up to multiples of constants) positive, distributional zero-energy solution.

Combining this classification and Lemma 5 we get

LEMMA 6. *Suppose* (H.2). *Then $H(t)$ is subcritical (resp. critical) for all $t \in \mathbb{R}$ iff $H(0)$ is. Now we are ready to state our main result.*

THEOREM 7. *Assume V satisfies* (H.1) *and the KdV-equation* (2). *Suppose $H(0) \geqslant 0$ and assume $0 < \psi_{\pm} \in C^{\infty}(\mathbb{R}^2)$ to satisfy $H(t)\psi_{\pm}(t) = 0$, $t \in \mathbb{R}$, and* (17). *Define*

$$\begin{aligned} \psi_\sigma(t,x) &:= 2^{-1}[1-\sigma(t)]\,\psi_-(t,x) + 2^{-1}[1+\sigma(t)]\,\psi_+(t,x), \\ \phi_\sigma(t,x) &:= -\psi_{\sigma,x}(t,x)/\psi_\sigma(t,x), \\ \tilde{V}_\sigma(t,x) &:= \phi_\sigma(t,x)^2 + \phi_{\sigma,x}(t,x); \qquad (t,x) \in \mathbb{R}^2, \end{aligned} \tag{32}$$

where $\sigma\colon \mathbb{R} \to [-1, 1]$, $\sigma \in C^{\infty}(\mathbb{R})$. Then ϕ_σ and $\tilde{V}_\sigma$ satisfy (H.1). *In addition,*

$$mKdV(\phi_\sigma) = 0,\ KdV(\tilde{V}_\sigma) = 0 \qquad \text{iff} \quad \dot{\sigma} = 0 \text{ or } W(\psi_-, \psi_+) = 0. \tag{33}$$

Proof. Let $\phi := -\psi_x/\psi$, $0 < \psi \in C^{\infty}(\mathbb{R}^2)$, $H(t)\psi(t) = 0$. Then a straightforward calculation yields the identity

$$mKdV(\phi) = \psi^{-2}\{\psi_t\psi_x - \psi\psi_{xt} - 6V\psi_x^2 + 3\psi_{xx}^2 + 4\psi_x\psi_{xxx} - \psi\psi_{xxxx}\}. \tag{34}$$

Taking $\psi = \psi_\sigma$ in (34) and taking into account (17) for $\psi_{\pm}$, finally gives

$$mKdV(\phi_\sigma) = -\psi_\sigma^{-2}\dot{\sigma}W(\psi_-, \psi_+)/2. \tag{35}$$

In particular, if $mKdV(\phi_\sigma) = 0$ then $KdV(\tilde{V}_\sigma) = 0$ since V and $\tilde{V}_\sigma$ are connected via commutation and hence (4) applies with $V_1 = V$, $V_2 = \tilde{V}_\sigma$, $\phi = \phi_\sigma$. ∎

Remark 8. Iff $H(0)$ is subcritical, Theorem 7 yields a one-parameter family of solutions ϕ_σ, $\sigma \in [-1, 1]$ of the *mKdV*-equation (1) (choose $\psi_{\pm}$ s.t. $W(\psi_-, \psi_+) \neq 0$). Iff $H(0)$ is critical, we necessarily have $W(\psi_-, \psi_+) = 0$ and hence (32) yields a unique solution of (1) (since ϕ_σ in (32) is then actually independent of σ). Moreover, since $\phi_\sigma = -\psi_{\sigma,x}/\psi_\sigma$, $\sigma \in [-1, 1]$ is the general solution of the Riccati equation $\phi_x - \phi^2 = -V$ on $\mathbb{R}$, the explicit construction (32) yields all smooth solutions ϕ of (1) related to V via $V = \phi^2 - \phi_x$.

Remark 9. Given ϕ_σ in (32), $H(t)$ is recovered from ϕ_σ via

$$\begin{aligned} H(t) &= -\partial_x^2 + V(t,\cdot) = A_\sigma(t)^* A_\sigma(t), && t\in\mathbb{R},\\ A_\sigma(t) &= \partial_x + \phi_\sigma(t,\cdot), && t\in\mathbb{R},\\ V(t,x) &= \phi_\sigma(t,x)^2 - \phi_{\sigma,x}(t,x), && (t,x)\in\mathbb{R}^2. \end{aligned} \tag{36}$$

Similarly, $\tilde{H}_\sigma(t) := -\partial_x^2 + \tilde{V}_\sigma(t,\cdot)$, $t\in\mathbb{R}$, is recovered from ϕ_σ via

$$\begin{aligned} \tilde{H}_\sigma(t) &= -\partial_x^2 + \tilde{V}_\sigma(t,\cdot) = A_\sigma(t)\, A_\sigma(t)^*, && t\in\mathbb{R},\\ \tilde{V}_\sigma(t,x) &= \phi_\sigma(t,x)^2 + \phi_{\sigma,x}(t,x), && (t,x)\in\mathbb{R}^2 \end{aligned} \tag{37}$$

illustrating again the role that commutation plays in our approach.

Remark 10. As shown in [9], Theorem 7 actually applies to the entire hierarchy of higher-order (m) KdV-equations.

Theorem 7 has been applied to soliton-like solutions in [9] in full details. Special cases of finite-zone, periodic solutions and solitons relative to such a periodic background together with an extension of the above framework to certain singular solutions of (1) resp. (2) are also contained in [9]. A complete treatment of these cases is in preparation.

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Note added in proof. The "if part" in Theorem 7 follows, e.g., from prolongation methods developed in H. D. Wahlquist and F. B. Estabrook, Prolongation structures of nonlinear evolution equations, *J. Math. Phys.* **16** (1975), 1–7. A different approach to Theorem 7, assuming rapidly decreasing solutions of the KdV-eq., can be found in Section 38 of R. Beals, P. Deift, C. Tomei, Direct and inverse scattering on the line, *in* "Mathematical Surveys and Monographs," Vol. 28, Amer. Math. Soc., Providence, RI, 1988.

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