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Explicit Construction of Solutions of the Modified Kadomtsev–Petviashvili Equation

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Given a solution of the Kadomtsev–Petviashvili equation we explicitly construct a solution of the modified Kadomtsev–Petviashvili equation related to one another by a generalized Miura transformation. The construction is modeled after a previous treatment of the modified Korteweg–de Vries case. As an illustration of our method we derive the soliton solutions of the modified Kadomtsev–Petviashvili equation. © 1991 Academic Press, Inc.

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1. Introduction

In this paper we study the Kadomtsev–Petviashvili-II (from now on simply abbreviated by KP) equation

$$\mathrm{KP}(V) := V_t - 6VV_x + V_{xxx} + 3\int_{\infty}^{x} dx'\, V_{yy} = 0 \qquad (1.1)$$

and its modified versions, the mKP_ε-equations as introduced, e.g., in [20, 22, 24, 25]

$$\mathrm{mKP}_\varepsilon(\phi) := \phi_t - 6\phi^2\phi_x + \phi_{xxx} + 3\int_{\infty}^{x} dx'\, \phi_{yy} + 6\varepsilon\phi_x\left(\int_{\infty}^{x} dx'\, \phi_y\right) = 0, \qquad \varepsilon = \pm 1. \qquad (1.2)$$

The following generalization of Miura's transformation and his identity [30] for the (m)KdV-case to the present (m)KP one (due to [24])

$$V_\varepsilon(t, x, y) = \phi(t, x, y)^2 + \phi_x(t, x, y) - \varepsilon\int_{\infty}^{x} dx'\, \phi_y(t, x', y), \qquad \varepsilon = \pm 1, \qquad (1.3)$$

$$\mathrm{KP}(V_\varepsilon) = \left[\partial_x + 2\phi - \varepsilon\int_{\infty}^{x} \partial_y\right]\mathrm{mKP}_\varepsilon(\phi), \qquad \varepsilon = \pm 1 \qquad (1.4)$$

shows that given a solution ϕ of $\mathrm{mKP}_\varepsilon(\phi) = 0$, the transformation (1.3) yields a solution V_ε of $\mathrm{KP}(V_\varepsilon) = 0$. Our main objective in this paper is to reverse this process, i.e., given a solution V of $\mathrm{KP}(V) = 0$ we shall develop a method to construct solutions ϕ_ε of $\mathrm{mKP}_\varepsilon(\phi_\varepsilon) = 0$, $\varepsilon = \pm 1$, such that (1.3) (with $\phi = \phi_\varepsilon$) holds. Our methods generalize a previous treatment of the analogous problem in the (modified) Korteweg–de Vries context [15] (see also [13, 14]) in the sense that in contrast to [15] we now consider general, complex-valued, and singular solutions of (1.1) and (1.2).

In Section 2 we recall basic facts in connection with the KP-equation; in particular, we briefly sketch the dressing method of Zakharov and Shabat [45] to construct special solutions of (1.1). In Section 3 we rederive the soliton solutions of the KP-equation (1.1) with the help of the dressing method. In Section 4 we accomplish our main goal and construct solutions of the MKP_ε-equations (1.2) in Theorem 4.5. In Lemma 4.3 we also derive a factorization of the linear operator $L_\varepsilon := -\partial_x^2 + \varepsilon\partial_y + V$, $\varepsilon = \pm 1$, in the Lax pair for (1.1) that resembles the special (supersymmetric) factorization used in [15] in the corresponding KdV-case. Finally, in Section 5 we apply this method to derive the soliton solutions of the mKP_ε-equations (1.2).

Without going into details we also remark that all our results extend to the KP-I-equation (replace $+3\int_{\infty}^{x} dx'\, V_{yy}$ by $-3\int_{\infty}^{x} dx'\, V_{yy}$ in (1.1)) and its modified versions.

2. Preliminaries on the KP-Equation

In this section we briefly introduce the Kadomtsev–Petviashvili (KP) equation and some basic material such as its Lax pair and the dressing method to construct special solutions of it.

Throughout this paper we shall assume hypothesis

(H.2.1). Suppose $V: \mathbf{R}^3 \to \mathbf{C}$ is of the type

$$V(t, x, y) = \sum_{j=1}^{M} \alpha_j [x - x_j(t, y)]^{-2} + v(t, x, y),$$

where $\alpha_j \in \{0, 2\}$, $x_j \in C^\infty(\mathbf{R}^2)$, $1 \leqslant j \leqslant M$, $M \in \mathbf{N}$, $x_j \neq x_{j'}$, for $j \neq j'$, and $v \in C^\infty(\mathbf{R}^3)$. Moreover, we assume that

$$\lim_{x_0 \to \infty} \int_{x_0}^{x} dx'\, \partial_y^l v(t, x', y) \in C^\infty(\mathbf{R}^3), \qquad l = 1, 2$$

(the integral taken in the Riemann sense).

The KP-equation (for solutions satisfying (H.2.1)) is then defined as [21]

$$\mathrm{KP}(V) := V_t - 6VV_x + V_{xxx} + 3(\partial_x^{-1} V_{yy}) = 0, \tag{2.1}$$

where we abbreviate

$$(\partial_x^{-1} f)(t, x, y) := \int_{\infty}^{x} dx'\, f(t, x', y) \tag{2.2}$$

for appropriate f's. Clearly (H.21) with $\alpha_j = 0, j = 1, ..., M$, is designated for smooth solutions V, whereas (H.2.1) with $\alpha_j = 2$, $j = 1, ..., M$, is chosen to accomodate singular solutions, the simplest of which is of the type [8]

$$V(t, x, y) = 2(x + ay - 3a^2 t + b)^{-2}, \qquad (a, b) \in \mathbf{C}^2. \tag{2.3}$$

Equation (2.1) admits the following Lax representation.

Lemma 2.1 [10] (see also [25, 45]). *Assume* (H.2.1). *Then*

$$L_{\varepsilon, t} - [B_{\lambda, \varepsilon}, L_\varepsilon] = \mathrm{KP}(V + \lambda), \qquad \lambda \in \mathbf{R}, \varepsilon = \pm 1 \tag{2.4}$$

on $C^\infty(\mathbf{R}^2\backslash X(t))$, *where*

$$L_\varepsilon(t) := -\partial_x^2 + \varepsilon\partial_y + V(t), \qquad \varepsilon = \pm 1,\ t\in\mathbf{R}, \tag{2.5}$$

$$B_{\lambda,\varepsilon}(t) := -4\partial_x^3 + 6[V(t)+\lambda]\,\partial_x + 3V_x(t) + 3\varepsilon[\partial_x^{-1}V_y(t)],$$
$$\lambda\in\mathbf{R},\ \varepsilon=\pm 1,\ t\in\mathbf{R}, \tag{2.6}$$

$$X(t) := \{(x_j(t,y), y)\in\mathbf{R}^2 \mid y\in\mathbf{R},\ \alpha_j = 2,\ j\in\{1,...,M\}\}, \qquad t\in\mathbf{R}. \tag{2.7}$$

For later purposes we isolate the case where $V\equiv 0$, i.e., introduce on $C^\infty(\mathbf{R}^2)$

$$L_\varepsilon^{(0)} := -\partial_x^2 + \varepsilon\partial_y, \qquad \varepsilon=\pm 1, \tag{2.8}$$

$$B_\lambda^{(0)} := -4\partial_x^3 + 6\lambda\partial_x, \qquad \lambda\in\mathbf{R}. \tag{2.9}$$

In order to construct special solutions (such as solitons) of the KP-equation (2.1), we briefly review the dressing method due to Zakharov and Shabat [45]. In this approach one constructs an integral operator $\hat{F}_\varepsilon(t,y)$

$$(\hat{F}_\varepsilon(t,y)f)(x) = \int_{\mathbf{R}} dx'\, F_\varepsilon(t,x,x',y)\, f(x'), \qquad \varepsilon=\pm 1 \tag{2.10}$$

satisfying

$$[\hat{F}_\varepsilon, L_\varepsilon^{(0)}] = 0 = [\hat{F}_\varepsilon, \partial_t - B_\lambda^{(0)}]. \tag{2.11}$$

In that case the integral kernel $F_\varepsilon(t,x,x',y)$ has to solve

$$F_{\varepsilon,xx} - F_{\varepsilon,x'x'} - \varepsilon F_{\varepsilon,y} = 0 = F_{\varepsilon,t} - 6\lambda(F_{\varepsilon,x} + F_{\varepsilon,x'}) + 4(F_{\varepsilon,xxx} + F_{\varepsilon,x'x'x'}). \tag{2.12}$$

In addition one introduces Volterra operators

$$(\hat{K}_{\varepsilon,\pm}(t,y)f)(x) = \pm\int_x^{\pm\infty} dx'\, K_{\varepsilon,\pm}(t,x,x',y)\, f(x'), \qquad \varepsilon=\pm 1 \tag{2.13}$$

and assumes

$$(1+\hat{K}_{\varepsilon,+})(1+\hat{F}_\varepsilon) = (1+\hat{K}_{\varepsilon,-}), \qquad \varepsilon=\pm 1. \tag{2.14}$$

In particular,

$$K_{\varepsilon,+}(t,x,x',y) + F_\varepsilon(t,x,x',y)$$
$$+\int_x^\infty dx''\, K_{\varepsilon,+}(t,x,x'',y)\, F_\varepsilon(t,x'',x',y) = 0, \qquad x'>x,$$

$$K_{\varepsilon,-}(t,x,x',y)$$
$$= F_\varepsilon(t,x,x',y) + \int_x^\infty dx''\, K_{\varepsilon,+}(t,x,x'',y)\, F_\varepsilon(t,x'',x',y), \qquad x'<x. \tag{2.15}$$

For the purpose we have in mind (viz., the construction of soliton solutions of (2.1)) it suffices to define $\hat{F}_\varepsilon$, $\hat{K}_{\varepsilon,\pm}$, $(1+\hat{K}_{\varepsilon,\pm})^{-1}$, etc. on $C_0^\infty(\mathbf{R}\backslash X(t,y))$ where

$$X(t,y) := \{x_j(t,y)\in\mathbf{R} \mid \alpha_j = 2, j\in\{1,...,M\}\}, \qquad (t,y)\in\mathbf{R}^2.$$

(For more general situations see, e.g., the treatment in [29, 35].) It then follows that

$$\begin{aligned} &(1+\hat{K}_{\varepsilon,\pm})\,L_\varepsilon^{(0)}(1+\hat{K}_{\varepsilon,\pm})^{-1} = L_\varepsilon, && \varepsilon=\pm 1,\\ &(1+\hat{K}_{\varepsilon,\pm})(\partial_t - B_\lambda^{(0)})(1+\hat{K}_{\varepsilon,\pm})^{-1} = \partial_t - B_{\lambda,\varepsilon}, && \lambda\in\mathbf{R}, \varepsilon=\pm 1, \end{aligned} \tag{2.16}$$

where L_ε, $B_{\lambda,\varepsilon}$ are given by (2.5) and (2.6), respectively, with V given by

$$V_\varepsilon(t,x,y) = -2\partial_x K_{\varepsilon,+}(t,x,x,y), \qquad \varepsilon=\pm 1. \tag{2.17}$$

(Observe that the right-hand side of (2.16) is independent of whether one uses the upper or lower Volterra operator $\hat{K}_{\varepsilon,\pm}$.)

Moreover, (2.10)–(2.15) imply

$$L_{\varepsilon,t} - [B_{\lambda,\varepsilon}, L_\varepsilon] = 0, \qquad \varepsilon=\pm 1 \tag{2.18}$$

and hence

$$\mathrm{KP}(V_\varepsilon+\lambda) = 0, \qquad \varepsilon=\pm 1 \tag{2.19}$$

by (2.4) with V_ε given by (2.17). Finally, let $\psi_\varepsilon^{(0)}(t,x,y)$ satisfy

$$L_\varepsilon^{(0)}\psi_\varepsilon^{(0)} = -\kappa^2\psi_\varepsilon^{(0)}, \quad (\partial_t - B_\lambda^{(0)})\,\psi_\varepsilon^{(0)} = 0, \quad \kappa\geqslant 0, \lambda\in\mathbf{R}, \varepsilon=\pm 1 \tag{2.20}$$

in the sense of distributions, i.e., in $C_0^\infty(\mathbf{R}^2)'$ and $C_0^\infty(\mathbf{R}^3)'$, respectively. Then

$$\psi_\varepsilon := (1+\hat{K}_{\varepsilon,+})\,\psi_\varepsilon^{(0)}, \qquad \varepsilon=\pm 1 \tag{2.21}$$

satisfies

$$L_\varepsilon\psi_\varepsilon = -\kappa^2\psi_\varepsilon, \qquad (\partial_t - B_{\lambda,\varepsilon})\,\psi_\varepsilon = 0, \qquad \kappa\geqslant 0, \lambda\in\mathbf{R}, \varepsilon=\pm 1 \tag{2.22}$$

in the sense of distributions, i.,e., in $C_0^\infty(\mathbf{R}^2\backslash X(t))'$ and $C_0^\infty(\mathbf{R}^3\backslash X)'$, respectively. Here

$$X := \{(t, x_j(t,y), y)\in\mathbf{R}^3 \mid (t,y)\in\mathbf{R}^2, \alpha_j = 2, j\in\{1,...,M\}\}.$$

A solution of (2.20) to be used in Section 3 is e.g., given by

$$\psi^0(t,x,y) = \exp[-\kappa x + 4(\kappa^3 - 6\lambda\kappa)t], \qquad \kappa\geqslant 0, \lambda\in\mathbf{R}. \tag{2.23}$$

Using this method we shall sketch a derivation of the soliton solutions of (2.1) originally obtained in [45] in the next section. Further applications of the dressing method (also to the KP-I-equation) can, e.g., be found in [27, 31, 35, 45].

Apart from the dressing method, other techniques have successfully been applied to (2.1). We mention e.g., the $\bar{\partial}$-approach to multidimensional inverse scattering [1, 3, 4, 42, 44] and Hirota's τ-function approach [2, 17, 18, 38] and its link with infinite dimensional Lie algebras [9, 16, 20, 22, 36, 39, 40] and Fredholm determinants [34, 35]. The problem of complete integrability and Hamiltonian structures for (2.1) was addressed, e.g., in [6, 19, 26, 43]. Bäcklund transformations for (2.1) were studied in [7, 41, 46]; also other particular solutions of (2.1) such as meromorphic ones [2, 8, 34, 35, 38] and periodic multiphase solutions [5] appeared in the literature. (This list is by no means complete; we only give some of the more recent accounts and encourage the reader to study the references cited therein.)

3. Soliton Solutions for the KP-Equation

We briefly sketch a derivation of the N-soliton solution of the KP-equation (2.1) with the help of the dressing method. Originally these solutions were derived in [45] (see also [11, 28, 31, 32, 35, 37]). We define

$$F_\varepsilon(t, x, x', y) := \sum_{n=1}^{N} \tilde{c}_{\varepsilon,n}(t, y)\, e^{-(p_n x + q_n x')},$$

$$\tilde{c}_{\varepsilon,n}(t, y) = \tilde{c}_n \exp\{\varepsilon(p_n^2 - q_n^2)\, y + [4(p_n^3 + q_n^3) - 6\lambda(p_n + q_n)]\, t\},$$

$$\lambda \in \mathbf{R}, \quad \tilde{c}_n \in \mathbf{R}\backslash\{0\}, \quad p_n > 0, \quad q_n > 0, \quad 1 \leqslant n \leqslant N, \quad \varepsilon = \pm 1. \tag{3.1}$$

Then F_ε satisfies (2.11). Next assume

$$\begin{aligned} \tilde{k}_\varepsilon(t, x, y) &= (\tilde{k}_{\varepsilon,1}(t, x, y), \ldots, \tilde{k}_{\varepsilon,N}(t, x, y)), \\ \tilde{d}_\varepsilon(t, x, y) &= (-\tilde{c}_{\varepsilon,1}(t, y)\, e^{-p_1 x}, \ldots, -\tilde{c}_{\varepsilon,N}(t, y)\, e^{-p_N x}), \\ \tilde{A}_{N,\varepsilon}(t, x, y) &= [\tilde{c}_{\varepsilon,n}(t, y)(p_n + q_m)^{-1}\, e^{-(p_n + q_m)x}]_{n,m=1}^{N} \end{aligned} \tag{3.2}$$

to satisfy

$$(1 + \tilde{A}_{N,\varepsilon})\, \tilde{k}_\varepsilon = \tilde{d}_\varepsilon, \qquad \varepsilon = \pm 1. \tag{3.3}$$

Then

$$K_{\varepsilon,+}(t, x, x', y) := \sum_{n=1}^{N} \tilde{k}_{\varepsilon,n}(t, x, y)\, e^{-q_n x'}, \qquad x' > x, \varepsilon = \pm 1 \tag{3.4}$$

satisfies (2.15). By Cramer's rule

$$\tilde{k}_{\varepsilon,n} = \det(\tilde{A}_{N,\varepsilon,n})/\det(1 + \tilde{A}_{N,\varepsilon}), \qquad 1 \leqslant n \leqslant N, \varepsilon = \pm 1, \tag{3.5}$$

where $\tilde{A}_{N,\varepsilon,n}$ is the matrix $1+\tilde{A}_{N,\varepsilon}$ with the nth column replaced by $\tilde{d}_\varepsilon$. (The positivity of $\det(1+A_{N,\varepsilon})$ will be discussed in Proposition 3.3.) By (2.17) and (2.19) this yields the N-soliton KP-solutions.

THEOREM 3.1 [11, 28, 31, 32, 35, 37, 45]. *Define*

$$V_{N,\varepsilon}(t,x,y) := -2\partial_x K_{\varepsilon,+}(t,x,x,y), \qquad N\in\mathbf{N},\ \varepsilon=\pm 1 \tag{3.6}$$

with $K_{\varepsilon,+}$ *defined by* (3.4). *Then*

$$V_{N,\varepsilon}(t,x,y) = -2\partial_x^2 \ln\{\det[1+\tilde{A}_{N\varepsilon}(t,x,y)]\} \tag{3.7}$$

and

$$\mathrm{KP}(V_{N,\varepsilon}+\lambda)=0, \qquad \lambda\in\mathbf{R},\ \varepsilon=\pm 1. \tag{3.8}$$

Moreover (*following* [12, 23]), *we have*

$$\sum_{n=1}^{N} \tilde{k}_{\varepsilon,n}(t,x,y)\, e^{-q_n x} = \partial_x \ln\{\det[1+\tilde{A}_{N,\varepsilon}(t,x,y)]\}, \tag{3.9}$$

$$L_\varepsilon \tilde{k}_{\varepsilon,n} = -q_n^2 \tilde{k}_{\varepsilon,n}, \qquad 1\leqslant n\leqslant N, \tag{3.10}$$

$$L_\varepsilon \psi_{N,\varepsilon} = -\kappa^2\psi_{N,\varepsilon}, \qquad (\partial_t - B_{\lambda,\varepsilon})\,\psi_{N,\varepsilon}=0, \tag{3.11}$$

where

$$\begin{aligned}\psi_{N,\varepsilon}(t,x,y) &= [1+\hat{K}_{\varepsilon,+}(t,y)]\,\psi^0(t,x,y)\\ &= \left\{1+\sum_{n=1}^{N}(q_n+\kappa)^{-1}\tilde{k}_{\varepsilon,n}(t,x,y)\,e^{-q_n x}\right\} e^{-\kappa x+(4\kappa^3-6\lambda\kappa)t}, \end{aligned} \tag{3.12}$$

$$\kappa\geqslant 0, \quad \lambda\in\mathbf{R}, \quad N\in\mathbf{N}, \quad \varepsilon=\pm 1$$

and ψ^0 *is given by* (2.23).

We also note that $\tilde{A}$ may be put in a more symmetrical form by introducing

$$\begin{aligned}
c_{\varepsilon,n}(t,y) &:= [\tilde{c}_{\varepsilon,n}(t,y)]^{1/2},\\
k_{\varepsilon,n}(t,x,y) &:= [\tilde{c}_{\varepsilon,n}(t,y)]^{-1/2}\,\tilde{k}_{\varepsilon,n}(t,x,y), \qquad 1\leqslant n\leqslant N,\ \varepsilon=\pm 1,\\
k_\varepsilon(t,x,y) &:= (k_{\varepsilon,1}(t,x,y),\ldots,k_{\varepsilon,N}(t,x,y)),\\
d_\varepsilon(t,x,y) &:= (-c_{\varepsilon,1}(t,y)\,e^{-p_1 x},\ldots,-c_{\varepsilon,N}(t,y)\,e^{-p_N x}),\\
A_{N,\varepsilon}(t,x,y) &:= [c_{\varepsilon,n}(t,y)\,c_{\varepsilon,m}(t,y)(p_n+q_m)^{-1}\,e^{-(p_n+q_m)x}]_{n,m=1}^{N}.
\end{aligned} \tag{3.13}$$

Then

$$(1+A_{N,\varepsilon})\,k_\varepsilon = d_\varepsilon, \qquad \varepsilon=\pm 1 \tag{3.14}$$

and hence

$$\det[1+\tilde{A}_{N,\varepsilon}(t,x,y)]=\det[1+A_{N,\varepsilon}(t,x,y)], \qquad \varepsilon=\pm 1 \tag{3.15}$$

by the similarity of $\tilde{A}_{N,\varepsilon}$ and $A_{N,\varepsilon}$.

Since (3.7) (in contrast to the KdV-solitons) for $N \geqslant 2$ also contains singular solutions for certain values of p_j, q_j (see below) it is worthwhile to specify conditions under which (3.7) describes regular (i.e., $C^\infty(\mathbf{R}^3)$) solutions of (2.1). For that purpose we need some definitions. Let M be any $N \times N$ matrix. We then define M^j to be M with the jth line and row removed, and in general, $M^{j_1,\ldots,j_m}$ to be the $(N-m)\times(N-m)$ matrix obtained from M by removing the $j_1, \ldots, j_m$th lines and rows.

The characteristic polynomial of M can then be expressed as

$$\det(z+M)=\det(M)+z\sum_{j=1}^{N}\det(M^j)+z^2\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{N}\det(M^{j_1,j_2})$$
$$+\cdots+z^{N-1}\sum_{\substack{j_1,\ldots,j_{N-1}=1\\ j_1<\cdots<j_{N-1}}}^{N}\det(M^{j_1,\ldots,j_{N-1}})+z^N, \qquad z\in\boldsymbol{C}, \tag{3.16}$$

where

$$\sum_{\substack{j_1,\ldots,j_{N-1}=1\\ j_1<\cdots<j_{N-1}}}^{N}\det(M^{j_1,\ldots,j_{N-1}})=\operatorname{Tr}(M), \tag{3.17}$$

etc. are the principal minors of M. Moreover, using [33, p. 92] we get

$$\det[A_{N,\varepsilon}^{j_1,\ldots,j_m}(t,x,y)]$$
$$=\prod_{l=1}^{m}c_{j_l}(t,y)^2\,e^{-(p_{j_l}+q_{j_l})x}\det[(p_{j_l}+q_{j_k})^{-1}]_{l,k=1}^{m}$$
$$=\prod_{l=1}^{m}c_{j_l}(t,y)^2\,e^{-(p_{j_l}+q_{j_l})x}\prod_{\substack{l,k=1\\ j_l<j_k}}^{m}(p_{j_l}-p_{j_k})(q_{j_l}-q_{j_k})\Big/\prod_{l,k=1}^{m}(p_{j_l}+q_{j_k}),$$
$$1\leqslant m\leqslant N-1. \tag{3.18}$$

Using (3.16) we obtain (cf. [32])

Proposition 3.2. *Let* $N>1$. *Then* $\det[1+A_{N,\varepsilon}(t,x,y)]>0$ (*and hence* $V_{n,\varepsilon}\in C^\infty(\mathbf{R}^3)$ *by* (3.7) *and* (3.15)) *if*

$$(p_j-p_k)(q_j-q_k)\geqslant 0 \qquad \text{for all} \quad j,k=1,\ldots,N,\ j<k. \tag{3.19}$$

Condition (3.19) *is also necessary in the sense that if* $(p_{j_0}-p_{k_0})(q_{j_0}-q_{k_0})<0$ *for some* $j_0, k_0\in\{1,\ldots,N\}$, $j_0<k_0$, *then one can choose* $c_j(t,y)^2$, $j\in\{1,\ldots,N\}\backslash\{j_0,k_0\}$, *in such a way that* $\det[1+A_{N,\varepsilon}(t,x,y)]$ *changes sign at least once as x runs from* $-\infty$ *to* $+\infty$.

Proof. Sufficiency of (3.19) is clear from (3.16) and (3.18) since every term in (3.16) (for $z=1$) will be nonnegative and at least $N+1$ of them will be strictly positive. To prove the necessity statement assume that

$$(p_{j_0}-p_{k_0})(q_{j_0}-q_{k_0})<0 \qquad \text{for some} \quad j_0, k_0 \in \{1, ..., N\}, j_0<k_0. \tag{3.20}$$

Then (suppressing the t, y variables) choose $x_0<0$ sufficiently negative such that

$$\begin{aligned} &c_{j_0}^2 c_{k_0}^2 e^{-(p_{j_0}+q_{j_0}+p_{k_0}+q_{k_0})x_0} \frac{(p_{j_0}-p_{k_0})(q_{j_0}-q_{k_0})}{(p_{j_0}+q_{j_0})(p_{j_0}+q_{k_0})(p_{k_0}+q_{j_0})(p_{k_0}+q_{k_0})} \\ &\quad+\frac{c_{j_0}^2}{p_{j_0}+q_{j_0}} e^{-(p_{j_0}+q_{j_0})x_0}+\frac{c_{k_0}^2}{p_{k_0}+q_{k_0}} e^{-(p_{k_0}+q_{k_0})x_0}+1 \leqslant -1. \end{aligned} \tag{3.21}$$

Next choose c_j^2, $j\in\{1, ..., N\}\backslash\{j_0, k_0\}$, small enough such that

$$\begin{aligned} &\det[1+A_{N,\varepsilon}(x_0)]-c_{j_0}^2 c_{k_0}^2 e^{-(p_{j_0}+q_{j_0}+p_{k_0}+q_{k_0})x_0} \\ &\quad\times\frac{(p_{j_0}-p_{k_0})(q_{j_0}-q_{k_0})}{(p_{j_0}+q_{j_0})(p_{j_0}+q_{k_0})(p_{k_0}+q_{j_0})(p_{k_0}+q_{k_0})} \\ &\quad-\frac{c_{j_0}^2}{p_{j_0}+q_{j_0}} e^{-(p_{j_0}+q_{j_0})x_0}-\frac{c_{k_0}^2}{p_{k_0}+q_{k_0}} e^{-(p_{k_0}+q_{k_0})x_0}-1 \leqslant 1/2. \end{aligned} \tag{3.22}$$

Then

$$\det[1+A_{N,\varepsilon}(x_0)] \leqslant -1/2. \tag{3.23}$$

On the other hand,

$$\det[1+A_{N,\varepsilon}(x)] \xrightarrow[x\to+\infty]{} 1. \tag{3.24}$$

By continuity of $\det[1+A_{N,\varepsilon}(x)]$ w.r.t. $x\in\mathbf{R}$ we conclude that $\det[1+A_{N,\varepsilon}(x)]$ cannot be of definite sign for all $x\in\mathbf{R}$.

The first part of Proposition 3.2 is due to [32] for $N=2, 3$. We conclude Section 3 with the

EXAMPLE 3.3. $N=1$.

$$\begin{aligned} V_{1,\varepsilon}(t, x, y) = &-\frac{(p_1+q_1)^2}{2}\cosh^{-2}\left\{\frac{1}{2}(p_1+q_1)x-\frac{\varepsilon}{2}(p_1^2-q_1^2)y\right. \\ &\left.+[3\lambda(p_1+q_1)-2(p_1^3+q_1^3)]t-\frac{1}{2}\ln\left[\frac{\tilde{c}_1}{p_1+q_1}\right]\right\}, \end{aligned} \tag{3.25}$$

$$\mathrm{KP}(V_{1,\varepsilon}+\lambda)=0, \qquad \lambda\in\mathbf{R},\ \varepsilon=\pm 1. \tag{3.26}$$

4. THE mKP-EQUATION

In this section we study the modified Kadomtsev–Petviashvili (mKP) equation as discussed, e.g., in [20, 22, 24, 25]. In particular, given a solution V of the KP-equation (2.1) we explicitly construct a corresponding solution ϕ_ε of the mKP$_\varepsilon$-equation (4.1) that are linked to each other by the generalized Miura transformation (4.5).

We introduce hypothesis

(H.4.1). Suppose $\phi\colon \mathbf{R}^3 \to \mathbf{C}$ is of the type

$$\phi(t, x, y) = \sum_{j=1}^{\tilde{M}} \beta_j [x - \tilde{x}_j(t, y)]^{-1} + \varphi(t, x, y)],$$

where $\beta_j \in \{0, 1, -1\}$, $\tilde{x}_j \in C^\infty(\mathbf{R}^2)$, $1 \leqslant j \leqslant \tilde{M}$, $\tilde{M} \in \mathbf{N}$, $\tilde{x}_j \neq \tilde{x}_{j'}$, for $j \neq j'$, and $\varphi \in C^\infty(\mathbf{R}^3)$. Moreover, we assume that

$$\lim_{x_0 \to \infty} \int_{x_0}^{x} dx' \, \partial_y^l \varphi(t, x', y) \in C^\infty(\mathbf{R}^3), \qquad l = 1, 2$$

(the integral taken in the Riemann sense).

The mKP$_\varepsilon$-equations (for solutions ϕ satisfying (H.4.1)) following [20, 22, 24, 25], are then defined by

$$\mathrm{mKP}_\varepsilon(\phi) := \phi_t - 6\phi^2\phi_x + \phi_{xxx} + 3(\partial_x^{-1}\phi_{yy}) + 6\varepsilon\phi_x(\partial_x^{-1}\phi_y), \qquad \varepsilon = \pm 1. \tag{4.1}$$

A (scalar) Lax pair for the mKP$_\varepsilon$-equations is provided by

PROPOSITION 4.1 [24, 25]. *Assume* (H.4.1). *Then*

$$\hat{L}_{\varepsilon,\sigma,t} - [\hat{B}_{\varepsilon,\sigma}, \hat{L}_{\varepsilon,\sigma}] = -2\varepsilon\sigma mKP_\varepsilon(\phi)\,\partial_x, \qquad \varepsilon, \sigma = \pm 1 \tag{4.2}$$

on $C^\infty(\mathbf{R}^2 \backslash \tilde{X}(t))$, *where*

$$\hat{L}_{\varepsilon,\sigma}(t) := -\partial_x^2 - 2\varepsilon\sigma\phi(t)\,\partial_x + \sigma\partial_y, \qquad t \in \mathbf{R}, \varepsilon, \sigma = \pm 1, \tag{4.3}$$

$$\begin{aligned}\hat{B}_{\varepsilon,\sigma}(t) := {} & -4\partial_x^3 - 12\varepsilon\sigma\phi(t)\,\partial_x^2 - 6\varepsilon\sigma\phi_x(t)\,\partial_x \\ & - 6\phi(t)^2\,\partial_x - 6\varepsilon(\partial_x^{-1}\phi_y(t))\,\partial_x, \qquad t \in \mathbf{R}, \varepsilon, \sigma = \pm 1\end{aligned} \tag{4.4}$$

$$\tilde{X}(t) := \{(\tilde{x}_j(t, y), y) \in \mathbf{R}^2 \mid y \in \mathbf{R}, \beta_j = \pm 1, j \in \{1, ..., \tilde{M}\}\}, \qquad t \in \mathbf{R}. \tag{4.5}$$

Next we recall the following generalized Miura transformation.

PROPOSITION 4.2 [24]. *Suppose* ϕ *satisfies* (H.4.1) *and that for* $(k=2, l=1)$, $(k=0, l=1)$, $(k=0, l=2)$

$$\lim_{x \to \infty} \partial_x^k \partial_y^l \phi(t, x, y) = 0 \qquad \textit{for all} \quad (t, y) \in \mathbf{R}^2.$$

Define

$$V_\varepsilon(t, x, y) := \phi(t, x, y)^2 + \phi_x(t, x, y) - \varepsilon(\partial_x^{-1}\phi_y)(t, x, y), \qquad \varepsilon = \pm 1. \tag{4.6}$$

Then

$$\mathrm{KP}(V_\varepsilon) = [\partial_x + 2\phi - \varepsilon\partial_x^{-1}\partial_y]\, \mathrm{mKP}_\varepsilon(\phi), \qquad \varepsilon = \pm 1. \tag{4.7}$$

In particular, whenever ϕ satisfies $\mathrm{mKP}_\varepsilon(\phi) = 0$, V_ε defined in (4.6) satisfies $\mathrm{KP}(V_\varepsilon) = 0$. In the rest of this section we shall develop a device to reverse this process; i.e., given a solution V of the KP-equation (2.1), we shall construct solutions ϕ_ε of the mKP_ε-equation (4.1) that are linked to each other by the generalized Miura transformation (4.6) (with $\phi = \phi_\varepsilon$).

Before that we mention a slight generalization of Proposition 4.2.

Define

$$V_\varepsilon(t, x, y) := \phi(t, x, y)^2 + \phi_x(t, x, y) - \varepsilon(\partial_x^{-1}\phi_y)(t, x, y) + \mu, \quad \mu \in \mathbf{R}, \varepsilon = \pm 1. \tag{4.8}$$

Then

$$\mathrm{KP}(V_\varepsilon) = [\partial_x + 2\phi - \varepsilon\,\partial_x^{-1}\partial_y][\phi_t - 6(\phi^2 + \mu)\,\phi_x + \phi_{xxx} + 3(\partial_x^{-1}\phi_{yy}) + 6\varepsilon\phi_x(\partial_x^{-1}\phi_y)], \qquad \mu \in \mathbf{R}, \varepsilon = \pm 1. \tag{4.9}$$

Moreover, let

$$\xi = x + 6\mu t, \qquad \mu \in \mathbf{R},$$

$$\tilde{V}(t, \xi, y) := V(t, x, y) - \mu, \qquad \tilde{\phi}(t, \xi, y) := \phi(t, x, y). \tag{4.10}$$

Then

$$V_t - 6VV_x + V_{xxx} + 3(\partial_x^{-1}V_{yy}) = \tilde{V}_t - 6\tilde{V}\tilde{V}_\xi + \tilde{V}_{\xi\xi\xi} + 3(\partial_\xi^{-1}\tilde{V}_{yy}), \tag{4.11}$$

$$\phi_t - 6(\phi^2 + \mu)\,\phi_x + \phi_{xxx} + 3(\partial_x^{-1}\phi_{yy}) + 6\varepsilon\phi_x(\partial_x^{-1}\phi_y) = \tilde{\phi}_t - 6\tilde{\phi}^2\tilde{\phi}_\xi + \tilde{\phi}_{\xi\xi\xi} + 3(\partial_\xi^{-1}\tilde{\phi}_{yy}) + 6\varepsilon\tilde{\phi}_\xi(\partial_\xi^{-1}\phi_y), \qquad \mu \in \mathbf{R}. \tag{4.12}$$

We also add the trivial observation that if $V(t, x, y)$ satisfies the KP-equation (2.1) then so does $V(t, x, -y)$. Similarly, if $\phi(t, x, y)$ satisfies the mKP_ε-equation then $\phi(t, x, -y)$ and $-\phi(t, x, y)$ satisfy the $\mathrm{mKP}_{-\varepsilon}$-equation.

As a warm up for our final goal in this section we shall now present a factorization of the operator L_ε defined in (2.5). To avoid technicalities we restrict ourselves to smooth functions $\phi \in C^\infty(\mathbf{R}^3)$:

For all $t \in \mathbf{R}$ we define the set of operators

$$\begin{aligned} A(t) &:= \partial_x + \phi(t) \qquad \text{on} \quad C^\infty(\mathbf{R}^2 \backslash \tilde{X}(t)), \\ A(t)^+ &:= -\partial_x + \phi(t) \qquad \text{on} \quad C^\infty(\mathbf{R}^2 \backslash \tilde{X}(t)), \end{aligned} \tag{4.13}$$

$$(G(t)f)(x, y) = \exp\left[\int_0^x dx' \phi(t, x', y) f(x, y)\right], \quad f \in C^\infty(\mathbf{R}^2 \backslash \tilde{X}(t)), \tag{4.14}$$

$$(D(t)f)(x, y) = \int_\infty^x dx' [f_y(x', y) - \Phi_y(t, y) f(x', y)], \qquad f \in C_0^\infty(\mathbf{R}^2 \backslash \tilde{X}(t)), \tag{4.15}$$

where

$$\Phi_y(t, y) := \int_\infty^0 dx\, \phi_y(t, x, y). \tag{4.16}$$

Then

$$A(t)^+ = -G(t)\, \partial_x G(t)^{-1}, \qquad t \in \mathbf{R}, \tag{4.17}$$

$$D(t)\, \partial_x = \partial_y - \Phi_y(t) \qquad \text{on} \quad C_0^\infty(\mathbf{R}^2 \backslash \tilde{X}(t)), t \in \mathbf{R}. \tag{4.18}$$

Combining (4.13)–(4.18) one obtains

PROPOSITION 4.3. *Define*

$$C(t) := -G(t)\, D(t)\, G(t)^{-1}, \qquad t \in \mathbf{R} \tag{4.19}$$

on $C_0^\infty(\mathbf{R}^2 \backslash \tilde{X}(t))$. *Then*

$$L_\varepsilon(t) = [A(t) + \varepsilon C(t)]\, A(t)^+, \qquad t \in \mathbf{R} \tag{4.20}$$

on $C_0^\infty(\mathbf{R}^2 \backslash \tilde{X}(t))$, *where* V *in* (2.5) *is given by* V_ε *in* (4.6).

Remark 4.4. Equation (4.19) resembles the well-known (m)KdV case studied in [13–15]. In fact, if $\partial_y \phi = 0$ and ϕ is real-valued then

$$L(t) := -\partial_x^2 + \phi(t, \cdot)^2 + \phi_x(t, \cdot) = A(t)\, A(t)^*, \tag{4.21}$$

where

$$A(t) := \partial_x + \phi(t, \cdot) \tag{4.22}$$

and

$$V = \phi^2 + \phi_x \tag{4.23}$$

is Miura's original transformation [30].

This similarity to the (m)KdV-case motivated our main result.

THEOREM 4.5. *Let* $\varepsilon = \pm 1$ *and assume that* V *satisfies* (H.2.1) *and* $\mathrm{KP}(V) = 0$. *In addition suppose that* $\psi_\varepsilon \in C^\infty(\mathbf{R}^3 \backslash X)$ *satisfies for some* $\tau \in \mathbf{R}$ (*independent of* t *and* y)

$$L_\varepsilon(t)\, \psi_\varepsilon(t) = \tau \psi_\varepsilon(t), \tag{4.24}$$

$$(\partial_t - B_\varepsilon(t))\, \psi_\varepsilon(t) = 0, \tag{4.25}$$

$$\lim_{x \to \infty} \partial_y^l \ln[\psi_\varepsilon(t, x, y)] = \varepsilon \tau \delta_{l1}, \qquad l = 1, 2,\ y \in \mathbf{R} \tag{4.26}$$

for all $t \in \mathbf{R}$, *where* $B_\varepsilon(t) := B_{0,\varepsilon}(t)$ (*i.e., we choose* $\lambda = 0$ *in* (2.6)),

$$X := \{(t, x_j(t, y), y) \in \mathbf{R}^3 \mid (t, y) \in \mathbf{R}^2, \alpha_j = 2, j \in \{1, ..., M\}\}, \tag{4.27}$$

and (4.24) *and* (4.25) *hold in the sense of distributions, i.e., in* $C^\infty(\mathbf{R} \backslash X(t))'$ *and* $C_0^\infty(\mathbf{R}^3 \backslash X)'$, *respectively. Assume that* ψ *has only poles resp. zeros of finite order at* $x_j(t, y)$, $j = 1, ..., M$. *Define*

$$\phi_\varepsilon(t, x, y) = \partial_x \ln[\psi_\varepsilon(t, x, y)]. \tag{4.28}$$

Then ϕ_ε *satisfies* (H.4.1). *Moreover,*

$$V(t, x, y) = \phi_\varepsilon(t, x, y)^2 + \phi_{\varepsilon,x}(t, x, y) - \varepsilon(\partial_x^{-1} \phi_{\varepsilon,y})(t, x, y) \tag{4.29}$$

and

$$\mathrm{mKP}_\varepsilon(\phi_\varepsilon) = 0, \qquad \varepsilon = \pm 1. \tag{4.30}$$

Proof. Equation (4.29) simply follows from (4.24), (4.26), and

$$\begin{aligned} \phi_{\varepsilon,x} &= \partial_x^2 \ln[\psi_\varepsilon] = \psi_\varepsilon^{-2}[\psi_\varepsilon \psi_{\varepsilon,xx} - \psi_{\varepsilon,x}^2] \\ &= V - \tau - \phi_\varepsilon^2 + \varepsilon\, \partial_y \ln[\psi_\varepsilon] = V - \phi_\varepsilon^2 + \varepsilon(\partial_x^{-1} \phi_{\varepsilon,y}). \end{aligned} \tag{4.31}$$

In order to prove (4.30), we may assume $\tau = 0$ (otherwise replace ψ_ε by $e^{-\varepsilon\tau y}\psi_\varepsilon$). Then one computes

$$\begin{aligned} &\phi_{\varepsilon,t} - 6\phi_\varepsilon^2 \phi_{\varepsilon,x} + \phi_{\varepsilon,xxx} \\ &\quad = \psi_\varepsilon^{-1} \psi_{\varepsilon,xt} - \psi_{\varepsilon,x} \psi_{\varepsilon,t} \psi_\varepsilon^{-2} + 6\psi_\varepsilon^{-3} \psi_{\varepsilon,x}^2 \psi_{\varepsilon,xx} \\ &\qquad + \psi_\varepsilon^{-1} \psi_{\varepsilon,xxxx} - 4\psi_\varepsilon^{-2} \psi_{\varepsilon,x} \psi_{\varepsilon,xxx} - 3\psi_\varepsilon^{-2} \psi_{\varepsilon,xx}^2 \end{aligned} \tag{4.32}$$

and, using (4.26)

$$(\partial_x^{-1}(\psi_\varepsilon^{-1}\psi_{\varepsilon,x})_{yy})(x) = \lim_{x_0\to\infty}\int_{x_0}^{x} dx'\, \partial_y^2\,\partial_{x'} \ln\psi_\varepsilon$$

$$= \partial_y^2 \ln[\psi_\varepsilon(x)] - \lim_{x_0\to\infty} \partial_y^2 \ln[\psi_\varepsilon(x_0)] = \partial_y^2 \ln[\psi(x)], \tag{4.33}$$

$$(\psi_\varepsilon^{-1}\psi_{\varepsilon,x})_x(x)(\partial_x^{-1}(\psi_\varepsilon^{-1}\psi_{\varepsilon,x})_y)(x)$$

$$= (\psi_\varepsilon^{-1}\psi_{\varepsilon,x})_x(x) \lim_{x_0\to\infty}\int_{x_0}^{x} dx'\,\partial_y\,\partial_{x'} \ln[\psi_\varepsilon]$$

$$= (\psi_\varepsilon^{-1}\psi_{\varepsilon,x})_x(x)\{\partial_y \ln[\psi_\varepsilon(x)] - \lim_{x_0\to\infty} \partial_y \ln[\psi_\varepsilon(x_0)]\}$$

$$= (\psi_\varepsilon^{-3}[\psi_\varepsilon\psi_{\varepsilon,xx} - \psi_{\varepsilon,x}^2]\,\psi_y)(x). \tag{4.34}$$

Thus we have expressed every term in $\mathrm{mKP}_\varepsilon(\phi_\varepsilon)$ in terms of ψ_ε and its partial derivatives. In the final step one adds up (4.32)–(4.34) and invokes (4.24), (4.25), and $\mathrm{KP}(V)=0$ in order to arrive at (4.30) after straightforward though lengthy computations.

Clearly $\phi_\varepsilon \in C^\infty(\mathbf{R}^3)$ if $0<\psi_\varepsilon \in C^\infty(\mathbf{R}^3)$. If ψ_ε has zeros or poles at $\tilde{x}_j$, $j\in\{1, ..., \tilde{M}\}$, for some $\tilde{M}\in\mathbf{N}$ then ϕ_ε, solving (4.30), can only have first order poles at $\tilde{x}_j$ with residue ± 1 in agreement with (4.29) and the fact that V, solving $\mathrm{KP}(V)=0$, can only have second order poles with coefficient 2 at $x_j=\tilde{x}_{l(j)}$, $j\in\{1, ..., M\}$, for some $M\leqslant\tilde{M}$.

We shall use Theorem 4.5 to compute the N-soliton of the mKP_ε-equations in the next section.

Remark 4.6. One can interchange $\int_\infty^x ....$ by $\int_{-\infty}^x ....$ in (1.1)–(1.4) by simply replacing ∞ by $-\infty$ in (H.2.1), (2.2), (H.4.1), Proposition 4.2, (4.15), (4.16), (4.26), and (4.33), (4.34).

5. Soliton Solutions for the mKP_ε-Equations

In this section we derive the soliton solutions for the mKP_ε-equations [20] (see also [18]) using Theorems 3.1 and 4.5.

Lemma 5.1. *Define* (cf. (3.12))

$$\phi_{N,\varepsilon}(t,x,y) := \partial_x \ln[\psi_{N,\varepsilon}(t,x,y)], \qquad N\in\mathbf{N},\ \varepsilon=\pm 1. \tag{5.1}$$

Then (cf. (3.3) *and* (4.13))

$$\phi_{N,\varepsilon}(t,x,y) = -\kappa - \sum_{n=1}^{N} [(\kappa+q_n)^{-1}(A^+(t)\,\tilde{k}_{n,\varepsilon})(t,x,y) + \tilde{k}_{n,\varepsilon}(t,x,y)]\,e^{-q_n x}, \qquad \kappa\geqslant 0. \tag{5.2}$$

LEMMA 5.2. *Define*

$$\begin{aligned}\tilde{\tilde{k}}_{n,\varepsilon}(t,x,y) &:= -(\kappa+q_n)^{-1}A^+(t)\,\tilde{k}_n(t,x,y), \qquad 1\leqslant n\leqslant N,\\ \tilde{\tilde{k}}_\varepsilon(t,x,y) &:= (\tilde{\tilde{k}}_{\varepsilon,1}(t,x,y),\dots,\tilde{\tilde{k}}_{\varepsilon,N}(t,x,y)),\\ \tilde{\tilde{d}}_\varepsilon(t,x,y) &:= \left(-\frac{\kappa-p_1}{\kappa+q_1}\tilde{c}_{\varepsilon,1}(t,y)\,e^{-p_1x},\dots,-\frac{\kappa-p_N}{\kappa+q_N}\tilde{c}_{\varepsilon,N}(t,y)\,e^{-p_Nx}\right),\end{aligned} \tag{5.3}$$

$$\tilde{\tilde{A}}_{N,\varepsilon}(t,x,y) := \left[\frac{\kappa-p_n}{\kappa+q_n}\tilde{c}_{\varepsilon,n}(t,y)(p_n+q_m)^{-1}e^{-(p_n+q_m)x}\right]_{n,m=1}^N.$$

Then

$$(1+\tilde{\tilde{A}}_{N,\varepsilon})\,\tilde{\tilde{k}}_\varepsilon = \tilde{\tilde{d}}_\varepsilon. \tag{5.4}$$

This finally yields the soliton solutions for the mKP_ε-equations

THEOREM 5.3. *We have*

$$\phi_{N,\varepsilon}(t,x,y) = -\kappa+\partial_x \ln\left\{\frac{\det[1+\tilde{\tilde{A}}_{N,\varepsilon}(t,x,y)]}{\det[1+\tilde{A}_{N,\varepsilon}(t,x,y)]}\right\}, \qquad \kappa\geqslant 0. \tag{5.5}$$

Moreover

$$\mathrm{mKP}_\varepsilon(\phi_{N,\varepsilon})=0, \qquad \varepsilon=\pm 1 \tag{5.6}$$

iff $\lambda=\kappa^2$.

Proof. Insertion of (3.9) and its analogue for $\tilde{\tilde{k}}_{\varepsilon,n}$, $\tilde{\tilde{A}}_{N,\varepsilon}$ into (5.2) yields (5.5). Comparison with (4.25) in Theorem 4.5 no shows that we still have to fix λ in (3.1) appropriately in order to conclude (5.6). By (3.11) $\lambda=\kappa^2$ turns out to the right choice.

In contrast to the original derivation of (5.5), (5.6) in [20] which uses the machinery of τ-functions and vertex operators, our approach is close in spirit to the KdV-soliton derivations in [12, 23].

EXAMPLE 5.4. $N=1$.

$$\phi_{1,\varepsilon}(t,x,y) = -\kappa+\frac{p_1+q_1}{2}\{\tan[\alpha(t,x,y)]-\tanh[\beta(t,x,y)]\} \tag{5.7}$$

with

$$\begin{aligned}\alpha(t,x,y) &= -\frac{1}{2}(p_1+q_1)x+\frac{\varepsilon}{2}(p_1^2-q_1^2)\,y\\ &\qquad + [2(p_1^3+q_1^3)-3\kappa^2(p_1+q_1)]\,t+\frac{1}{2}\ln\left[\frac{\tilde{c}_1}{p_1+q_1}\right],\\ \beta(t,x,y) &= \alpha(t,x,y)+\frac{1}{2}\ln\left(\frac{\kappa-p_1}{\kappa+q_1}\right), \qquad \kappa\geqslant 0.\end{aligned} \tag{5.8}$$

If $\kappa \neq p_1$ then

$$\lim_{x \to \pm\infty} \phi_{1,\varepsilon}(t, x, y) = -\kappa, \tag{5.9}$$

whereas if $\kappa = p_1$ we get $\beta(t, x, y) = -\infty$ and hence

$$\lim_{x \to \infty} \phi_{1,\varepsilon}(t, x, y) = -p_1, \qquad \lim_{x \to -\infty} \phi_{1,\varepsilon}(t, x, y) = q_1, \qquad (t, y) \in \mathbf{R}^2. \tag{5.10}$$

Remark 5.5. We note that the sufficient condition (3.19) for regularity of the KP-solitons also represents a sufficient condition for the regularity of the mKP$_\varepsilon$-solitons as is evident by combining (3.19) with (5.4) and (5.5).

Remark 5.6. If one wishes to use $\int_{-\infty}^{x} \ldots$ instead of $\int_{\infty}^{x} \ldots$ in (1.1)–(1.4) (see Remark 4.6), one needs to replace $K_{\varepsilon,+}(t, x, x', y)$ by $K_{\varepsilon,-}(t, x, x', y)$ in Sections 3 and 5.

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