JOURNAL OF FUNCTIONAL ANALYSIS **91**, 117–142 (1990)

Relativistic Schrödinger Operators: Asymptotic Behavior of the Eigenfunctions

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Communicated by L. Gross

Received December 19, 1988; received March 10, 1989

Nonrelativistic Schrödinger operators are perturbations of the negative Laplacian and the connection with stochastic processes (and Brownian motion in particular) is well known and usually goes under the name of Feynman and Kac. We present a similar connection between a class of relativistic Schrödinger operators and a class of processes with stationary independent increments. In particular, we investigate the decay of the eigenfunctions of these operators and we show that not only exponential decay but also polynomial decay can occur. © 1990 Academic Press, Inc.

I. INTRODUCTION

The motivation for the present study is to be found in the desire of a better understanding of the spectral properties of some pseudo-differential operators which occur naturally when one tries to include relativistic corrections to the mathematical theory based on the Schrödinger operator,

$$H = H_0 + V, \tag{I.1}$$

where

$$H_0 = -\Delta \tag{I.2}$$

* Partially supported by NSF Grant DMS-8701320.

[†] Partially supported by NSF Grant DMS 84-16049.

0022-1236/90 $3.00

and where V is the operator of multiplication by a function V. The latter corresponds to the quantization of the potential energy. For example, in the case of the N-body problem one has

$$V(x) = \sum_{1 \leqslant j < k \leqslant N} \frac{1}{|x_j - x_k|} + \sum_{1 \leqslant j < k \leqslant M} \frac{Z^2}{|y_j - y_k|} - \sum_{\substack{1 \leqslant j \leqslant N \\ 1 \leqslant k \leqslant M}} \frac{Z}{|x_j - y_k|}, \tag{I.3}$$

where Z is the charge of the nuclei, N is the number of electrons, M is the number of nuclei, $y_1, ..., y_M$ are the positions of the M nuclei in $\mathbf{R}^3$, $x_1, ..., x_N$ are the positions of the N electrons also in $\mathbf{R}^3$, and $x = (y_1, ..., y_M, x_1, ..., x_N)$. Most of the mathematical treatments deal with the case of nuclei at rest (or with infinite mass). In this case, one uses $x = (x_1, ..., x_n)$ and one thinks of $y_1, ..., y_n$ as parameters.

The operator H_0 corresponds to the quantization of the kinetic energy and is usually called the free Hamiltonian. It is the sum of the kinetic energies of the various electrons, but for all mathematical purposes we can assume that it is the negative Laplacian in n dimensions where $n = 3N$. The operators H and H_0 are well defined on the space of smooth functions with compact supports and they are investigated as unbounded operators in the Hilbert space $L^2(\mathbf{R}^n)$. The classical form of the kinetic energy of a particle is $p^2/2m$, where p denotes the momentum of the particle in question and m its mass. The quantization procedure takes the classical momentum p into the differential operator $-i\nabla$. This explains why the Laplacian is, up to a multiplicative constant, the quantum analog of the kinetic energy. In other words, $H_0 = F(-i\nabla) = F(p)$ with $F(p) = p^2 = p_1^2 + \cdots + p_n^2$. Relativity theory tells us that such a choice for the kinetic energy is appropriate at low energies only. Moreover, the kinetic energy should be proportional to the modulus of p rather than its square for high energies. Actually, the classical energy for a relativistic particle of mass m is

$$E = \sqrt{p^2 + m^2},$$

and this justifies the choice:

$$H_0 = \sqrt{p^2 + m^2} - m = \sqrt{-\Delta + m^2} - m, \tag{I.4}$$

for its quantum analog. The point of the subtraction of the constant m is to make sure that the spectrum of the operator H_0 is $[0, \infty)$. This explains the terminology of *relativistic Schrödinger operators* for the operators of the form (I.1) with H_0 given by (I.4). Note that we will abuse this convention and use this terminology for operators given by functions F more general than the one we just discussed.

There exists an important literature on the spectral properties of such relativistic Hamiltonians. Most of it has been strongly influenced by Lieb's investigations on the stability of matter. See, for example, Lieb [21], Weder [33, 34], Herbst [17], Daubechies and Lieb [13], Conlon [10], Daubechies [12], Fefferman [15], Fefferman and de la Llave [14], and also Lieb and Yau [22].

One of the fundamental mathematical problems in proving the stability (or instability) of matter is to estimate the infimum of the spectrum of the operator H when the numbers N and/or M become large. In this asymptotic regime, the free Hamiltonian $H_0 = F(p)$ given by (I.4) can be abandoned and replaced by $H_0 = |p|$. Indeed, the difference between these two operators remains bounded and the asymptotic result needed to prove the stability of matter can be proved using either one of these free Hamiltonians. Obviously, the scaling properties of the function $F(p) = |p|$ attracted early investigations of the corresponding pseudo-differential operator H_0, especially because its scaling is related to the scaling of the Coulomb potential. Also, very fine estimates on its Green's function are available. See, for example, Stein [32]. These are the technical reasons why $F(p) = |p|$ is preferred to $\sqrt{p^2 + m^2} - m$.

A deep understanding of the mathematical properties of the operator H for N and M fixed is of crucial importance in proving the estimates relevant to the stability of matter. But it is also a very interesting mathematical problem of its own. In this respect, if one is ready to ignore temporarily the important scaling property, the operator $H_0 = \sqrt{p^2 + m^2} - m$ is in many ways more regular and more attractive than the operator $H_0 = |p|$. Indeed, its Green's function decays exponentially instead of polynomially, and as we are about to demonstrate in this paper, the eigenfunctions of the corresponding Schrödinger operator (I.1) decay exponentially instead of polynomially.

In fact the exponential decay of the eigenfunctions corresponding to isolated eigenvalues played a crucial role in many of the investigations of the spectral properties of the nonrelativistic Schrödinger operators (I.1), (I.2) and one expects the same to happen for its relativistic counterpart (I.1), (I.4). Surprisingly enough, except for the isolated work of Nardini, the problem of the decay of the eigenfunctions of these operators has not been investigated. See Nardini [23, 24]. One of the goals of the present study is to fill this gap.

We now describe the connection of the above problems and the theory of stochastic processes. The notations and the terminology we use throughout the paper are introduced along the way. As explained above, there are functions $F(p)$ which are natural candidate for the definition of the quantum analog of the classical kinetic energy. But it is very interesting from a mathematical point of view to investigate the largest possible class of such functions $F(p)$. See for example Herbst and Sloan [18] for a functional analytic study in this spirit. A natural requirement that one should have on a free Hamiltonian $H_0 = F(p)$ is that it generates a semigroup $\{e^{-tH_0}; t \geqslant 0\}$ of positivity preserving operators on $L^2(\mathbf{R}^n)$. See, for example, Reed and Simon [27]. It is proven in Appendix 2 of Section 12 of Chapter XIII that $F(p)$ is such a function if and only if for each cut-off function $h: \mathbf{R}^n \to \mathbf{R}^n$ with compact support and which satisfies $h(x) = x$ in a neighborhood of the origin one has the representation:

$$F(p) = a + ib \cdot p + p \cdot C_p - \int_{\mathbf{R}^n} [e^{ip \cdot x} - 1 - ip \cdot h(x)] \, \nu(dx), \tag{I.5}$$

for some real constant a, some vector b in $\mathbf{R}^n$, some nonnegative definite matrix C, and some nonnegative measure ν satisfying $\int_{\mathbf{R}^n} \min(1, |x|^2) \, \nu(dx) < \infty$. Note that if $F(p)$ has a representation of the above type for some cut-off function h, it has a similar representation for any other cut-off function. The choice of the cut-off function only affects the constant a and the vector b. The latter is called the drift, C is called the covariance, and the measure ν is called the Lévy measure. Formula (I.5) is the famous *Lévy–Kintchine formula* and the function F is called the exponent function by probabilists because of (I.8) below. Well-known examples are

$$F(p) = \sqrt{p^2 + m^2} - m, \tag{I.6}$$

for $m > 0$ which we already encountered and the stable case of

$$F^{(\alpha)}(p) = |p|^\alpha, \tag{I.7}$$

for $0 < \alpha \leqslant 2$. The additive constant a will be chosen to be equal to zero. This normalization is very convenient for it implies the existence of a convolution semigroup $\{\mu_t; t \geqslant 0\}$ of (infinitely divisible) probability measures on $\mathbf{R}^n$ such that

$$\hat{\mu}_t(p) = e^{-tF(p)}, \tag{I.8}$$

for all $t > 0$ and p in $\mathbf{R}^n$. In other words, the free semigroup $\{e^{-tH_0}; t \geqslant 0\}$ is a convolution semigroup given by

$$[e^{-tH_0} f](x) = \int_{\mathbf{R}^n} f(x+y) \, \mu_t(dy). \tag{I.9}$$

But it is well known that such semigroups are generated by *stochastic processes with stationary independent increments* which we will call *Lévy processes* from now on. More precisely, on the canonical path space $\Omega = D([0, \infty), \mathbf{R}^n)$ of right continuous functions with left limits from

$[0, \infty)$ into $\mathbf{R}^n$ endowed with the smallest σ-field $\mathscr{F}$ for which all the coordinate functions $X_t: \Omega \ni \omega \to X_t(\omega) = \omega(t) \in \mathbf{R}^n$ are measurable, one can define for each $x \in \mathbf{R}^n$ a probability measure $\mathbf{P}_x$ such that

$$\mathbf{P}_x\{X_0 = x\} = 1, \tag{I.10}$$

and the random variables $X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are independent with distributions $\mu_{t_1 - t_0}, ..., \mu_{t_n - t_{n-1}}$, respectively, whenever $0 = t_0 < t_1 < \cdots < t_n < \infty$. Note that each probability measure $\mathbf{P}_x$ is the image of the unique probability $\mathbf{P}_0$ under the mapping $\omega(\cdot) \hookrightarrow x + \omega(\cdot)$ and that all the probabilistic quantities could be written in terms of the single measure $\mathbf{P} = \mathbf{P}_0$. If we use the notation $\mathbf{E}_x$ for the expectation with respect to the probability $\mathbf{P}_x$ (we will also use the notation $\mathbf{E}$ for the expectation with respect to the measure $\mathbf{P}$), the formula (I.9) can be rewritten as

$$[e^{-tH_0}f](x) = \mathbf{E}_x\{f(X_t)\}. \tag{I.11}$$

We will explain in Section III below how the semigroup $\{e^{-tH}; t \geqslant 0\}$ generated by the full Hamiltonian (I.1) is given by the Feynman–Kac formula:

$$[e^{-tH}f](x) = \mathbf{E}_x\{f(X_t)\, e^{-\int_0^t V(X_s)\, ds}\}. \tag{I.12}$$

We will use the idea introduced in Carmona [6] and extended in Carmona and Simon [8] to extract the decay of the eigenfunctions of H from this formula. Note that the case $\alpha = 2$ corresponds to the nonrelativistic Hamiltonians and, up to a variance multiplicative factor, to the process of Brownian motion. The latter is essentially the only Lévy process with continuous sample paths, and consequently the space $\Omega = C([0, \infty), \mathbf{R}^n)$ can be used as the canonical path space in this case.

The technical properties of Lévy processes which we need are recalled in Section II below. We prove only those for which we could not find a proof in print. There is an immense literature on the subject of Lévy processes, but unfortunately we do not know a self contained easy text to refer the reader to. We will simply single out the survey paper Fridstedt [16] and the forthcoming monograph Carmona [9] for reference purposes.

The Lévy processes corresponding to the Hamiltonian function (I.7) are the symmetric stable processes and their properties are well known. The fundamentals of their potential theory has been worked out in the works of Port and Stone [25] and we will refer to the above mentionned paper of Fridstedt for a comprehensive survey of their properties. Surprisingly enough, the process corresponding to our relativstic Hamiltonian (I.6) has been basically ignored by the probabilists. The very recent works of Bakry [2, 3] are the only ones we could locate. In particular, it is shown that, like the symmetric stable processes, the process can be obtained by a time change of Brownian motion and that, like in the case $\alpha = 1$, the time change can be explicitly given as the first time a one-dimensional Brownian motion hits a certain level. But now the Brownian motion has a drift m. This simple generalization of the classical argument due to Lévy gives an explicit formula for the transition density of the process. See (III.7) below.

Section III contains the extension to a very large class of *relativistic Schrödinger operators* of the characterization of the relevent family of potentials given in Aizenman and Simon [1] for the usual nonrelativistic Schrödinger operators. From a probabilistic point of view, this family of potentials has a very natural definition in terms of expectations of additive functionals of the process. One of the results of Aizenman and Simon was to find an equivalent analytic form and to show that it was natural as well. They called this family the Kato class because T. Kato was the first to use successfully this assumption. We will follow this terminology. Section IV contains the results on the decay of the eigenfunctions which we annnounced above. They give both upper and lower bounds. We recover Nardini's result and we show that exponential decay does not hold in general. In particular, we prove that the eigenfunctions decay polynomially in the stable case of $F^{(\alpha)}(p) = |p|^{\alpha}$.

Finally, we would like to make a remark about the nature of the proofs. As we already pointed out, they follow the lines of Carmona [6] and Carmona and Simon [8]. Nevertheless they are embarrassingly simpler. Indeed, we use a cleaner stopping argument and a formula from probabilistic potential theory and we reduce the proofs to the estimation of the free Green's function. This fact corroborates a simple remark made to one of us (R.C.) by S. Agmon: the decay of the free Green's function should govern the decay of the bound states in the regions where the potential can be shown to be negligeable. This statement is obviously correct when the free Hamiltonian H_0 is a local operator. Unfortunately, the usual Laplacian is the only free relativistic Hamiltonian being local. Our proofs actually provide an indirect justification of Agmon's remark.

In the last section we show the equivalence between the recurrence of the Lévy process and the existence of bound states for all the possible choices of negative square wells. This connection is thoroughly discussed in Simon [30] in the nonrelativistic case of the Laplacian and the process of Brownian motion, and a more probabilistic but more complicated proof was later given in Ruelle [28]. While we are extending this equivalence to the more general setting of relativistic Schrödinger operators and Levy processes, we at the same time simplify them.

II. Preliminaries on Lévy Processes

We assume that a triplet (b, C, ν) is fixed once for all and we consider the corresponding function $F(p)$ defined by (I.5). The following assumption will be made throughout the paper:

$$\int_{\mathbf{R}^n} e^{-tF(p)}\,dp < \infty, \qquad t > 0. \tag{A}$$

Under this assumption, for each $t > 0$ the function

$$p_t(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{-tF(p)} e^{ip \cdot x}\,dx, \qquad x \in \mathbf{R}^n, \tag{II.1}$$

is the density of the measure μ_t defined by (I.9). Note that p_t is a bounded continuous function such that $\lim_{|x| \to \infty} p_t(x) = 0$. Note also that, for each fixed $x \in \mathbf{R}^n$, $p_t(x)$ is an analytic function of t on $(0, \infty)$. Consequently, the transition probability $p_t(x, dy)$ of the Lévy process introduced in Section I above has a density $p_t(x, y) = p_t(x - y)$ which is jointly continuous in x and y for each $t > 0$. Moreover, it is easy to see that $0 < p_t(x, y) \leqslant p_t(0, 0) < \infty$. See, for example, Carmona [9].

We will need sharp estimates of the transition probability $p_t(x)$. In the stable case corresponding to $F^{(\alpha)}(p) = |p|^\alpha$ one can use the scaling relation

$$p_t^{(\alpha)}(x) = t^{-n/\alpha} p_1^{(\alpha)}(t^{-1/\alpha} x) \tag{II.2}$$

to reduce the problem to estimating $p_1(x)$, and then use the existence of positive constants c_1 and c_2 such that

$$\frac{c_1}{|x|^{n+\alpha}} \leqslant p_1^{(\alpha)}(x) \leqslant \frac{c_2}{|x|^{n+\alpha}}, \qquad |x| \geqslant 1, \tag{II.3}$$

if $\alpha \neq 2$. See, for example, Blumenthal and Getoor [4]. The exponential bounds of the case $\alpha = 2$ are well known and are not recalled here. In the case of the relativistic Hamiltonian $F^{(r)}(p) = \sqrt{p^2 + m^2} - m$, the transition density has the expression

$$p_t^{(r)}(x) = (2\pi)^{-n} \frac{t}{\sqrt{|x|^2 + t^2}} \int_{\mathbf{R}^n} e^{mt} e^{-\sqrt{(|x|^2 + t^2)(p^2 + m^2)}}\,dp; \tag{II.4}$$

see Herbst and Sloan [18]. This gives

$$\begin{aligned} p_t^{(r)}(x) &\geqslant (2\pi)^{-n} e^{-m|x|} \frac{t}{\sqrt{|x|^2 + t^2}} \int_{\mathbf{R}^n} e^{-\sqrt{|x|^2 + t^2}\,|p|}\,dp \\ &\geqslant c_n t e^{-m|x|} \sqrt{|x|^2 + t^2}^{\,-n-1}, \end{aligned} \tag{II.5}$$

for some positive constant c_n. One can also remark that the density $p_t^{(r)}(x)$ can be evaluated in closed form in terms of a Bessel function. See Eq. (2.13) of Lieb and Yau [22]. On the other hand, the density $p_t^{(r)}(x)$ is subordinated to the classical Brownian motion density and as explained in Bakry [2, 3] one has

$$p_t^{(r)}(x) = e^{mt} \int_0^{\infty} e^{-m^2 u} p_u^{(2)}(x)\, \theta_t(du), \tag{II.6}$$

where $p_t^{(2)}(x) = (4\pi t)^{-n/2} \exp(-|x|^2/4t)$ is the transition density of Brownian motion and where:

$$\theta_t(du) = \pi^{-1/2} t u^{-3/2} \exp(-t^2/4u)\, du. \tag{II.7}$$

The above formulae and estimates will be extremely useful in the particular cases of the Hamiltonians given by the functions $F^{(r)}$ and $F^{(\alpha)}$. The situation is more difficult for general Lévy processes. Nevertheless, we will be able to obtain similar results whenever we are able to connect the decay properties of the Lévy measure to the integrability of the paths of the process. This explains our motivation for the following results culminating in Proposition II.5 below. We believe that the idea is very classical in nature but we could not find this result in the published literature.

PROPOSITION II.1. *Let $F(p)$ be given by* (I.5). *Then, the following are equivalent*:

(i) *$F(p)$ has an analytic continuation to a strip $\{\zeta \in \mathbf{C}^n;\ \zeta = p + i\eta,\ |\eta| < a\}$ for some $a > 0$ with the property that for each $t > 0$, each $\eta \in \mathbf{R}^n$ with $|\eta| < a$, $e^{-tF(\cdot + i\eta)} \in L^2(\mathbf{R}^n)$ and for any $b \in (0, a)$,*

$$\sup_{|\eta| \leqslant b} \|e^{-tF(\cdot + i\eta)}\|_2 < \infty, \tag{II.8}$$

(ii) *ν is exponentially localized in the sense that $e^{b|x|}\nu \in L^2$ for all $b < a$.*

Proof. We first notice that for each $t > 0$ the exponential decay of the Lévy measure ν is equivalent to the exponential decay of the transition probability p_t (i.e., the Fourier transform of the function $e^{-tF(\cdot)}$). See, for example, Jurek and Smalarala [19].

(i) $\Rightarrow$ (ii) If $F(p)$ has an analytic continuation to a strip so does $e^{-tF(p)}$ for each t. Assumption (A) and Theorem IX.13 of Reed and Simon [26] imply that p_t decays exponentially. So does ν according to our first remark.

(ii) $\Rightarrow$ (i) The exponential localization of ν implies the exponential decay of p_t for each $t > 0$, and this implies (A) and the existence of an analytic continuation for $e^{-tF(p)}$ because of the same Theorem IX.13 of

Reed and Simon [26]. The fact that $F(p)$ has an analytic continuation to a strip, can be seen directly. It is obvious that it is sufficient to show that

$$\int_{\mathbf{R}^n} [e^{ip\cdot x} - 1 - ip\cdot h(x)]\, \nu(dx),$$

has an analytic continuation to a strip. One has

$$\begin{aligned}
&\int_{\mathbf{R}^n} [e^{ip\cdot x} - 1 - ip\cdot h(x)]\, \nu(dx)\\
&\quad = \int_{\mathbf{R}^n} [e^{ip\cdot x} - 1 - ip\cdot x + ip\cdot x - ip\cdot h(x)]\, \nu(dx)\\
&\quad = ip\cdot \int_{\mathbf{R}^n} [x - h(x)]\, \nu(dx) + \sum_{k=2}^{\infty} \frac{i^k}{k!} \int_{\mathbf{R}^n} (p\cdot x)^k\, \nu(dx),
\end{aligned} \tag{II.9}$$

which is obviously analytic for $|p| < r$ for some $r > 0$ because ν decays exponentially. In fact the above argument gives analyticity in any ball $|p - p_0| < r$ as long as $p_0 \in \mathbf{R}^n$ and this concludes the proof. ∎

Our interest in the above result lies in the following:

COROLLARY II.2. *The Lévy measure corresponding to the exponent $F(p) = \sqrt{|p|^2 + m^2} - m$ is exponentially localized.*

Next we relate the exponential localization of the Lévy measure to the existence of exponential moments for the position of the Lévy process. The ideas used below, and especially Lemma II.3, are well known. We nevertheless give detailed proofs because we could not find the results we need in the published literature.

LEMMA II.3. *Let us assume that the Lévy process $\mathbf{X} = \{X_t; t \geq 0\}$ has jumps of size no greater than one. Then there exist constants $\delta_0 > 0$ and $c_1 > 0$ such that*

$$\mathbf{E}\{e^{\delta \sup_{0\leq s\leq t}|X_s|}\} \leq c_1 e^t, \tag{II.10}$$

for all $\delta \in (0, \delta_0)$.

Proof. Let us define the stopping times T_n by $T_1 = \inf\{t > 0;\ |X_t| > 1\}$ and $T_k = \inf\{t > T_{k-1};\ |X_t - X_{T_{k-1}}| > 1\}$ for $k \geq 2$, and let us set

$a=\mathbf{E}\{e^{-T_1}\}$. Notice that $T_1>0$ $\mathbf{P}$-a.s. by the right continuity of the paths and this implies that $a<1$. Moreover,

$$\begin{aligned}
\mathbf{E}\{e^{-T_k}\} &= \mathbf{E}\{e^{-(T_k-T_{k-1})}e^{-(T_{k-1}-T_{k-2})}\cdots e^{-T_1}\} \\
&= \prod_{j=1}^{k} \mathbf{E}\{e^{-(T_j-T_{j-1})}\} \\
&= a^k, \qquad\qquad \text{(II.11)}
\end{aligned}$$

because of the strong Markov property. Notice also that $\sup_{0\leqslant s\leqslant T_k}|X_s|\leqslant 2k$ $\mathbf{P}$-a.s. so that $\{\sup_{0\leqslant s\leqslant t}|X_s|>2k\}\subset\{t>T_k\}$ for each $t>0$. Hence,

$$\begin{aligned}
\mathbf{P}\{\sup_{0\leqslant s\leqslant t}|X_s|>2k\} &\leqslant \mathbf{P}\{t>T_k\} \\
&\leqslant e^t\mathbf{E}\{e^{-T_k}\} \\
&= e^t a^k, \qquad\qquad \text{(II.12)}
\end{aligned}$$

because of (II.11) and, consequently,

$$\begin{aligned}
\mathbf{E}\{e^{\delta\sup_{0\leqslant s\leqslant t}|X_s|}\} &= \mathbf{E}\{e^{\delta\sup_{0\leqslant s\leqslant t}|X_s|}\mathbf{1}_{\{\sup_{0\leqslant s\leqslant t}|X_s|<2\}}\} \\
&\quad+\sum_{k=1}^{\infty}\mathbf{E}\{e^{\delta\sup_{0\leqslant s\leqslant t}|X_s|}\mathbf{1}_{\{2k\leqslant\sup_{0\leqslant s\leqslant t}|X_s|<2(k+1)\}}\} \\
&\leqslant e^{2\delta}+\sum_{k=1}^{\infty}e^{2(k+1)\delta}\mathbf{P}\{\sup_{0\leqslant s\leqslant t}|X_s|\geqslant 2k\} \\
&\leqslant e^{2\delta}\left[\sum_{k=0}^{\infty}e^{(2\delta+\log a)k}\right]e^t
\end{aligned}$$

because of (II.12). This gives the desired result provided $\delta>0$ is small enough, i.e., $2\delta+\log a<0$. ∎

LEMMA II.4. *With the above notations, the following holds*:

$$\mathbf{E}\{e^{\delta\sup_{0\leqslant s\leqslant t}|X_s|}\}\leqslant c_1e^{t(1+\int_{|x|>1}(e^{\delta|x|}-1)\nu(dx))}. \qquad \text{(II.13)}$$

Proof. For each time t we denote by $\Delta X_t=X_t-X_{t-}$ the possible jump of the process. Then we let $Y_t=\sum_{0\leqslant s\leqslant t}(\Delta X_s)\mathbf{1}_{\{|\Delta X_s|>1\}}$ be the sum of the jumps of size larger than one and $X_t^1=X_t-Y_t$. It is well known that $\{Y_t;t\geqslant 0\}$ and $\{X_t^1;t\geqslant 0\}$ are independent Lévy processes. Consequently we have

$$\begin{aligned}
\mathbf{E}\{e^{\delta\sup_{0\leqslant s\leqslant t}|X_s|}\} &\leqslant \mathbf{E}\{e^{\delta[\sup_{0\leqslant s\leqslant t}|X_s^1|+\sup_{0\leqslant s\leqslant t}|Y_s|]}\} \\
&= \mathbf{E}\{e^{\delta\sup_{0\leqslant s\leqslant t}|X_s^1|}\}\,\mathbf{E}\{e^{\delta\sup_{0\leqslant s\leqslant t}|Y_s|}\}. \qquad \text{(II.14)}
\end{aligned}$$

Moreover the jumps of X_t^1 are not greater than one and according to Lemma II.3 above this implies that the first factor in the right-hand side of (II.14) is bounded above by $c_1 e^t$. In order to control the second factor we rewrite the process $\mathbf{Y}$ in a more convenient way. For each Borel set A whose closure does not contain the origin we define the stopping times S_A^n as the nth instant of a jump with amplitude in A. More precisely we set

$$S_A^1 = \inf\{t > 0;\ |X(t) - X(t-)| \in A\}$$

and

$$S_A^n = \inf\{t > S_A^{n-1};\ |X(t) - X(t-)| \in A\}$$

whenever $n \geqslant 2$. The quantity

$$N_t(A) = N([0, t] \times A) = \sum_{n \geqslant 1} \mathbf{1}_{\{S_A^n \leqslant t\}}$$

is the number of jumps of amplitude in A before time t. It is well known that it is a Poisson random measure on $\mathbf{R}^n \backslash \{0\}$ with intensity $dt\, \nu(dx)$. $\mathbf{Y}$ has the representation:

$$Y_t = \int_{\{|x| > 1\}} x N_s(dx).$$

Note that for each $t > 0$, one has

$$\begin{aligned} \sup_{0 \leqslant s \leqslant t} |Y_s| &\leqslant \sup_{0 \leqslant s \leqslant t} \int_{\{|x| > 1\}} |x|\, N_s(dx) \\ &= \int_{\{|x| > 1\}} |x|\, N_t(dx), \end{aligned} \tag{II.15}$$

$\mathbf{P}$-a.s. because N_s is a monotone increasing function of s. Consequently, (II.15) implies that

$$\begin{aligned} \mathbf{E}\{e^{\delta \sup_{0 \leqslant s \leqslant t} |Y_s|}\} &\leqslant \mathbf{E}\{e^{\delta \int_{\{|x| > 1\}} |x|\, N_t(dx)}\} \\ &= e^{t \int_{\{|x| > 1\}} (e^{\delta x} - 1)\, \nu(dx)}, \end{aligned}$$

which completes the proof of (II.13). ∎

The following is now an immediate consequence of formula (II.13) above.

PROPOSITION II.5. *Let us assume that the Lévy measure ν of the Lévy process $\mathbf{X} = \{X_t; t \geqslant 0\}$ is exponentially localized. Then there exist positive constants c_1, c_2, and δ_0 such that*

$$\mathbf{E}\{e^{\delta \sup_{0 \leqslant s \leqslant t} |X_s|}\} \leqslant c_1 e^{c_2 t}, \tag{II.16}$$

whenever $0 < \delta < \delta_0$.

We will also use the λ-potentials of the process for $\lambda \geqslant 0$. The λ-potential kernel is defined as the function:

$$g_\lambda(x) = \int_0^\infty e^{-\lambda t} p_t(x)\, dt. \tag{II.17}$$

In the stable case given by $F^{(\alpha)}(p) = |p|^\alpha$ one can use the scaling property to see that

$$g_\lambda(x) = \int_0^\infty t^{-n/\alpha} e^{-\lambda t} p_1(t^{-1/\alpha} x)\, dt,$$

and then the estimate (II.3) to obtain easily the existence for each $\lambda > 0$ of positive constants $c_{1,\lambda}$ and $c_{2,\lambda}$ such that

$$\frac{c_{1,\lambda}}{|x|^{n+\alpha}} \leqslant g_\lambda(x) \leqslant \frac{c_{2,\lambda}}{|x|^{n+\alpha}}, \qquad |x| \geqslant 1. \tag{II.18}$$

This estimate will play a crucial role in the sequel. In the case of the relativistic Hamiltonian $F(p) = \sqrt{p^2 + m^2} - m$ one can use estimate (II.5) and obtain

$$\begin{aligned} g_\lambda(x) &= \int_0^\infty e^{-\lambda t} p_t(x)\, dt \\ &\geqslant c_n e^{-m|x|} \int_0^\infty \frac{t e^{-\lambda t}}{\sqrt{|x|^2 + t^2}^{\,n+1}}\, dt \\ &= c_n \phi(|x|)\, e^{-m|x|}, \end{aligned} \tag{II.19^1}$$

for some positive constant $c_n > 0$ and all x, where $\phi(x)$ is a function which is equivalent to $|x|^{-n}$ when $x \to \infty$. Moreover, this lower bound can be improved in the case $\lambda \leqslant m$; using formulae (II.6) and (II.7), one can show that

$$g_\lambda(x) \geqslant c_n \phi(|x|)\, e^{-\sqrt{2m\lambda - \lambda^2}\,|x|}, \tag{II.19^2}$$

for some positive constant $c_n > 0$ and all x where $\phi(x)$ is a function which is equivalent to $|x|^{-n}$ when $x \to \infty$. Also, using only (II.6) and (II.7) it is very easy to derive the upper bound,

$$g_\lambda(x) \leqslant \frac{c'_n}{|x|^{n/2-1}} \begin{cases} e^{-\sqrt{2m\lambda-\lambda^2}\,|x|}, & \text{if } \lambda \leqslant m; \\ e^{-m|x|}, & \text{otherwise,} \end{cases} \tag{II.20}$$

holds for some constant $c'_n > 0$ and all $x \in \mathbf{R}^n$ for which $|x| \geqslant 1$.

For each Borel subset B of $\mathbf{R}^n$ we use the notation

$$T_B = \inf\{t > 0;\, X_t \in B\}, \tag{II.21}$$

for the first hitting time of B. We will be particularly interested in estimates of the Laplace transform of this stopping time. In particular, we will use the classical formula from the potential theory of Markov processes (see, for example, Blumenthal and Getoor [5, p. 285],

$$\mathbf{E}_x\{e^{-\lambda T_B}\} = \int_{\mathbf{R}^n} g_\lambda(x-y)\, d\mu_B^\lambda(y), \tag{II.22}$$

for some nonnegative measure μ_B^λ called the λ-capacitory measure of B. This measure is concentrated on the closure of B and its total mass is the λ-capacity of the set B. In particular, it is finite when B is compact.

III. The Kato Class for Relativistic Schrödinger Operators

This section is devoted to the introduction of the Schrödinger operators and to the discussion of the hypotheses we make on the potential function.

We assume that H_0 is the nonnegative self-adjoint operator on $L^2(\mathbf{R}^n)$ generating a convolution semigroup $\{\mu_t;\, t \geqslant 0\}$ satisfying the assumption (A) of the preceding section. We perturb this operator by the multiplication operator by a locally integrable function V such that

$$\lim_{t \searrow 0} \sup_{x \in \mathbf{R}^n} \mathbf{E}_x\left\{\int_0^t V^-(X_s)\, ds\right\} = 0 \tag{III.1}$$

and such that the positive part V^+ satisfies locally the same assumption. In other words (III.1) holds with $V\mathbf{1}_K$, instead of V^-, for each compact subset K of $\mathbf{R}^n$. These assumptions imply that the operator $H = H_0 + V$ is essentially self-adjoint on the space $C_c^\infty(\mathbf{R}^n)$ of C^∞-functions with compact supports in $\mathbf{R}^n$ and that the corresponding semigroup can be expressed by the Feynman–Kac formula,

$$[e^{-tH}f](x) = \mathbf{E}_x\{f(X_t)\, e^{-\int_0^t V(X_s)\, ds}\}, \tag{III.2}$$

where here and throughout the paper we use the convention $e^{-\infty} = 0$ and where $\{X_t; t \geqslant 0\}$ is the Lévy process associated with the convolution semigroup $\{\mu_t; t \geqslant 0\}$. It is easy to prove using this formula that the function $e^{-tH}f$ is a bounded continuous function whenever $t > 0$ and f is in some L^p-space. See, for example, Carmona [7], Simon [31], or Carmona [9]. In particular, this implies that all the eigenfunctions of the operator H are bounded and continuous. In fact, we will make a crucial use of the interaction between the operator H and the Lévy process at the level of the eigenfunctions. Indeed, if $\psi \in L^2(\mathbf{R}^n)$ is an eigenfunction of the operator H with eigenvalue E, then the process $\{M_t; t \geqslant 0\}$, defined by

$$M_t = \psi(X_t)\, e^{Et} e^{-\int_0^t V(X_s)\, ds}, \tag{III.3}$$

is a martingale. The estimates proven later in this paper are extracted from relations obtained by stopping such martingales at appropriate first-hitting times.

But before we get to the investigation of the decay properties of the eigenfunctions, we give an analytic equivalent to the fundamental assumption (III.1) on the potentials.

DEFINITION III.1. A measurable function f on $\mathbf{R}^n$ is said to be in $l^1(L^\infty)$ if

$$\|f\|_{l^1(L^\infty)} = \sum_{\alpha \in \mathbf{Z}^n} \sup_{x \in C_\alpha} |f(x)| < \infty, \tag{III.4}$$

where C_α denotes the cube centered at $\alpha \in \mathbf{Z}^n$ with sides of length 1.

Let us assume that, instead of the partition $\{C_\alpha; \alpha \in \mathbf{R}^n\}$ by cubes of side 1, we have another partition $\{C'_\iota; \iota \in I\}$, where I is a countable index set and the cells C'_ι are bounded subsets of $\mathbf{R}^n$ such that there exist two integers n_1 and n_2 such that each of the cells C_α is covered by at most n_1 cells C'_ι and, conversely, each of the cells C'_ι is covered by at most n_2 cells C_α. Then it is easy (though cumbersome) to see that the norm $\|\cdot\|'_{l^1(L^\infty)}$, defined by

$$\|f\|'_{l^1(L^\infty)} = \sum_{\iota \in I} \sup_{x \in C'_\iota} |f(x)|, \tag{III.5}$$

is equivalent to the norm $\|\cdot\|_{l^1(L^\infty)}$ defined by (III.4), in the sense that there exists a constant $c > 0$ such that

$$c^{-1}\|f\|'_{l^1(L^\infty)} \leqslant \|f\|_{l^1(L^\infty)} \leqslant c\|f\|'_{l^1(L^\infty)}, \tag{III.6}$$

for all the functions f. This simple remark will give us a convenient device to show that some density functions are of the $l^1(L^\infty)$-class. Also, the following two lemmas show how appropriate this class is.

LEMMA III.1. *If $g \in L^1(\mathbf{R}^n)$ and $f \in l^1(L^\infty)$, then $g * f \in l^1(L^\infty)$ and*

$$\|g * f\|_{l^1(L^\infty)} \leqslant \|g\|_{L^1} \|f\|_{l^1(L^\infty)}. \tag{III.7}$$

Proof.

$$\begin{aligned} |(g * f)(x)| &= \left| \sum_{\alpha \in \mathbf{Z}^n} \int_{C_\alpha} g(x-y) f(y)\, dy \right| \\ &\leqslant \sum_{\alpha \in \mathbf{Z}^n} \left(\int_{C_\alpha} |g(x-y)|\, dy \right) \sup_{y \in C_\alpha} |f(y)| \end{aligned}$$

and, consequently,

$$\begin{aligned} \|g * f\|_{l^1(L^\infty)} &\leqslant \sum_{\alpha' \in \mathbf{Z}^n} \sum_{\alpha \in \mathbf{Z}^n} \sup_{x \in C_{\alpha'}} \left(\int_{x - C_\alpha} |g(y)|\, dy \right) \sup_{y \in C_\alpha} |f(y)| \\ &\leqslant \|g\|_{L^1} \|f\|_{l^1(L^\infty)}. \quad \blacksquare \end{aligned}$$

LEMMA III.2. *For each $\delta > 0$ there exists a constant $C_\delta > 0$ such that*

$$\|\mathbf{1}_{\{|y| > \delta\}} f\|_{l^1(L^\infty)} \leqslant C_\delta \|f\|_{L^1}, \tag{III.8}$$

for any spherically symmetric function f (i.e., for which $f(x) = f(x')$ whenever $|x| = |x'|$) whose modulus is radially nonincreasing in the sense that for all $x \in \mathbf{R}^n$ the function $r \hookrightarrow |f(rx)|$ is nonincreasing on $[0, \infty)$.

Notice that this result implies, in particular, that any such function is of the $l^1(L^\infty)$-class.

Proof. The proof is trivial for $n = 1$ if one chooses the covering formed by the $C'_\iota = (\iota\delta, \iota(\delta + 1)]$ when $\iota \in \mathbf{Z}$, because the function $x \hookrightarrow |f(x)|$ is even and nonincreasing on $[0, \infty)$. In the general case $n > 1$ one chooses a covering by cells $C'_\iota = C'_{n,k}$, where $C_{0,0} = \{x \in \mathbf{R}^n;\ |x| < \delta\}$ and where for each fixed integer $n \geqslant 1$, the $C'_{n,k}$ form when k varies from 0 to k_n, a partition of the annulus $\{x \in \mathbf{R}^n;\ n\delta \leqslant |x| < (n+1)\delta\}$, by solid blocks having a volume approximately equal to δ^n. The proof then mimics the one-dimensional case. $\blacksquare$

The following result follows immediately from the above lemma.

COROLLARY. *If for each fixed $t > 0$, the transition density $p_t(\cdot)$ is spherically symmetric and radially nonincreasing, then it is of the $l^1(L^\infty)$-class and, moreover, for each fixed $\delta > 0$ one has*

$$\sup_{t > 0} \|\mathbf{1}_{\{|y| > \delta\}} p_t\|_{l^1(L^\infty)} < \infty. \tag{III.9}$$

This result is of great importance to us because most of the transition densities of the Lévy processes we are interested in actually satisfy the above assumptions. Indeed, the spherical symmetry is implied by the fact that we are dealing with exponent functions $F(p)$ which depend only on the norm $|p|$ of p, and the radial nonincrease can be checked by inspection in the case of the process of Brownian motion corresponding to the exponent function $F^{(2)}$ and in the relativistic case of $F^{(r)}$ as well. Moreover, the results of Kanter [20] and Wolfe [35] imply that it is also true for all the symmetric stable cases $F^{(\alpha)}$ and more generally for all the Lévy processes of the so-called class L. See, for example, Wolfe [35] for the definition of the class L.

The next lemma gives another technical result which we will need. It concerns the divergence at the origin of the Green's function at different energies (i.e., of the resolvent kernel g_β for various values of β).

LEMMA III.3. *If*

$$\lim_{|x| \to 0} g_{\beta_0}(x) = \infty, \tag{III.10}$$

for some fixed $\beta_0 > 0$, then for all $\beta > 0$ one has

$$\lim_{|x| \to 0} \frac{g_1(x)}{g_\beta(x)} = 1. \tag{III.11}$$

Proof. Let us fix $\beta > 0$. We notice that, for each fixed $t_0 > 0$ one has

$$\sup_{x \in \mathbf{R}^n} \int_{t_0}^{\infty} e^{-\beta t} p_t(x)\, dt \leqslant \int_{t_0}^{\infty} e^{-\beta t} p_t(0)\, dt < \infty,$$

because of the analyticity in t of the function $p_t(0)$ on $(0, \infty)$. Consequently, the ratio $g_1(x)/g_\beta(x)$ will have the same limit (if any) as the limit of the ratio,

$$\frac{\int_0^{t_0} e^{-t} p_t(x)\, dt}{\int_0^{t_0} e^{-\beta t} p_t(x)\, dt}. \tag{III.12}$$

We prove only the result in the case $\beta > 1$ because the proof in the case $\beta < 1$ is similar. In this case $\beta > 1$, the ratio in (III.12) is always larger than or equal to one. On the other hand, this ratio is also less than or equal to $e^{(\beta - 1)t_0}$ independently of x. This concludes the proof because the choice of $t_0 > 0$ was arbitrary. ∎

Note that the above lemma covers the case $n \geqslant 2$. The situation is easier in the one-dimensional case where the resolvent densities $g_\beta(x)$ are defined and continuous at the origin $x=0$. In this case, the limit (III.10) exists and is equal to a finite number. The main result of this section is the following:

THEOREM III.1. *Let us assume that* V *is a nonnegative function on* $\mathbf{R}^n$ *which is uniformly locally integrable and let us assume that the assumption* (A) *and the condition* (III.9) *are satisfied by the Lévy process. Then, the following three conditions are equivalent*:

(i) $\lim_{t\searrow 0}\sup_{x\in\mathbf{R}^n}\int_0^t \mathbf{E}_x\{V(X_s)\}\,ds=0,$

(ii) $\lim_{\beta\to\infty}\sup_{x\in\mathbf{R}^n}[(H_0+\beta)^{-1}V](x)=0,$

(iii) $\lim_{\delta\searrow 0}\sup_{x\in\mathbf{R}^n}\int_{|x-y|<\delta} g_1(x-y)V(y)\,dy=0.$

Proof. The assumption on the function V means that the constant $c(V)$, defined by

$$c(V)=\sup_{x\in\mathbf{R}^n}\int_{x+C_0} V(y)\,dy,$$

is finite. This assumption implies that, for each fixed $t>0$, we have

$$\sup_{s\geqslant t}\sup_{x\in\mathbf{R}^n}(p_s*V)(x)\leqslant c(V)\sup_{s\geqslant t}\|p_s\|_{l^1(L^\infty)},$$

which is finite because of our assumption (III.9). Indeed,

$$\begin{aligned}(p_s*V)(x)&=\int_{\mathbf{R}^n}p_s(y)\,V(x-y)\,dy\\&=\sum_{\alpha\in\mathbf{Z}^n}\int_{C_\alpha}p_s(y)\,V(x-y)\,dy\\&\leqslant\sum_{\alpha\in\mathbf{Z}^n}\sup_{y\in C_\alpha}p_s(y)\int_{C_\alpha}V(x-y)\,dy\\&\leqslant c(V)\,\|p_s\|_{l^1(L^\infty)}.\end{aligned}$$

Next we notice that for each $t>0$, $\beta>0$ and for each $x\in\mathbf{R}^n$ we have

$$\int_0^t\mathbf{E}_x\{V(X_s)\}\,ds\leqslant e^{\beta t}\int_0^\infty e^{-\beta s}\mathbf{E}_x\{V(X_s)\}\,ds$$

so that,

$$\sup_{x\in\mathbf{R}^n}\int_0^t\mathbf{E}_x\{V(X_s)\}\,ds\leqslant e^{\beta t}\sup_{x\in\mathbf{R}^n}[(H_0+\beta)^{-1}V](x). \qquad \text{(III.13)}$$

Now, if we assume (ii) and if $\varepsilon>0$, we can choose $\beta>0$ large enough so that

$$\sup_{x\in\mathbf{R}^n}\,[(H_0+\beta)^{-1}V](x)<\varepsilon/2, \tag{III.14}$$

and then choose t small enough so that $e^{\beta t}<2$ and (i) follows easily from (III.13).

Conversely, let us assume (i). Now, if $\varepsilon>0$, we can choose $t>0$ small enough so that

$$\sup_{x\in\mathbf{R}^n}\int_0^t \mathbf{E}_x\{V(X_s)\}\,ds<\varepsilon/2,$$

and, consequently,

$$\begin{aligned}\sup_{x\in\mathbf{R}^n}[(H_0+\beta)^{-1}V](x) &\leqslant \varepsilon/2+\sup_{x\in\mathbf{R}^n}\int_t^\infty e^{-\beta s}\mathbf{E}_x\{V(X_s)\}\,ds\\ &<\varepsilon/2+\sup_{s\geqslant t}\,\sup_{x\in\mathbf{R}^n}(p_s*V)(x)\int_t^\infty e^{-\beta s}\,ds\\ &\leqslant \varepsilon/2+c(V)\sup_{s\geqslant t}\|p_s\|_{L^1(L^\infty)}\,\beta^{-1}e^{-\beta t},\end{aligned}$$

which can be made arbitrarily small provided β is chosen large enough. This completes the proof of the equivalence of properties (i) and (ii).

Let us assume (ii) once more. As before, if $\varepsilon>0$ is arbitrary, one can choose $\beta>0$ large enough so that (III.14) holds. Then, using Lemma III.3 if $n\geqslant 2$ and the remark following it in the one-dimensional case, one finds constants $\delta>0$ and $c>0$ such that

$$|z|\leqslant\delta\Rightarrow g_1(z)\leqslant c g_\beta(z).$$

Consequently, we have

$$\begin{aligned}&\sup_{x\in\mathbf{R}^n}\int_{|x-y|\leqslant\delta} g_1(x-y)\,V(y)\,dy\\ &\quad\leqslant c\sup_{x\in\mathbf{R}^n}\int_{|x-y|\leqslant\delta} g_1(x-y)\,V(y)\,dy\\ &\quad\leqslant\varepsilon,\end{aligned}$$

because of our choice of β. This proves that (ii) implies (iii). Finally, we prove that (iii) implies (ii). If $\varepsilon>0$ is fixed, assumption (iii) gives the existence of a number $\delta>0$ sufficiently small so that

$$\sup_{x\in\mathbf{R}^n}\int_{|x-y|\leqslant\delta} g_1(x-y)\,V(y)\,dy<\varepsilon/2.$$

Then, for any $\beta \geqslant 1$, one also has

$$\sup_{x \in \mathbf{R}^n} \int_{|x-y| \leqslant \delta} g_\beta(x-y)\, V(y)\, dy < \varepsilon/2, \tag{III.15}$$

because the function $\beta \hookrightarrow g_\beta(z)$ is nonincreasing for each fixed $z \in \mathbf{R}^n$. Moreover, for this fixed δ, one also has

$$\begin{aligned}
&\sup_{x \in \mathbf{R}^n} \int_{|x-y| > \delta} g_\beta(x-y)\, V(y)\, dy \\
&\quad = \sup_{x \in \mathbf{R}^n} \int_0^\infty e^{-\beta s} \int_{|y| > \delta} p_s(y)\, V(x-y)\, dy\, ds \\
&\quad \leqslant c(V) \sup_{s > 0} \| \mathbf{1}_{\{|y| > \delta\}} p_s \|_{L^q(L^r)} \int_0^\infty e^{-\beta s}\, ds \\
&\quad < \varepsilon/2,
\end{aligned} \tag{III.16}$$

provided β is large enough because of the assumption (III.9). The proof is now complete because we have

$$\sup_{x \in \mathbf{R}^n} [(H_0 + \beta)^{-1} V](x) \leqslant \varepsilon,$$

by putting together (III.15) and (III.16). ∎

A potential function $V = V^+ - V^-$ will be said to be of the Kato class whenever for each compact subset K of $\mathbf{R}^n$ the functions $\mathbf{1}_K V^+$ and V^- satisfy any one of the three equivalent conditions of the above theorem. As we already pointed out, this is the right kind of assumption. This fact was demonstrated in Simon [31] in the nonrelativistic case. Most of the abstract results discussed in his Section A.2 are still valid in the present situation. In particular, this is the case for the paragraph (2) on page 459. It is argued there that condition (ii) of the above theorem implies that V^- is H_0-form bounded with relative bound 0 and that, consequently, $H_0 + V$ can be defined in the sense of quadratic forms if V is in the Kato class.

We close this section with the following elementary consequence of Theorem III.1.

Remark. The definition of the Kato class depends only on the behaviour of the exponent function $F(p)$ when $p \to \infty$. Indeed the above equivalence shows that it depends only on the behaviour of the Green's kernel $[H_0 + 1]^{-1}(x, y)$ at small distances, and since one has the explicit formula,

$$[H_0 + 1]^{-1}(x, y) = g_1(x-y),$$

with

$$g_1(x) = \int_{\mathbf{R}^n} \frac{e^{-ipx}}{1+F(p)}\, dp$$
$$= c_R + \int_{|p| \geqslant R} \frac{e^{-ipx}}{1+F(p)}\, dp,$$

the behaviour of g_1 at the origin is determined by the behaviour of $F(p)$ at ∞. In particular, this implies that the Kato class is the same whether one deals with $F^{(1)}(p) = |p|$ or $F^{(r)}(p) = \sqrt{|p|^2 + m^2} - m$.

IV. The Eigenfunction Fall-Off

IV.1. *The case* $V(x) \to 0$

Throughout the rest of the paper we will use the notation T_r for the first hitting time $T_{B(0,r)}$ of the ball of radius r centered at the origin. We study the decay properties of a L^2-eigenfunction ψ corresponding to a negative eigenvalue E. The strategy for proving upper bounds is the following. Since M_t defined by (III.3) is a martingale, we have

$$|\psi(x)| = |\mathbf{E}\{\psi(X_{t \wedge T_r})\, e^{Et \wedge T_r} e^{-\int_0^{t \wedge T_r} V(X_s)\, ds}\}|$$
$$\leqslant \|\psi\|_\infty\, \mathbf{E}_x\{e^{Et \wedge T_r} e^{\int_0^{t \wedge T_r} V^-(X_s)\, ds}\}$$

because the function ψ is bounded and, consequently,

$$|\psi(x)| \leqslant \|\psi\|_\infty\, \mathbf{E}_x\{e^{-\beta t \wedge T_r}\},$$

whenever, for $\varepsilon > 0$ fixed, r is chosen large enough so that $V^-(y) < \varepsilon$ for $|y| > r$. We will choose ε small enough so that $\beta \equiv -(E+\varepsilon) > 0$. By the monotone convergence theorem we have

$$\lim_{t \to \infty} \mathbf{E}_x\{e^{-\beta t \wedge T_r}\} = \mathbf{E}_x\{\lim_{t \to \infty} e^{-\beta t \wedge T_r}\} = \mathbf{E}_x\{e^{-\beta T_r}\}$$

and, hence,

$$|\psi(x)| \leqslant \|\psi\|_\infty\, \mathbf{E}_x\{e^{-\beta T_r}\}. \tag{IV.1}$$

We are now ready to prove the easy

Proposition IV.1. *If* $F^{(\alpha)}(p) = |p|^\alpha$ *with* $\alpha < 2$ *there exists a constant* $c > 0$ *such that*

$$|\psi(x)| \leqslant \frac{c}{1+|x|^{n+\alpha}}, \tag{IV.2}$$

for all $x \in \mathbf{R}^n$. *For* $F^{(r)}(p) = \sqrt{p^2 + m^2} - m$, *for each* $\varepsilon > 0$, *there exists a constant* $c_\varepsilon > 0$ *such that*

$$|\psi(x)| \leqslant c_\varepsilon e^{-m_\varepsilon |x|}, \tag{IV.3}$$

for all $x \in \mathbf{R}^n$, *where* $m_\varepsilon = m$ *if* $|E| > m$ *and* $\sqrt{2m|E| - E^2} - \varepsilon$ *whenever* $|E| \leqslant m$.

Proof. The above estimates trivially follow Eq. (IV.1), the potential theory formula (II.22), and the estimates (II.18) and (II.20), respectively. ▮

Remark. The above argument also gives a proof of the well-known result for the classical case of $F^{(2)}(p) = |p|^2$.

In the general case of a Lévy process with an exponentially localized Lévy measure, one has the same result as in the case of the relativistic Hamiltonian, namely:

PROPOSITION IV.2. *Assume that the Lévy measure* ν *decays exponentially. Then there exist positive constants* c_1, c_2, *such that*

$$|\psi(x)| < c_1 e^{-c_2|x|}, \tag{IV.4}$$

for all $x \in \mathbf{R}^n$.

The strategy for proving lower bounds is the following. We consider only the case of the ground state eigenfunction; i.e., we assume that E is the infimum of the spectrum of H. The ground state eigenfunction is positive everywhere and continuous, hence locally bounded away from zero. Consequently, once more using the martingale M_t defined by (III.3) we obtain

$$\begin{aligned}
\psi(x) &= \mathbf{E}_x\{\psi(X_{t \wedge T_r})\, e^{Et \wedge T_r} e^{-\int_0^{t \wedge T_r} V(X_s)\,ds}\} \\
&\geqslant \mathbf{E}_x\{\psi(X_{t \wedge T_r})\, e^{Et \wedge T_r} e^{-\int_0^{t \wedge T_r} V^+(X_s)\,ds}\} \\
&= \mathbf{E}_x\{\psi(X_{T_r})\, e^{ET_r} e^{-\int_0^{T_r} V^+(X_s)\,ds}\} \\
&\geqslant \inf_{y \in B(0,r)} \psi(y)\, \mathbf{E}_x\{e^{-\beta T_r}; T_r < \infty\},
\end{aligned}$$

where we used the usual convention $e^{-\infty} = 0$ and Fatou's lemma to take the limit $t \to \infty$ and where we assume that $\varepsilon > 0$ is small enough so that $\beta \equiv -(E - \varepsilon) > 0$ and if r is large enough so that $V^+(x) < \varepsilon$ for $|x| > r$. Consequently, one has

$$\psi(x) \geqslant \text{cst}\, \mathbf{E}_x\{e^{-\beta T_r}\}, \tag{IV.5}$$

for some positive constant cst. As above, one easily proves the following:

PROPOSITION IV.3. *In the case* $F^{(\alpha)}(p) = |p|^{\alpha}$ *there exists a constant* $c' > 0$ *such that*

$$\psi(x) \geqslant \frac{c'}{1 + |x|^{n+\alpha}}, \tag{IV.6}$$

for all $x \in \mathbf{R}^n$. *In the case* $F^{(r)}(p) = \sqrt{p^2 + m^2} - m$, *for each* $\varepsilon > 0$ *there exists a constant* $c'_\varepsilon > 0$ *such that*

$$\psi(x) \geqslant c'_\varepsilon e^{-m_\varepsilon |x|}, \tag{IV.7}$$

for all $x \in \mathbf{R}^n$, *where now* $m_\varepsilon = m$ *if* $|E| > m$ *and* $\sqrt{2m|E| - E^2} - \varepsilon$ *whenever* $|E| \leqslant m$.

Proof. The above estimates trivially follow Eq. (IV.5), the potential theory formula (II.22), and the estimate (II.18) and (II.19), respectively. ∎

Remark. The assumption $\lim_{|x| \to \infty} V(x) = 0$ is slightly stronger than what we actually need. An inspection of the above proofs shows that one only needs $\lim_{|x| \to \infty} V^{+}(x) = 0$ to obtain the lower bound and the assumption $\lim_{|x| \to \infty} V^{-}(x) = 0$ in order to prove the upper bound.

IV.2. *The Case* $V(x) \to \infty$

PROPOSITION IV.4. *Let us assume that* $\lim_{|x| \to \infty} V(x) = \infty$ *and that the Lévy measure* ν *is exponentially localized* (*i.e.*, $\exists b > 0$, *such that* $\int e^{b|x|} \nu(dx) < \infty$). *Then for every positive constant* δ, *there exists a positive constant* δ', *such that*

$$|\psi(x)| \leqslant \delta' e^{-\delta |x|}, \tag{IV.8}$$

for all $x \in \mathbf{R}^n$.

Proof. If one uses again the martingale M_t defined in (III.3) and if for each $r > 0$ we set $\tau = \inf\{t > 0;\ |X_t - x| > r\}$, we have

$$\psi(x) = \mathbf{E}_x\{\psi(X_{t \wedge \tau})\, e^{t \wedge \tau \lambda}\, e^{-\int_0^{t \wedge \tau} V(X_s)\, ds}\} \tag{IV.9}$$

and, consequently

$$\begin{aligned} |\psi(x)| &\leqslant \|\psi\|_\infty \mathbf{E}_x\{e^{t \wedge \tau \lambda} e^{-\int_0^{t \wedge \tau} V(X_s)\, ds}\} \\ &\leqslant \|\psi\|_\infty \mathbf{E}_x\{e^{-(v - \lambda) t \wedge \tau}\} \end{aligned}$$

provided we set $v = \inf_{|y - x| \leqslant r} V(y)$,

$$|\psi(x)| \leqslant \|\psi\|_\infty [e^{-(v - \lambda)t} + e^{-\alpha r} \mathbf{E}_0\{e^{\alpha \sup_{0 \leqslant s \leqslant t} |X_s|}\}]$$

for any $\alpha > 0$. By Lemmas II.3 and II.5, choose $0 < \alpha < \min(b, -1/2 \log a)$, a as in Lemma II.3. We thus have

$$|\psi(x)| \leqslant \|\psi\|_{\infty} [e^{-(v-\lambda)t} + e^{-\alpha r} c_1 e^{c_2 t}], \tag{IV.10}$$

where c_1 is as in Lemma II.3 and $c_2 = 1 + \int (e^{\alpha |x|} - 1)\, v(dx)$. Take $r = p_1 |x|$, $p_1 \in (0, 1)$, $t = p_2 |x|$, $p_2 > 0$ small enough such that $c_2 p_2 < p_1 \alpha$. We now have the result. ∎

V. Recurrence and the Existence of Bound States

Nonrelativistic Schrödinger operators with a negative potential have been known for a long time to always have bound states (i.e., negative eigenvalues) in one and two dimensions, a fact which is no longer true in higher dimensions. The purpose of this section is to give a general criterion to compute the critical dimension above which a general Schrödinger operator, of the type we discussed so far, need not have a bound state if the negative potential is not deep enough. Theorem V.1 below implies that this critical dimension is also two for the relativistic case corresponding to the exponent function $F^{(r)}$ but that it is one for the stable cases corresponding to the exponent function $F^{(\alpha)}$ with $\alpha < 2$.

In order to prove this result, we need an extra technical assumption on top of assumption (A):

$$\lim_{|p| \to \infty} F(p) = \infty.$$

Obviously, we did not aim at the greatest generality and there are presumably weaker conditions under which the following proof holds. We chose the above condition for convenience. One of its side effects is to rule out the lattice cases.

Theorem V.1. *The following three properties are equivalent*:

(i) *The Lévy process $\{X_t : t \geqslant 0\}$ is recurrent,*

(ii) *The exponent function F satisfies*

$$\int_{\{|p| \leqslant 1\}} \frac{dp}{F(p)} = \infty, \tag{V.1}$$

(iii) *The Schrödinger operator $H_0 + V$ has at least one negative bound state whenever V is a nonpositive, nonidentically zero, and bounded potential with compact support.*

Proof. The equivalence of (i) and (ii) is a classical result of the theory of Lévy processes. See, for example, Port and Stone [25] or Carmona [9]. Before proving the equivalence of (ii) and (iii) we should point out that any bounded potential function V with compact support is in the Kato class. In fact, it is a relatively compact form perturbation of H_0 under the assumption (V.1), and consequently, the essential spectrum of the perturbed operator $H = H_0 + V$ is still $[0, \infty)$. To prove that (ii) implies (iii) it suffices to consider the case $V = -\lambda \mathbf{1}_{B(\delta)}$, where

$$B(\delta) = \{x = (x_1, ..., x_n);\ |x_j| \leqslant \delta,\ j = 1, ..., n\}$$

for some $\lambda > 0$ and some $\delta > 0$. We use the Birman–Schwinger principle (see, for example, Simon [31]) to show the existence of negative eigenvalues of $H_0 + V$. If $E < 0$ and if $\phi \in L^2$ is such that $\hat{\phi}$ is continuous near the origin, then we have

$$\begin{aligned}
&\langle \phi, |V|^{1/2} (H_0 + E)^{-1} |V|^{1/2} \phi \rangle \\
&\quad = \lambda \int_{\mathbf{R}^n} \frac{\hat{\phi}(p)^2}{E + F(p)} \prod_{j=1}^{n} \left(\frac{\sin \delta p_j}{p_j} \right)^2 dp_1 \ldots dp_n \\
&\quad \geqslant \lambda \int_{\{|p| \leqslant 1\}} \frac{\hat{\phi}(p)^2}{E + F(p)} \prod_{j=1}^{n} \left(\frac{\sin \delta p_j}{p_j} \right)^2 dp_1 \ldots dp_n \\
&\quad \geqslant c |\hat{\phi}(0)|^2 \int_{\{|p| \leqslant 1\}} \frac{dp}{E + F(p)}
\end{aligned}$$

for some constant $c > 0$. Consequently, property (ii) implies that

$$\liminf_{E \searrow 0} \langle \phi, |V|^{1/2} (H_0 + E)^{-1} |V|^{1/2} \phi \rangle = \infty,$$

which in turn implies the existence of bound states. Conversely, if property (ii) does not hold, we have that

$$\int_{\mathbf{R}^n} \frac{1}{F(p)} \prod_{j=1}^{n} \left(\frac{\sin \delta p_j}{p_j} \right)^2 dp_1 \ldots dp_n < \infty$$

and, since for each $E > 0$ we have

$$\begin{aligned}
&\langle \phi, |V|^{1/2} (H_0 + E)^{-1} |V|^{1/2} \phi \rangle \\
&\quad \leqslant \lambda \|\hat{\phi}\|_{\infty}^2 \int_{\mathbf{R}^n} \frac{1}{F(p)} \prod_{j=1}^{n} \left(\frac{\sin \delta p_j}{p_j} \right)^2 dp_1 \ldots dp_n,
\end{aligned} \tag{V.2}$$

it is possible to choose a strictly positive function ϕ having one for L^2-norm and such that its Fourier transform $\hat{\phi}$ has a small enough L^∞-norm so that the right-hand side of (V.2) is less than or equal to one. For such a function ϕ one has

$$\limsup_{E \searrow 0} \langle \phi, |V|^{1/2} (H_0 + E)^{-1} |V|^{1/2} \phi \rangle \leqslant 1,$$

and this implies (using again the Birman–Schwinger principle) that the operator $H_0 + V$ does not have negative eigenvalues. ∎

The above result proves that the critical dimension is also two for the relativistic case corresponding to the exponent function $F^{(r)}$ but that it is one for the stable cases corresponding to the exponent functions $F^{(\alpha)}$ with $\alpha < 2$.

Acknowledgments

We thank E. Lieb for his comments on a first version of the paper. Moreover, the first named author (R.C.) thanks D. Bakry and S. Port for enlightening discussions on some of the facts of the theory of Lévy processes.

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