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SPECTRAL PROPERTIES OF NEUMANN LAPLACIAN OF HORNS

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Abstract

We study the Neumann Laplacian of unbounded regions in $\mathbb{R}^n$ with cusps at infinity so that the corresponding Dirichlet Laplacian has compact resolvent. Typical of our results is that of the region $\{(x,y)\in\mathbb{R}^2 \,|\, |xy|<1\}$ the Neumann Laplacian has absolutely continuous spectrum $[0,\infty)$ of uniform multiplicity four and an infinity of eigenvalues $E_0<E_1\leq\ldots\to\infty$ and that for the region $\{(x,y)\in\mathbb{R}^2 \,|\, |y|\leq e^{-|x|}\}$, it has absolutely continuous spectrum $[1/4,\infty)$ of uniform multiplicity 2 and an infinity of eigenvalues $E_0=0<E_1\leq \ldots\to\infty$. We use the Enss theory with a suitable asymptotic dynamics.

1. Introduction

Let $C_0^2[1,\infty)$ denote the twice continuously differentiable functions of compact support on $[1,\infty)$. Let f_1 and f_2 be two functions in $C^2[1,\infty)$ with $f\equiv \frac{1}{2}(f_2-f_1)>0$ everywhere and let

$$\Omega = \left\{(x,y) \,\middle|\, x>1 \quad \text{and} \quad f_1(x)<y<f_2(x)\right\} .$$

We define $H=-\Delta_\Omega^N$ on $L^2(\Omega)$, so that

$$\langle Hg,g\rangle = \int_\Omega |\nabla g|^2 dx\, dy$$

and $C_0^2(\overline{\Omega})$ is a quadratic form core. We let $h=\frac{1}{2}(f_1+f_2)$ so $f_1=h-f$, $f_2=h+f$.

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In the ensuing discussion, we put $h = 0$ for simplicity, and suppose that $|f(x)|+|f'(x)| \to 0$ and $|f|^{-1}|f'| \to 0$ as $x \to \infty$. It is a result of Rellich that the Dirichlet Laplacian, $-\Delta_\Omega^D$, has compact resolvent and several authors have studied the asymptotics of its eigenvalues [4],[12],[14],[15],[16].

For many f, the Laplacian $-\Delta_\Omega^N$ will not have compact resolvent; for example if $|\Omega| = \infty$, then it is easy to see that $0 \in \sigma_{\text{ess}}(-\Delta_\Omega^N)$ (see [7] or the appendix). For general f, Evans and Harris [8] have given necessary and sufficient conditions of $0 \in \sigma_{\text{ess}}(-\Delta_\Omega^N)$ and for $-\Delta_\Omega^N$ to have compact resolvent:

THEOREM (Evans-Harris [8]).
(a) $0 \in \sigma_{\text{ess}}(-\Delta_\Omega^N)$ *if and only if either*

$$\int_1^\infty f(s)ds = \infty \quad \text{or} \quad \varlimsup_{t\to\infty} \left(\int_1^t f(s)^{-1}ds\right)\left(\int_t^\infty f(s)ds\right) = \infty \tag{1.1}$$

(b) $-\Delta_\Omega^N$ *has compact resolvent if and only if*

$$\lim_{t\to\infty} \left(\int_1^t f(s)^{-1}ds\right)\left(\int_t^\infty f(s)ds\right) = 0 \tag{1.2}$$

Remarks: 1. If $\mathbb{R}^2$ is replaces by $\mathbb{R}^{d+1}$ with $(x_1, x_\perp) \in \mathbb{R}^{d+1}$; $x_\perp \in \mathbb{R}^d$ and $\Omega = \{x \mid |x_\perp| \le f(x_1)\ ;\ 1 < x_1 < \infty\}$, then the above holds if $f(s)$ is replaced by $f(s)^d$ and $f(s)^{-1}$ by $f(s)^{-d}$.

2. [8] deals with a variety of other regions.

3. If $f(x) = x^{-\alpha}$, then (1.1) holds and similarly if $f(x) = \exp(-x^\alpha)$ for $0 < \alpha < 1$. On the other hand (1.2) holds for $f(x) = \exp(-x^\alpha)$, $1 < \alpha$.

A key role is played in our analysis by the function

$$V(x) = \frac{1}{4}\left(\frac{f'}{f}\right)^2 + \frac{1}{2}\left(\frac{f'}{f}\right)' \tag{1.3}$$

$$k_1(x) = |f_1'(x)| + |f_2'(x)| \tag{1.4}$$

$$k_2(x) = \tfrac{1}{2}|f'(x)|f(x)^{-1}k_1(x)\ . \tag{1.5}$$

Why this plays the role of a Schrödinger operator potential will become clear shortly. One of the main theorems in this paper is

THEOREM 1.1. *Suppose that $V(x), k_1(x)$ and $k_2(x)$ are all $0(|x|^{-1-\varepsilon})$ at infinity. Then*

(i) $\sigma_{\text{ac}}(-\Delta_\Omega^N) = [0,\infty)$ *of uniform multiplicity*

(ii) $\sigma_{\text{sing}}(-\Delta_\Omega^N) = \emptyset$

(iii) *If $h = 0$, then $-\Delta_\Omega^N$ has an infinity of eigenvalues, E_n, each of finite multiplicity with $E_n \to \infty$. In general, each eigenvalue has finite multiplicity and they are isolated points in $[0,\infty)$.*

At first sight, the embedded eigenvalues may seem surprising but they are there for a simple reason. Ω is invariant under the symmetry $y \to -y$ so $-\Delta_\Omega^N$ is also. Its spectrum on the odd function is discrete. We conjecture that on the even space where there is a.c. spectrum, there are no eigenvalues.

EXAMPLES: Take $h = 0$. If $f(x) = x^{-\alpha}$, then $f' \sim x^{-\alpha-1}$, $(f')^2/f \sim x^{-\alpha-2}$ and $V(x) \sim x^{-2}$ so the theorem applies for any $\alpha > 0$. If $f(x) = \exp(-x^\alpha)$, then it is easy to see that $f', (f')^2/f = O(\exp(-\frac{1}{2}x^\alpha))$ so we need only look at $V(x) \sim x^{2(\alpha-1)}$. If $\alpha < \frac{1}{2}$, the theorem applies. If $\alpha > 1$, $V(x) \to \infty$ and we will be able to show with our methods that $-\Delta_\Omega^N$ has compact resolvent (consistent with the theorem of Evans and Harris). For $\frac{1}{2} \le \alpha < 1$, $V(x) \sim x^{-\gamma}$ with $0 < \gamma < 1$ and it is likely that one could modify our arguments by using Enss theory for long range potentials [6],[9]. In any event our methods show that $\sigma(-\Delta_\Omega^N) = [0,\infty)$ in this case. For $f(x) = \exp(-x)$, $V(x) = \frac{1}{4}$ and our methods easily imply that the conclusions of Theorem 1.1 hold if $[0,\infty)$ is replaced by $[\frac{1}{4},\infty)$. This is related to the Laplacian on a hyperbolic manifold with cusp, see e.g. [11].

The basic idea behind the proof is to look at $-d^2/dy^2$ on $(f_1(x), f_2(x))$ with Neumann boundary conditions. This has 0 as its lowest eigenvalue but the next eigenvalue is $(\pi/2)^2\ f(x)^{-2}$ which goes to infinity as $x \to \infty$. As a result, it seems reasonable to imagine that any non-discrete spectrum can only come from functions which are nearly constant in the y direction. Thus, it is natural to consider the subspaces of $L^2(\Omega)$ consisting of functions $u(x,y)$ independent of y with the Neumann form restricted to it.

We let $\mathcal{H}_1 = L^2(1,\infty; 2f\,dx)$ the define the isometry $J_1 : \mathcal{H}_1 \to L^2(\Omega)$ by

$$(J_1 u)(x,y) = u(x)$$

so that

$$(J_1^* g)(x) = \frac{1}{2f}\int_{f_1}^{f_2} g(x,y)dy\ .$$

We then define the quadratic form Q_1 on $C_0^2[1,\infty) \subseteq \mathcal{H}_1$ by

$$Q_1(u) = 2\int_1^\infty \left|\frac{du}{dx}\right|^2 f\,dx + c\big|u(1)\big|^2$$

where

$$c = -f'(1)\ .$$

This form is associated with the operator H_1 defined on the domain

$$D_1 = \left\{u \in C_0^2[0,\infty) \mid u'(1) = -\frac{f'(1)}{2f(1)}u(1)\right\}$$

by

$$\begin{aligned} H_1 u &= -f^{-1}\frac{d}{dx}\left\{f\frac{du}{dx}\right\} \\ &= -\frac{d^2u}{dx^2} - f'f^{-1}\frac{du}{dx}\ . \end{aligned}$$

We next define the operator H_2 on $\mathcal{H}_2 = L^2(1,\infty;dx)$ by

$$H_2 = W^{-1}H_1W$$

where the unitary operator W from $\mathcal{H}_2$ to $\mathcal{H}_1$ is given by

$$Wv = \gamma v$$

for $\gamma = (2f)^{-1/2}$ in $C^2[1,\infty)$. One sees that $D_2 = W^{-1}D_1$ is given by

$$D_2 = \{v \in C_0^2[1,\infty)\ ;\ v'(1) = 0\}$$

and that H_2 is given on D_2 by

$$\begin{aligned} H_2 v &= -\gamma\frac{d}{dx}\left\{\gamma^{-2}\frac{d}{dx}(\gamma v)\right\} \\ &= -\frac{d^2v}{dx^2} + Vv \end{aligned}$$

where

$$\begin{aligned} V &= \frac{\gamma''}{\gamma} - 2\left(\frac{\gamma'}{\gamma}\right)^2 \\ &= \frac{1}{4}\left(\frac{f'}{f}\right)^2 + \frac{1}{2}\left(\frac{f'}{f}\right)'\ . \end{aligned}$$

We summarize the main results of the paper in the case $h = 0$. In section 2, we will prove the basic estimate which verifies the fact that it looks like the Dirichlet form. It says that for $u \in D(H_2)$:

$$\|(H+1)^{-1/2}(HJ_2 - J_2H_2)u\| \leq 2\left\|f'\frac{du}{dx}\right\| + \|(f')^2f^{-1}u\| + C_4|u(1)| \quad (1.6)$$

where the norm on the left is an $L^2(\Omega, d^2x)$ norm and on the right $L^2(1,\infty;dx)$ and where $J_2 = J_1W$. (1.6) is somewhat subtle in that there is a careful cancellation of $0(|x|^{-1})$ needed.

In section 3, we use (1.6) to show that so long as f' and $(f')^2f^{-1}$ go to zero at infinity $[1 - J_2^*J_2](H+1)^{-1}$ is compact verifying the notion that any essential spectrum must come from the form H on Ran J. Then we use the Enss theory in section 4 to prove Theorem 1.1. While our analysis is for two dimensional regions, it is easy to extend it to suitable horn shaped regions in $\mathbb{R}^n$.

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2. The Technical Heart of the Matter

Our goal is to prove the estimate (1.6). We shall always assume that V is bounded below. This implies that H_2 is essentially self-adjoint on D_2, and hence by unitary equivalence on H_1 is essentially self-adjoint on D_1.

PROPOSITION 2.1. *If $u \in D_1$ then*

$$\|(H+1)^{-1/2}(HJ_1 - J_1H_1)u\|_2 \leq \left\|\frac{du}{dx}(|f_1'| + |f_2'|)\right\|_2 + c_3|u(1)| .$$

Proof: If $g \in C_0^2(\overline{\Omega})$ then

$$\begin{aligned}\langle g, HJ_1u\rangle &= \iint_\Omega \frac{du}{dx}\frac{\partial g}{\partial x}dx\,dy \\ &= \int_1^\infty \frac{du}{dx}\left\{\int_{f_1}^{f_2}\frac{\partial g}{\partial x}dy\right\}dx\end{aligned}$$

and

$$\begin{aligned}\langle g, J_1 H_1 u\rangle &= \langle J_1^* g, H_1 u\rangle \\ &= \int_1^\infty \frac{du}{dx}\frac{d}{dx}\left\{\frac{1}{f}\int_{f_1}^{f_2} g\,dy\right\} f\,dx + \frac{cu(1)}{2f(1)}\int_{f_1(1)}^{f_2(1)} g(1,y)dy\ .\end{aligned}$$

Therefore,

$$\begin{aligned}&\left|\,\langle g,(HJ_1 - J_1H_1)u\rangle\,\right| \\ &\le \int_1^\infty \left|\frac{du}{dx}\right|\left|-\frac{f'}{f^2}\int_{f_1}^{f_2} g\,dy + \frac{f_2'}{f}g(x,f_2) - \frac{f_1'}{f}g(x,f_1)\right| f\,dx \\ &\quad + \frac{|cu(1)|}{2f(1)}\int_{f_1(1)}^{f_2(1)} |g(1,y)|dy \\ &\le \int_1^\infty \left|\frac{du}{dx}\right|\left|\frac{f_2'}{2f^2}\int_{f_1}^{f_2}\{g(x,y) - g(x,f_2(x))\}dy\right| f\,dx \\ &\quad + \int_1^\infty \left|\frac{du}{dx}\right|\left|\frac{f_1'}{2f^2}\int_{f_1}^{f_2}\{g(x,y) - g(x,f_1(x))\}dy\right| f\,dx \\ &\quad + \frac{|cu(1)|}{2f(1)}\int_{f_1(1)}^{f_2(1)} |g(1,y)|dy \\ &\le \int_1^\infty \left|\frac{du}{dx}\right|(|f_1'| + |f_2'|)\left\{\int_{f_1}^{f_2}\left|\frac{\partial g}{\partial y}\right| dy\right\}dx \\ &\quad + \frac{|cu(1)|}{2f(1)}\int_{f_1(1)}^{f_2(1)} |g(1,y)|dy \\ &= \left\langle J_1\left\{\left|\frac{du}{dx}\right|(|f_1'| + |f_2'|)\right\}, \left|\frac{\partial g}{\partial y}\right|\right\rangle \\ &\quad + c_1'|u(1)|\int_{f_1(1)}^{f_2(1)} |g(1,y)|dy\ .\end{aligned}$$

Now $\|\frac{\partial g}{\partial y}\|_2 \le \|(H+1)^{1/2}g\|_2$ and $\int_{f_1(1)}^{f_2(1)} |g(1,y)|dy \le c_2\|(H+1)^{1/2}g\|_2$. Hence

$$\left|\,\langle g,(HJ_1 - J_1H_1)u\rangle\,\right| \le \left\{\left\|\frac{du}{dx}(|f_1'| + |f_2'|)\right\|_2 + c_3|u(1)|\right\}\|(H+1)^{1/2}g\|_2$$

for all $g \in C_0^2(\overline{\Omega})$. This implies the estimate of the proposition, since $C_0^2(\overline{\Omega})$ is a core of $H^{1/2}$. □

PROPOSITION 2.2. *If $v \in D_2$ and $J_2 = J_1 W$ then*

$$\left\|(H+1)^{-1/2}(HJ_2 - J_2H_2)v\right\|_2 \le \left\|k_1 \frac{dv}{dx}\right\|_2 + \|k_2 v\|_2 + c_4\left|v(1)\right|$$

where k_1 and k_2 are defined in (1.4) and (1.5), respectively.

Proof: Since H_1 is unitarily equivalent to H_2 under W, Proposition 2.1 yields

$$\begin{aligned}\left\|(H+1)^{-1/2}(HJ_2 - J_2H_2)v\right\|_2 &\le \left\|\gamma^{-1}k_1 \frac{d}{dx}(\gamma v)\right\|_2 + c_3\left|\gamma(1)v(1)\right| \\ &\le \left\|k_1 \frac{dv}{dx}\right\|_2 + \left\|\frac{k_1\gamma'}{\gamma}v\right\|_2 + c_4\left|v(1)\right| . \quad \square\end{aligned}$$

3. Relative Compactness

As our first application of Proposition 2.2., we intend to prove two distinct but related results that say that continuous spectrum is only associated to Ran J_2. Let $P =$ projection on a Ran J_2 and let $Q = 1 - P$, i.e. $P = J_2J_2^*$. Throughout this section, we assume that f, k_1, k_2 vanish as $x \to \infty$. We will prove

THEOREM 3.1. *$Q(H+1)^{-1/2}$ is compact.*

THEOREM 3.2. *If g is a continuous function on $\mathbf{R} \cup \{\infty\}$, then $g(H)J_2 - J_2g(H_2)$ is compact.*

THEOREM 3.3. *H has compact resolvent if and only if $-\frac{d^2}{dx^2} + V(x)$ has compact resolvent on $L^2(1,\infty)$.*

Proof of Theorem 3.1: This is very close in spirit to the proof of Rellich, that the Dirichlet Laplacian of such horns has compact resolvent. It clearly suffices to prove that $Q(H+1)^{-1}Q$ is compact. Let H_L^- (resp. H_L^+) be the operator on $\{(x,y) \in \Omega \mid x < L\}$ (resp. $\{(x,y) \in \Omega \mid x > L\}$) with an additional Neumann boundary condition at $x = L$. Then

$$(H+1)^{-1} \le (H_L^- + 1)^{-1} \oplus (H_L^+ + 1)^{-1}$$

by Dirichlet-Neumann bracketing and $Q = Q_L^- \oplus Q_L^+$. Each $(H_L^- + 1)^{-1}$ is compact as the Neumann Laplacian of a region with piecewise smooth boundary (see [3], Theorem 1.7.12), so it suffices to prove that

$$\|Q_L^+(H_L^+ + 1)^{-1}Q_L^+\| \to 0$$

as $L \to \infty$. But

$$Q_L^+(H_L^+ + 1)^{-1}Q_L^+ \le Q_L^+ \left(-\frac{d^2}{dy^2} + 1\right)^{-1} Q_L^+$$
$$\le \sup_{x \ge L} \left[\left(\frac{\pi}{2}\right)^2 f(x)^{-2} + 1\right]^{-1} \to 0$$

since $\left(\frac{\pi}{2}\right)^2 f(x)^{-2}$ is the lowest eigenvalue of $-d^2/dy^2$ on $\operatorname{Ran} Q_L^+$. □

Proof of Theorem 3.2: By the Stone-Weierstrass theorem, we need only prove the result for $g(x) = (x+1)^{-n}$ and then by induction, only for $g(x) = (x+1)^{-1}$, i.e. that

$$(H+1)^{-1}J_2 - J_2(H_2+1)^{-1}$$

is compact.

Since V is bounded, H_2 has the same domain as the operator $H_0 = -\frac{d^2}{dx^2}$ on $\mathcal{H}_2$ subject to Neumann boundary conditions at $x = 1$. Also

$$\begin{aligned}
&(H+1)^{-1}J_2 - J_2(H_2+1)^{-1} \\
&= (H+1)^{-1}(J_2H_2 - HJ_2)(H_2+1)^{-1} \\
&= (H+1)^{-1/2}(H+1)^{-1/2}(J_2H_2 - HJ_2)(H_0+1)^{-1}(H_0+1)(H_2+1)^{-1}
\end{aligned}$$

and it suffices to prove that

$$A = (H+1)^{-1/2}(J_2H_2 - HJ_2)(H_0+1)^{-1}$$

is compact. According to Proposition 2.2

$$\|Av\|_2 \le \left\|k_1 \frac{d}{dx}(H_0+1)^{-1}v\right\|_2 + \left\|k_2(H_0+1)^{-1}v\right\|_2 + c_4\left|(H_0+1)^{-1}v(1)\right| .$$

Our assumptions on k_1 and k_2 imply that $k_1\frac{d}{dx}(H_0+1)^{-1}$ and $k_2(H_0+1)^{-1}$ are compact. Moreover

$$(H_0+1)^{-1}v(1) = \langle v, w\rangle$$

where $w \in \mathcal{H}_2$. The compactness of A follows by a general argument. □

Proof of Theorem 3.3: By Theorem 3.1., $(H+1)^{-1}$ is compact if and only if $(H+1)^{-1}P = (H+1)^{-1}J_2J_2^*$ is compact. By Theorem 3.2, this is compact if and only if $J_2(H_2+1)^{-1}J_2^*$ is compact. Since J_2 is an isometry, this operator is compact if and only if $(H_2+1)^{-1}$ is compact. But as noted, H_2 is unitarily equivalent to $-\frac{d^2}{dx^2}+V(x)$. □

COROLLARY 3.4. *If f, k and k_2 vanish and $V(x)\to\infty$ as $x\to\infty$, then $-\Delta_\Omega^N$ has compact resolvent.*

Of course, since the theorem of Evans and Harris is necessary and sufficient, this is a weaker theorem but it may be easier to check that $V(x)\to\infty$ than their condition. In particular, this easily gives the case $f(x) = \exp(-x^\alpha)$; $\alpha > 1$.

4. The Enss Theory

In this section, we will prove Theorem 1.1. It is a fairly straightforward application of the version of Enss theory [5] as expanded in [10] using the method of Mourre [9] for identifying incoming and outgoing subspaces with a simplification of Davies [2] (see also [1]).

Let H_0 be the Neumann Laplacian, $-\frac{d^2}{dx^2}$ on $L^2(1,\infty;dx)$. Let A be the scale transformation about 1, i.e.

$$A = \tfrac{1}{2}\left[(x-1)p + p(x-1)\right]$$

on $L^2(-\infty,\infty;dx)$. Interpret $L^2(1,\infty;dx)$ as the functions on $L^2(-\infty,\infty;dx)$ even under reflection in $x=1$. A leaves that space invariant, so it induces an operator we will also call A on $L^2(1,\infty;dx)$. Because of the view of H_0 as $-\frac{d^2}{dx}$ on $L^2(-\infty,\infty;dx)$ restricted to a subspace, we have the following (see e.g.[1]).

(a) Let $P_\pm$ be the spectral projections for A on $\pm(0,\infty)$. Let $P_\pm^a$ be the spectral projections for A for $\pm(\pm a,\infty)$. Then $P_+^a + P_-^a = 1$.

(b) Let $\operatorname{supp} g \subset \{p^2 \mid \alpha < |p| < \beta\}$. Then for each $\varepsilon > 0$ and a:

$$\left\|\chi(x < 2(\alpha-\varepsilon)t \quad \text{or} \quad x > 2(\beta+\varepsilon)t)e^{-itH_0}g(H_0)P_\pm^a\right\| = 0(t^{-N})$$

for all N as $t\to\pm\infty$ where we use the Enss notation $\chi(\text{set})$ to indicate the characteristic function of the set. (Note: since $H_0 = p^2$, the velocity is $2p$.)

(c) For each a, $\operatorname*{s-lim}_{t\to\pm\infty} P_\mp^a e^{-itH_0} = 0$.

With these preliminaries we prove Theorem 1.1 as follows:

Step 1. $\underset{t\to\mp\infty}{s\text{-}\lim}\,(H+1)^{-1/2}e^{itH}J_2e^{-itH_0}$ exists. This follows by the standard Cook estimate from (1.6). To be explicit, we note that if

$$\int^{\infty}\left\|\frac{d}{ds}w(s)\right\|ds<\infty$$

for a vector valued function, then w is Cauchy and so $\lim_{s\to\infty} w(s)$ exists. Since $\bigcup_{a,g}\operatorname{Ran}\left(g(H_0)P_+^a\right)$ is dense we need only show for all such g and a:

$$\int_0^{\infty}\left\|(H+1)^{-1/2}(HJ_2-J_2H_0)e^{itH_0}g(H_0)P_+^a u\right\|dt<\infty\,.$$

This follows from Proposition 2.1 and (b) above.

Step 2. $\Omega^{\pm} = \underset{t\to\mp\infty}{s\text{-}\lim}\ e^{itH}J_2e^{-itH_0}$ exists. For by Theorem 3.2 $(H+1)^{-1/2}J_2 - J_2(H_2+1)^{-1/2}$ is compact and by the hypothesis on V, $(H_2+1)^{-1/2}-(H_0+1)^{-1/2}$ is compact so

$$\underset{t\to\mp\infty}{s\text{-}\lim}\ e^{itH}\left[(H+1)^{-1/2}J_2-J_2(H_0+1)^{-1/2}\right]e^{-itH_0}=0\,.$$

Thus by Step 1, $\Omega^{\pm}u$ exists if $u\in\operatorname{Ran}(H_0+1)^{-1/2}$ which is dense.

Step 3. $(H+1)^{-1/2}(\Omega^{\pm}-J_2)g(H_0)P_{\pm}$ is compact.

This follows from two notes. First, by the proof of Step 1,

$$(H+1)^{-1/2}(e^{itH}J_2e^{-itH_0}-J_2)g(H_0)P_{\pm}$$

converges in operator norm as $t\to\infty$ so it is sufficient that each of these operators is compact. Second, such operators are a finite integral of the form

$$(H+1)^{-1/2}e^{itH}(HJ_2-J_2H_0)e^{-itH_0}g(H_0)P_+$$

and these are compact by the estimates in Proposition 2.1.

Step 4. If $u_n \in \mathcal{H}_{\rm ac}(H)^\perp$ with $\|(H+1)u_n\|$ bounded and $(H+1)^{1/2}u_n \to 0$ weakly, then $\|u_n\| \to 0$. For $u_n \in \mathcal{H}_{ac}^\perp$ implies $(\Omega^\pm)^* u_n = 0$ and so by Step 3 for any g:

$$P_\pm g(H_0) J_2^* u_n \to 0 \ .$$

Since $P_+ + P_- = 1$, $g(H_0)J_2^* u_n \to 0$. Now

$$g(H_0)J_2^* - J_2^* g(H) = \{g(H_0) - g(H_2)\}J_2^* + \{g(H_2)J_2^* - J_2^* g(H)\}$$

the first term on the RHS being compact by the hypothesis on V and the second being compact by Theorem 3.2. Therefore $J^* g(H)u_n \to 0$. Since $\|(H+1)u_n\|$ is uniformly bounded we can make $\|g(H)u_n - u_n\|$ uniformly small for all n for an appropriate choice of g. It follows that $J^* u_n \to 0$. Thus

$$Pu_n = J_2 J_2^* u_n \to 0 \ .$$

By Theorem 3.1 $Qu_n \to 0$ so $u_n \to 0$.

Step 5. $\sigma_{\rm sing}(H) = \emptyset$ and in any finite interval H has only finitely many eigenvalues, each of finite multiplicity. For, if not, then it is easy to construct an orthonormal sequence u_n obeying $u_n \in \mathcal{H}_{\rm ac}^\perp$, $\|(H+1)u_n\|$ bounded and $(H+1)^{1/2}u_n \to 0$ weakly. But then $u_n \to 0$ by Step 4 contradicting the fact that u_n is normalized.

Step 6. $\operatorname{Ran}\Omega^+ = \mathcal{H}_{\rm ac}(H)$; in particular $\sigma_{\rm ac}$ has multiplicity 1. Let $u \in \mathcal{H}_{\rm ac} \cap (\operatorname{Ran}\Omega^+)^\perp$ with $u \in D(H)$. Let $u_n = e^{-inH}u$. By (c), $P_-(\Omega_-^*)u_n = P_- e^{-inH_0}(\Omega^-)^* u \to 0$, so as in Step 4, $\|u_n\| \to 0$, i.e. $u = 0$.

Step 7. H actually has embedded eigenvalues if $h = 0$. For let $\mathbf{R}^\pm$ be the projection onto those functions in $L^2(\Omega)$ which are even/odd under $y \to -y$. R^+ commutes with H and $\operatorname{Ran} R_- \subset \operatorname{Ran} Q$ so $H \restriction \operatorname{Ran} R^-$ has compact resolvent by Theorem 3.2. Thus it has the infinity of eigenvalues. □

For an example like $\{(xy) \mid (x,y) \le 1\}$ there are four horns so by localizing our arguments, we get absolutely continuous spectrum of multiplicity four. Embedding eigenvalues come from the space odd under both $x \to -x$ and $y \to -y$.

Appendix

Here is a quick proof of the following known result (see [3], Theorem 5.2.10 and [7]).

THEOREM . *Let $\Omega \subset \mathbb{R}^\nu$ be open with $|\Omega| = \infty$. Then $O \in \sigma_{\rm ess}(-\Delta_\Omega^N)$.*

Proof: Let $\Omega_n = \{x \in \Omega \mid \|x\| < n\}$ and let

$$u_n = \begin{cases} 1 & \|x\| \leq n-1 \\ n - \|x\| & n-1 \leq \|x\| \leq n \\ 0 & \|x\| \geq n \end{cases}$$

so $\alpha_n \equiv (u_n, -\Delta_\Omega^N u_n)/(u_n, u_n) \leq |\Omega_n \backslash \Omega_{n-1}|/|\Omega_{n-1}|$. We claim that

$$\underline{\lim} \alpha_n = 0 \tag{A.1}$$

so that there is a subsequence of unit vectors $w_n \equiv u_n/\|u_n\|$ with $w_n \to 0$ weakly (since $|\Omega_{n-1}| \to \infty$ and $(w_n, -\Delta_\Omega^N w_n) \to 0$). For if (A.1) fails, $|\Omega_n \backslash \Omega_{n-1}|/|\Omega_{n-1} \geq (1+\alpha)$ for some $\alpha > 0$ and all n large so $\Omega_n \geq (1+\alpha)^n$ violating $|\Omega_n| \geq t_\vee n^\vee$. □

The reader may notice this is just an extension of Schnol's argument [13],[1].

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