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On the Glimm–Jaffe Linear Lower Bound in $P(\phi)_2$ Field Theories

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Let $H(g) = H_0 + H_I(g)$ be the Hamiltonian of a $:P(\phi):_2$ quantum field theory with spatial cutoff, g. For $g \geqslant 0$, with $\int_{|x-y|\leqslant 1} |g(y)|^2 \, dy \leqslant C$ independent of x, we discuss the Glimm–Jaffe linear lower bound, $H(g) \geqslant -D$ [size of supp g]. We show that it is a fairly elementary consequence of the localizability of the interaction and the lower bound for $H(g)$.

1. INTRODUCTION

In this brief note, we will consider the Hamiltonian $H(g) = H_0 + H_I(g)$ where H_0 is the free Boson Hamiltonian of mass $m_0 > 0$ in two-dimensional space time and $H_I(g) = \int g(x) : P(\phi(x)) : dx$ where $g \in L^1 \cap L^2$; $g \geqslant 0$ and : : is Wick ordering. $P(X)$ is a polynomial bounded below for $X \in \mathbf{R}$. Fock space and the meaning of these various terms is reviewed in [3] and in Section III of [9]. We are particularly interested in the lower bound $E(g) = \inf \sigma(H(g))$. By a theorem of Nelson and Glimm [6, 2] whose proof has been simplified by Segal [8], $E(g) > -\infty$. We are concerned here with the stronger result of Glimm and Jaffe [4], that if $\| g_n \|_\infty$ and $\| dg_n/dx \|_\infty$ are bounded $E(g) \geqslant -D\mu(\text{supp}\, g_n)$ for a sequence g_n with $g_n \to 1$ (say).

In an interesting recent paper [5], Glimm and Jaffe have given (among other things) a new proof of the lower boundedness of Nelson–Glimm–Segal. While their new method of proof is (in our opinion) of greater difficulty than the NGS proof in its simplest form, it provides a much simplified proof of the linear lower bound $E(g) \geqslant -D\mu(\text{supp}\, g_n)$. This note had its genesis in trying to understand why the linear lower bound seemed to have a simple proof in this new Glimm–Jaffe setting but not in the hypercontractive setting [6, 2, 8, 9]. In fact, we will see that the linear lower bound is a consequence of the locality of H_I, the structure of Fock space and the NGS bound. Our proof will not depend on hypercontractive estimates, higher-order estimates or the new Glimm–Jaffe technique. *Any method of proving the* NGS *bound immediately yields the linear lower bound upon application of our result.*

The basic technical device we use is the notion of localized number operators similar to those of [4, 5]. However, *we localize in the relatavistic x-space (Dirac space) rather than in Newton–Wigner space as Glimm and Jaffe do.* One can understand why the operator N_{loc} in [5] has to have a Newton–Wigner kernel falling off no faster than exponentially; what was really critical is that it be strictly positive on an interval in Dirac space.

2. Local Number Operators

Our goal in this section is to introduce local number operators, and to prove an estimate $N_{\text{loc}} + H_I(g) \geqslant c$ as long as g has support in the region where N is localized.

We use the notation of [9]. It will be convenient to translate the one-particle space of [9, Section III.3] to x-space so we think of $\mathscr{H}$, the one-particle space, as functions of x with inner product

$$\langle f, g\rangle = \int dk\, \overline{\hat{f}(k)}\, \hat{g}(k)\, \omega(k)^{-1} \qquad \text{with} \quad \omega(k) = (k^2 + m_0{}^2)^{1/2}$$

and ^ the Fourier transform. For any interval, $J = [\alpha, \beta]$, of $\mathbf{R}$ with $-\infty < \alpha < \beta < \infty$, let $\mathscr{H}_J$ be the closure in $\mathscr{H}$ of the functions in $\mathscr{S}$ with support in $[\alpha, \beta]$. Let $\mathscr{F}$ be the Fock space built with one-particle space $\mathscr{H}$ and $\mathscr{F}_J$ be the one-particle space built on $\mathscr{H}_J$ [9, Section III.1]. Corresponding to the breakup of $\mathscr{H} = \mathscr{H}_J \oplus \mathscr{H}_J{}^\perp$ there is a tensor product decomposition $\mathscr{F} = \mathscr{F}_J \otimes \mathscr{F}_{(J,\perp)}$ of $\mathscr{F}$ (see [9, Section IV.2]).

Definition. $N_{\text{loc}}^{(J)} = d\Gamma(P_J)$.

These local number operators are different from the local number operators introduced by Glimm and Jaffe in [4]. Their local number operators N_J are localized in Newton–Wigner space [7]. Explicitly, one introduces the Newton–Wigner transform $\mathscr{N}\mathscr{W}(f) = \mathscr{F}^{-1}(\omega(k)^{1/2})\mathscr{F}$ where $\mathscr{F}$ is the Fourier transform and $\omega(k)^{1/2}$ is multiplication by $\omega(k)^{1/2}$. $\mathscr{N}\mathscr{W}$ maps $\mathscr{H}$ unitarily onto $L^2(x, dx)$ (with Lebesgue measure). $P_J^{(G-J)}$ is the projection onto those $f \in \mathscr{H}$ for which $\mathscr{N}\mathscr{W}(f)$

has support in J and the Glimm–Jaffe local number operators are $d\Gamma(P_J^{(G-J)}) = N_J$. The advantage of our localization (in Dirac x-space) is that $H_I(g)$ is nonlocal in Newton–Wigner space. The disadvantage of the $N_J^{(\mathrm{loc})}$ we use is that $P_K P_J \neq 0$ even if $K \cap J = \phi$. Thus one must be wary of using too intuitive an idea of localization.

The first crucial property of $N_{\mathrm{loc}}^{(J)}$ is

PROPOSITION II.1. *Under the decomposition* $\mathscr{F} = \mathscr{F}_J \otimes \mathscr{F}_{J,\perp}$, $N_{\mathrm{loc}}^{(J)} = (N \restriction \mathscr{F}_J) \otimes 1$. *Here N is the number operator, $N = d\Gamma(I)$.*

Proof. An elementary Fock space fact.

We are thus able to prove

THEOREM II.2. *Suppose $H_I(g) = \int dx\, g(x) :P(\phi(x)):$ where $P(x)$ is a polynomial which is bounded below (see* [9, *Section* III.3]), *where $g \in L^1 \cap L^2$. If g has support in $(\alpha, \beta) = J^{\mathrm{interior}}$ and $g \geqslant 0$, then $N_{\mathrm{loc}}^{(J)} + H_I(g)$ is bounded below.*

Remark. This is related to a result of Glimm–Jaffe [5]. Because of the differences of our local number operators, these results are not identical.

Proof. It is known that when $\operatorname{supp} g \subset J^{\mathrm{int}}$, then $H_I(g)$ is an unbounded operator affiliated with the algebra generated by the $\phi(h)$ with $h \in \mathscr{H}_J$ (see, e.g., [9, Theorem III.15(d)]). Thus $H_I(g)$ also decomposes under the tensor product $\mathscr{F}_J \otimes \mathscr{F}_{(J,\perp)}$ into $H_I(g) \restriction \mathscr{F}_J \otimes 1$. We conclude $N_{\mathrm{loc}}^{(J)} + H_I(g) = (N + H_I(g)) \restriction \mathscr{F}_J \otimes 1$. Since it is known that $N + H_I(g)$ is bounded from below [9, Theorem IV.1], [2, 6, 8], this local bound follows from the hypercontractive bound.

3. SUMS OF LOCAL NUMBER OPERATORS

Now let $J_n = [(n/2) - 1, (n/2) + 1]$ and $P^{(n)} \equiv P_{J_n}$, $N_{\mathrm{loc};n} \equiv N_{\mathrm{loc}}^{(J_n)}$. We first note the critical exponential falloff which is basically the strong cluster property of Araki, Hepp and Ruelle [1]

LEMMA 3.1. $\| P^{(n)} P^{(m)} \| \leqslant c_1 e^{-c_2 |n-m|}$ *with* $c_2 > 0$.

Remark. Actually c_2 may be chosen equal to $(1/2)\, m_0$, with m_0 the bare mass. This technical result is the heart of our proof of the linear lower bound. We defer the proof to Appendix 2—the basic input there is the exponential falloff of the Newton–Wigner transform kernel.

THEOREM III.1. *There is a constant, c_3, with $\sum_{i=-n}^{m} N_{\mathrm{loc};i} \leqslant c_3 N$ for all n, m.*

Proof. Because $d\Gamma$ is order preserving, we need only prove $\sum_{i=-n}^{m} P^{(i)} \leqslant c_3 1$. By Theorem A.2 (Appendix 1), it is sufficient to prove $\| P^{(i)} P^{(j)} \| = d_{ij}$ is the matrix of a bounded operator on $l_2(-\infty, \infty)$. But by Lemma 3.1, $d_{ij} \leqslant c_1 e^{-c_2|i-j|}$. Thus d_{ij} is the matrix of a bounded operator [for $e_{ij} = f(i-j)$ is a bounded operator on all $l_p(-\infty, \infty)$ by Young's inequality if $\sum_n |f(n)| < \infty$].

4. The Linear Lower Bound

It is now child's play to prove

THEOREM IV.1. *Let $P(X)$ be a polynomial which is bounded below. Let g_l be the characteristic function of $[-(l/2), (l/2)]$. Then for some constant c_4*

$$H_0 + H_I(g_l) \geqslant c_4 l.$$

More generally, if $\omega \geqslant d_1 1$ is a one-particle operator with $d_1 > 0$ and a number d_2 is given one can find $c(d_1, d_2)$ so that

$$d\Gamma(\omega) + H_I(g) \geqslant cl$$

whenever g is positive with support in $[-(l/2), (l/2)]$ and

$$\sup_{n=-l,\ldots,l-1} \int_{n/2}^{(n+1)/2} |g(x)|^2 \, dx \leqslant d_2 .$$

Proof. Let h_k be the characteristic function of $[(n/2) - (1/2), (n/2) + (1/2)]$. By Theorem II.2 for any $\alpha > 0$, there is a c_6 with $N_{\mathrm{loc}:0} + \alpha H_I(h_0) \geqslant c_6$. By translation invariance $N_{\mathrm{loc}:k} + \alpha H_I(h_k) \geqslant c_6$ also, so

$$\left(\sum_{k=-l+1}^{k=l-1} N_{\mathrm{loc};k} \right) + \alpha H_I(g_l) \geqslant (2l-1)\, c_6 .$$

By Theorem III.1

$$c_3 N + \alpha H_I(g_l) \geqslant (2l-1)\, c_6 ,$$

so taking $\alpha = c_3$,

$$N + H_I(g_l) \geqslant (2 c_6 c_3^{-1}) l.$$

The general theorem is a consequence of this argument and the fact that for a fixed interval of support, the bound of $N_0 + \alpha H_I(g)$ is only a function of $\| g \|_2$ and $\| g \|_1$ (see [9, Theorem III.20]).

APPENDIX 1: A Strong Bessel's Inequality

If $\{P_i\}_{i=-\infty}^{\infty}$ is a family of pairwise orthogonal projections on a Hilbert space, we know, of course, that $\sum_{i=-n}^{m} P_i \leqslant 1$ for any n, m. This is just a consequence of Bessel's inequality. We prove our generalization of this pairwise orthogonal theorem by mimicing the proof of Bessel's inequality.

Lemma A.1. *Let $\{\eta_i\}_{i=-\infty}^{\infty}$ be a family of vectors in a Hilbert space $\mathscr{H}$, with $|\langle \eta_i , \eta_j \rangle| = c_{ij}$, where c_{ij} is the matrix of a bounded operator c on $l_2(-\infty, \infty)$. Then, for any $f \in \mathscr{H}$:*

$$\sum_{i=-\infty}^{\infty} |\langle \eta_i , f \rangle|^2 \leqslant \| c \| \, \| f \|^2$$

Proof. Let $\alpha_i = \langle \eta_i , f \rangle$ and $\beta = \| c \|$. Then

$$0 \leqslant \left\| \beta f - \sum_{i=-n}^{m} \alpha_i \eta_i \right\|^2 = \beta^2 \| f \|^2 - 2\beta \sum_{i=-n}^{m} | \alpha_i |^2 + \sum_{i,j=-n}^{m} \bar{\alpha}_i \alpha_j \langle \eta_i , \eta_j \rangle.$$

But

$$\left| \sum_{i,=j-n}^{m} \bar{\alpha}_i \alpha_j \langle \eta_i , \eta_j \rangle \right| \leqslant \sum_{i,j=-n}^{m} c_{ij} \, | \bar{\alpha}_i | \, | \alpha_j | \leqslant \| c \| \sum_{i=-n}^{m} | \alpha_i |^2.$$

Thus $0 \leqslant \beta^2 \| f \|^2 - \beta \sum_{i=-n}^{m} | \alpha_i |^2$.

Theorem A.2. *Let $\{P_i\}_{i=-\infty}^{\infty}$ be a family of projections on a Hilbert space, $\mathscr{H}$. If $d_{ij} = \| P_i P_j \|$ is the matrix of a bounded operator, D, on $l_2(-\infty, \infty)$, then $\| \sum_{i=-n}^{m} P_i \| \leqslant \| D \|$ for all n, m (so $w - \lim \sum_{i=-\infty}^{\infty} P_i$ exists).*

Proof. Let η_i be defined by

$$\eta_i = \begin{cases} \dfrac{P_i f}{\| P_i f \|} & \text{if} \quad P_i f \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $|\langle\eta_i, \eta_j\rangle| = |\langle\eta_i, P_iP_j\eta_j\rangle| \leqslant \|\eta_i\| \|\eta_j\| d_{ij} = d_{ij}$. Since $|\langle\eta_i, \eta_j\rangle| = c_{ij}$ and d_{ij} are matrices with positive coefficients, C is a bounded operator and $\|C\| \leqslant \|D\|$.

By Lemma A.1, $\sum_{i=-\infty}^{\infty} |\langle\eta_i, f\rangle|^2 \leqslant \|D\| \|f\|^2$. Now $\alpha_i \equiv \langle\eta_i, f\rangle = \|P_if\|$ so $\sum_{i=-m}^{n} P_if = \sum_{i=-m}^{n} \|P_if\| \eta_i = \sum_{i=-m}^{n} \alpha_i\eta_i$. Thus

$$\left\| \sum_{i=-m}^{n} P_if \right\|^2 \leqslant \sum_{i,j=-m}^{n} |\langle\eta_i, \eta_j\rangle| \, |\alpha_i| \, |\alpha_j|,$$

$$\leqslant \|D\| \sum_{i=-m}^{n} |\alpha_i|^2 \leqslant \|D\|^2 \|f\|^2.$$

APPENDIX 2: Proof of Lemma 3.1

(1) Let us first note

Lemma A.3. *If $f \in \mathscr{H}$, i.e., $\int \overline{\hat{f}(k)} \hat{f}(k)\, \omega(k)^{-1}\, dk < \infty$ and f has support in an interval $[\alpha, \beta] - \infty < \alpha < \beta < \infty$, then $\hat{f}$ is an entire function and for any fixed real b, $F_b(k) = \hat{f}(k + ib)$ obeys $\int |F_b(k)|^2 \omega(k)^{-1}\, dk$ is finite and bounded uniformly for b in any compact.*

Proof. f is a distribution of compact support so the Payley–Weiner theorem [11] implies $\hat{f}$ is entire. Moreover, if $g \in \mathscr{S}$ and $|g(x)| \leqslant Ce^{-x^2}$; $g \equiv 1$ on $[\alpha, \beta]$, then

$$F_b = (\widehat{ge^{+bx}})\hat{f}.$$

It is thus enough to prove multiplication by $h \in \mathscr{S}$ leaves $\mathscr{H}$ invariant or equivalently, that convolution with $h \in \mathscr{S}$ takes $L^2(\mathbf{R}, \omega^{-1}\, dk)$ into itself. Convolution with h clearly takes L^∞ into itself. Let $g \in L^1(\mathbf{R}, \omega^{-1}\, dk)$ and $f = h*g$. Since $\omega(k)/\omega(p) \leqslant C\omega(k - p)$ for suitable C,

$$\int |f(p)|\, \omega(p)^{-1}\, dp \leqslant \int |h(p - k)| \frac{\omega(k)}{\omega(p)} g(k)^{-1}\, dk\, dp,$$

$$\leqslant C \|g\|_{L^1(\mathbf{R}, \omega^{-1} dk)}.$$

Thus, by the Reisz–Thorn theorem, $F_b \in L^2(\mathbf{R}, \omega^{-1}\, dk)$. This completes the proof of Lemma A.3.

(2) Let U_n be translation in $\mathscr{H}$ by $n/2$, i.e., $(U_nf)(x) =$

$f[x - (n/2)]$. Then U_n is unitary and $U_n^{-1} P^{(1)} U_n = P^{(n)}$. Thus to prove $\| P^{(i)} P^{(j)} \| \leqslant C e^{-D(i-j)}$ it is enough to prove $\| P_1 U_n P_1 \| \leqslant C e^{-D|n|}$.

(3) Applying the uniform boundedness principle to $P_1 U_n P_1 e^{+D|n|}$, it is enough to prove for all $\phi, \psi \in \mathscr{H}$ that $\langle \psi, P_1 U_n P_1 \phi \rangle \leqslant C_{\psi,\phi} e^{-D|n|}$.

(4) By polarization, it is enough to show

$$\langle \psi, U_n \psi \rangle \leqslant C_\psi e^{-D|n|}$$

for all $\psi \in \operatorname{Ran} P_1$.

(5) Let $\psi \in \operatorname{Ran} P_1$. Let $\phi = \mathscr{N}\mathscr{W}\psi$ where $\mathscr{N}\mathscr{W}$ is the Newton–Wigner transform $\hat{\phi} = \omega^{-1/2}\hat{\psi}$ introduced in Section 2. Let $\langle , \rangle_{\mathscr{N}\mathscr{W}}$ be the ordinary $L^2(\mathbf{R}, dx)$ inner product ($\langle , \rangle$ is the $\mathscr{H}$ inner product) so $\langle \mathscr{N}\mathscr{W}\psi, \mathscr{N}\mathscr{W}\psi \rangle_{\mathscr{N}\mathscr{W}} = \langle \psi, \psi \rangle$. If U_n also represents translation on Newton–Wigner space by $(n/2)$, $U_n(\mathscr{N}\mathscr{W}) = (\mathscr{N}\mathscr{W}) U_n$. To prove the result of (4) we need only prove $\langle \phi, U_n \phi \rangle_{\mathscr{N}\mathscr{W}} \leqslant C_\phi e^{-D|n|}$ for all $\phi \in \mathscr{N}\mathscr{W}(\operatorname{Ran} P_1)$.

(6) If $\psi \in \operatorname{Ran} P_1$, $\phi = \omega^{-1/2}\psi$ is analytic in $\{k \mid |\operatorname{Im} k| < m_0\}$ and each $\hat{\phi}(\cdot + ib) \in L^2(\mathbf{R}, dk)$ if $|b| \leqslant m_0$. Thus, by a simple theorem [10], $e^{\pm bx}\phi(x) \in L^2(\mathbf{R}, dx)$ if $|b| \leqslant m_0$. As a consequence $e^{b|x|}\phi \in L^2$ if $0 \leqslant b \leqslant m_0$.

(7) Finally pick $D = m_0$ fixed. Let $\phi \in \mathscr{N}\mathscr{W}(\operatorname{Ran} P_1)$. By (6), $\phi = e^{-D|x|}\eta$ for $\eta \in L^2$. Thus

$$\begin{aligned} \langle \phi, U_n \phi \rangle_{\mathscr{N}\mathscr{W}} &\leqslant \int e^{-D|x|} e^{-D|x-(n/2)|} \, |\eta(x)| \, |\eta[x - (n/2)]|, \\ &\leqslant e^{-(D/2)|n|} \|\eta\|_{\mathscr{N}\mathscr{W}} \|U_n \eta\|_{\mathscr{N}\mathscr{W}} \\ &\leqslant e^{-(D/2)n} \|\eta\|_{\mathscr{N}\mathscr{W}}. \end{aligned}$$

This completes the proof of Lemma 3.1.

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Note added in proof. Results related to Theorem A.2 have been previously published by M. Cotlar, *Rev. Math. Cuyana*, **1** (1955), 41–55 and A. W. Knapp and E. M. Stein, *Proc. Nat. Acad. Sci.* (*U.S.A.*) **63** (1969), 281–284. While these results are neither weaker nor stronger than Theorem A.2, the Knapp-Stein lemma can be used in the proof of Theorem III.1 in place of Theorem A.2.

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