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A SHORT PROOF OF ZHELUDEV'S THEOREM

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ABSTRACT. We give a short proof of Zheludev's theorem that states the existence of precisely one eigenvalue in sufficiently distant spectral gaps of a Hill operator subject to certain short-range perturbations. As a by-product we simultaneously recover Rofe-Beketov's result about the finiteness of the number of eigenvalues in essential spectral gaps of the perturbed Hill operator. Our methods are operator theoretic in nature and extend to other one-dimensional systems such as perturbed periodic Dirac operators and weakly perturbed second order finite difference operators. We employ the trick of using a selfadjoint Birman-Schwinger operator (even in cases where the perturbation changes sign), a method that has already been successfully applied in different contexts and appears to have further potential in the study of point spectra in essential spectral gaps.

Our main hypothesis reads:

(I) Let $V \in L^1_{\text{loc}}(\mathbb{R})$ be real-valued and of period $a > 0$, and suppose $W \in L^1(\mathbb{R}, (1+|x|)\,dx)$ to be real-valued, $W \neq 0$ on a set of positive Lebesgue measure.

Given V, one defines the Hill operator H_0 in $L^2(\mathbb{R})$ as the form sum of the Laplacian in $L^2(\mathbb{R})$,

$$-\frac{d^2}{dx^2} \quad \text{on } H^2(\mathbb{R}), \tag{1}$$

and the operator of multiplication by V,

$$H_0 := -\frac{d^2}{dx^2} \dot{+} V. \tag{2}$$

(To be more precise, since V is not assumed to be continuous, we should define H_0 as a direct integral over reduced operators on $L^2([0, a])$, see [12, §XIII.16].) Similarly, the perturbed Hill operator H_g is defined as the form sum in $L^2(\mathbb{R})$

$$H_g := H_0 \dot{+} gW, \qquad g > 0. \tag{3}$$

Standard spectral theory [2, 10, 11, 12] then yields that

$$\sigma(H_0) = \sigma_{\text{ac}}(H_0) = \bigcup_{n\in\mathbb{N}} [E_{2(n-1)}, E_{2n-1}], \qquad -\infty < E_0 < E_1 \leq E_2 < E_3 \leq E_4 < \cdots, \tag{4}$$

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$$\begin{aligned} &\sigma_p(H_0) = \sigma_{sc}(H_0) = \varnothing, \\ &\sigma_{ess}(H_g) = \sigma_{ac}(H_g) = \sigma(H_0), \qquad \sigma_{sc}(H_g) = \varnothing. \end{aligned} \tag{5}$$

The spectral gaps of H_0 (the essential spectral gaps of H_g) are denoted by

$$\rho_0 := (-\infty, E_0), \qquad \rho_n := \begin{cases} (E_{2n-1}, E_{2n}), & E_{2n-1} < E_{2n}, \\ \varnothing, \ E_{2n-1} = E_{2n}, & n \in \mathbb{N}. \end{cases} \tag{6}$$

Moreover one has

$$\sigma_p(H_g) \subset \bigcup_{n \in \mathbb{N}_0} \rho_n \tag{7}$$

and all eigenvalues of H_g are simple. (Here $\sigma(\cdot)$, $\sigma_{ac}(\cdot)$, $\sigma_{sc}(\cdot)$, and $\sigma_p(\cdot)$ denote the spectrum, absolutely continuous spectrum, singularly continuous spectrum, and point spectrum (the set of eigenvalues) respectively.) Following the usual terminology we call ρ_n an open spectral gap whenever $\rho_n \neq \varnothing$.

The purpose of this paper is to give a short proof of the following theorem that summarizes results of Firsova, Rofe-Beketov, and Zheludev:

Theorem 1 [3, 4, 6, 13, 14, 17, 18]. *Assume Hypothesis* (I). *Then*

(i) H_g *has finitely many eigenvalues in each open gap* ρ_n, $n \geq 0$.
(ii) H_g *has at most two eigenvalues in every open gap* ρ_n *for* n *large enough.*
(iii) *If* $\int_{\mathbb{R}} dx\, W(x) \neq 0$, H_g, $g > 0$ *has precisely one eigenvalue in every open spectral gap* ρ_n *for* n *sufficiently large.*

Remark 2. Parts (i) and (ii) are due to Rofe-Beketov [13]. Part (iii), under the additional conditions $\mathrm{sgn}(W) = \text{constant}$, $W \in L^1(\mathbb{R}; (1+x^2)\,dx)$, V piecewise continuous and W bounded is due to Zheludev [17]. In [18] the condition $\mathrm{sgn}(W) = \text{constant}$ has been replaced by $\int_{\mathbb{R}} dx\, W(x) \neq 0$ but it has been left open as to whether there are one or two eigenvalues in sufficiently distant spectral gaps ρ_n. The present version of (iii) was first proved by Firsova [3, 4] (see also [6]) and Rofe-Beketov [14] on the basis of ODE methods. The case of a perturbed Hill operator on the halfline $(0, \infty)$ has also been studied in [8].

Before we give a short proof of Theorem 1 based on operator theoretic methods we need to prepare various well-known results on Hill operators and establish some further notation.

The Green's function $G_0(z, x, x')$ (the integral kernel of the resolvent $(H_0 - z)^{-1}$) reads

$$\begin{aligned} G_0(z, x, x') = {} & W(\psi_+(z, \cdot, x_0), \psi_-(z, \cdot, x_0))^{-1} \\ & \times \begin{cases} \psi_-(z, x, x_0)\psi_+(z, x', x_0), & x \leq x', \\ \psi_+(z, x, x_0)\psi_-(z, x', x_0), & x \geq x', \end{cases} \\ & x_0 \in [0, a],\ z \in \mathscr{R}. \end{aligned} \tag{8}$$

Here $W(f, g)$ denotes the Wronskian of f and g,

$$W(f, g)(x) := f(x)g'(x) - f'(x)g(x), \tag{9}$$

and $\psi_\pm$ are the Floquet solutions of H_0 defined by

$$\begin{aligned} &\psi_\pm(z, x, x_0) := c(z, x, x_0) + \phi_\pm(z, x_0)s(z, x, x_0), \qquad z \in \mathscr{R},\ x \in \mathbb{R}, \\ &\psi_\pm(z, x_0, x_0) = 1, \qquad z \in \mathscr{R}, \end{aligned} \tag{10}$$

$$\phi_{\pm}(z, x_0) := \{\Delta(z) \pm [\Delta(z)^2 - 1]^{1/2} - c(z, x_0 + a, x_0)\}s(z, x_0 + a, x_0)^{-1}, \quad z \in \mathscr{R}, \tag{11}$$

where Δ denotes the discriminant (Floquet determinant) of H_0,

$$\Delta(z) := [c(z, x_0 + a, x_0) + s'(z, x_0 + a, x_0)]/2, \qquad z \in \mathbb{C}, \tag{12}$$

and s, c is a fundamental system of distributional solutions of $H_0 f = zf$, $z \in \mathbb{C}$, with

$$\begin{aligned} &s(z, x_0, x_0) = 0, \quad s'(z, x_0, x_0) = 1, \\ &c(z, x_0, x_0) = 1, \quad c'(z, x_0, x_0) = 0, \qquad z \in \mathbb{C}. \end{aligned} \tag{13}$$

Moreover, $\psi_{\pm}$ are meromorphic functions on the two-sheeted Riemann surface $\mathscr{R}$ of $[\Delta(z)^2 - 1]^{1/2}$ obtained by joining the upper and lower rims of two copies of the cut plane $\mathbb{C}\backslash\sigma(H_0)$ (or $\mathbb{C}\backslash\overline{[\rho(H) \cap \mathbb{R}]}$, $\rho(\cdot)$ the resolvent set) in the usual (crosswise) way. $\mathscr{R}$ is assumed to be compactified if only finitely many spectral gaps of H_0 are open, otherwise $\mathscr{R}$ is noncompact. Since we do not need this Riemann surface explicitly in the following considerations we assume that a suitable choice of cuts has been made and omit further details.

We note that s, c, and Δ are entire with respect to $z \in \mathbb{C}$, and Δ and G_0 are independent of the chosen reference point $x_0 \in [0, a]$. Especially, by considering a particular open gap $\rho_n = (E_{2n-1}, E_{2n})$, $n \geq 1$, one can always choose x_0 in such a way that the zeros of $s(z, x_0+a, x_0)$ (there is precisely one simple zero in each $\overline{\rho_n}$, $n \geq 1$, they constitute the Dirichlet eigenvalues of H_0 restricted to (x_0, x_0+a)) are not at $\partial_{\rho_n} = \{E_{2n-1}, E_{2n}\}$. (This fact is relevant in (11) and will be needed later on in (20).) From now on, when considering a particular gap ρ_n, we always assume that ρ_n is open, i.e., $\rho_n \neq \varnothing$. For simplicity we shall also assume that $E_0 \geq 1$ and for notational convenience we introduce $E_{-1} = 1$ (in order not to distinguish $n = 0$ and $n \geq 1$ in the following).

We also note that

$$W(\psi_+(z, \cdot, x_0), \psi_-(z, \cdot, x_0)) = -2[\Delta(z)^2 - 1]^{1/2} s(z, x_0 + a, x_0)^{-1}, \quad z \in \mathscr{R}, \tag{14}$$

and

$$-2[\Delta(z)^2 - 1]^{1/2} G_0(z, x, x) = s(z, x + a, x), \qquad z \in \mathbb{C}, \ x \in \mathbb{R}. \tag{15}$$

Moreover, restricting z to the upper sheet $\mathscr{R}_+$ of $\mathscr{R}$ from now on, the Floquet solutions $\psi_{\pm}$ have the particular structure

$$\begin{aligned} &\psi_{\pm}(z, x, x_0) = e^{\mp\alpha(z)(x-x_0)} p_{\pm}(\alpha(z), x, x_0), \\ &p_{\pm}(\alpha(z), x + a, x_0) = p_{\pm}(\alpha(z), x, x_0), \qquad z \in \mathscr{R}_+, \ x \in \mathbb{R}, \end{aligned} \tag{16}$$

where $\alpha(z)$ is given by

$$\begin{aligned} &\alpha(z) := a^{-1} \ln\{\Delta(z) + [\Delta(z)^2 - 1]^{1/2}\}, \qquad z \in \mathscr{R}_+, \\ &\cosh[\alpha(z)a] = \Delta(z), \qquad \sinh[\alpha(z)a] = [\Delta(z)^2 - 1]^{1/2}, \end{aligned} \tag{17}$$

and the branch of $[\Delta(z)^2 - 1]^{1/2}$ on $\mathscr{R}_+$ is chosen such that

$$\psi_{\pm}(z, \cdot, x_0) \in L^2(0, \pm\infty), \qquad z \in \mathscr{R}_+\backslash\sigma(H_0). \tag{18}$$

α (resp. $\alpha - \pi i$) is positive on open gaps ρ_{2n} (resp. ρ_{2n+1}), $n \in \mathbb{N}_0$, and monotonic near E_0, E_{4n-1}, E_{4n} (resp. E_{4n-3}, E_{4n-2}), $n \in \mathbb{N}$.

We also note the asymptotic relations

$$s(\lambda, x_0 + a, x_0) \underset{\lambda\uparrow\infty}{=} \lambda^{-1/2}\sin[\lambda^{1/2}a] + O(\lambda^{-1}), \tag{19}$$

and [18]

$$\begin{aligned} &p_\pm(\alpha(E_{r(n)}), x, x_0)^2 \\ &\underset{n\to\infty}{=} \frac{1}{2}\left[1 + \frac{a^2}{4n^2\pi^2}\frac{c'(E_{r(n)}, x_0 + a, x_0)}{s(E_{r(n)}, x_0 + a, x_0)}\right] \\ &\quad \cdot \{1 - \cos[(4n\pi/a)(x - x_0) + 2\delta_{r(n)}] + O(n^{-1})\}, \end{aligned} \tag{20}$$

$$\begin{aligned} &\delta_{r(n)} := \arctan\left\{(2n\pi/a)\left|\frac{s(E_{r(n)}, x_0 + a, x_0)}{c'(E_{r(n)}, x_0 + a, x_0)}\right|^{1/2}\right\}, \\ &\qquad\qquad r(n) = 4n - 1, 4n, \end{aligned} \tag{21}$$

and similarly for the odd open gaps ρ_{2n+1}, $n \in \mathbb{N}_0$. (In order to avoid that $s(E_{r(n)}, x_0 + a, x_0) = 0$ in (20), we tacitly made use of the fact that we may choose $x_0 = x_0(n)$ appropriately without affect Δ and the Green's function G_0 in (8). Such a choice will always be assumed in the following.)

Given these preliminaries we can split the Green's function G_0 into two parts as follows. For simplicity we only consider even open gaps ρ_{2n}, $n \in \mathbb{N}_0$, in details. The analysis for odd gaps ρ_{2n+1}, $n \in \mathbb{N}_0$, is completely analogous.

$$\begin{aligned} G_0(\lambda, x, x') &= -[s(\lambda, x_0 + a, x_0)/2\sinh\alpha(\lambda)]p(\alpha(\underset{(4n-1)}{E_{4n}}), x, x_0) \\ &\quad \cdot p(\alpha(\underset{(4n-1)}{E_{4n}}), x', x_0) + R_0(\lambda, x, x'), \\ p(\alpha(\underset{(4n-1)}{E_{4n}}), x, x_0) &:= p_+(\underset{(4n-1)}{E_{4n}}), x, x_0) \\ &= p_-(\alpha(\underset{(4n-1)}{E_{4n}}), x, x_0), \qquad x \in \mathbb{R}, \end{aligned} \tag{22}$$

for $\lambda \in [E_{4n} - \varepsilon_n, E_{4n}]$ ($\lambda \in [E_{4n-1}, E_{4n-1} + \varepsilon_n]$) with $\varepsilon_n > 0$ sufficiently small, $n \in \mathbb{N}_0$. One has the bound [13, 17]

$$\begin{aligned} &|R_0(\lambda, x, x')| \le C|E_{4n-1}|^{-1/2}(1 + |x| + |x'|), \\ &\qquad \lambda \in \overline{\rho_{2n}}, \ \alpha(\lambda) \in [0, \varepsilon_n], \ x, x' \in \mathbb{R}, \end{aligned} \tag{23}$$

with C independent of $n \in \mathbb{N}_0$. Since Zheludev [17, 18] relies on the estimate (23), he is forced to assume $W \in L^1(\mathbb{R}; (1 + x^2)\,dx)$ in order to make the integral kernel $|W(x)|^{1/2}R_0(\lambda, x, x')|W(x')|^{1/2}$ to be the integral kernel of a bounded (in fact Hilbert-Schmidt) operator in $L^2(\mathbb{R})$. In order to avoid this limitation we shall employ instead a device from [1] and use a different splitting of G_0:

$$\begin{aligned} G_0(z, x, x') &= G_0(z, x_0, x_0)^{-1}G_0(z, x, x_0)G_0(z, x_0, x') \\ &\quad + G^D_{0,x_0}(z, x, x') \\ &:= \gamma(z)P_{x_0}(z, x, x') + G^D_{0,x_0}(z, x, x'), \\ \gamma(z) &:= -\{s(z, x_0 + a, x_0)/2\sinh[\alpha(z)a]\}, \end{aligned} \tag{24}$$

where $G^D_{0,x_0}(z, x, x')$ denotes the integral kernel of the resolvent of the Dirichlet operator H^D_{0,x_0} obtained from H_0 by imposing an additional Dirichlet boundary condition at x_0. Explicitly we have

(25)
$$P_{x_0}(\lambda, x, x') = \begin{Bmatrix} \psi_-(\lambda, x, x_0), & x \le x_0 \\ \psi_+(\lambda, x, x_0), & x \ge x_0 \end{Bmatrix} \begin{Bmatrix} \psi_-(\lambda, x', x_0), & x' \le x_0 \\ \psi_+(\lambda, x', x_0), & x' \ge x_0 \end{Bmatrix}, \quad \lambda \in \overline{\rho_{2n}}, \; n \in \mathbb{N}_0,$$

and, similar to (3.7) in [1],

(26)
$$|G^D_{0,x_0}(\lambda, x, x')| \le C|E_{2n-1}|^{-1/2}|x_<| \le C|E_{2n-1}|^{-1/2}|x|^{1/2}|x'|^{1/2}, \quad \lambda \in \overline{\rho_{2n}}, \; n \in \mathbb{N}_0, \; \alpha(\lambda) \ge 0 \text{ small enough},$$

where C is independent of n and

(27)
$$|x_<| := \begin{cases} 0, & x \le x_0 \le x' \text{ or } x' \le x_0 \le x, \\ \min(|x - x_0|, |x' - x_0|) & \text{otherwise.} \end{cases}$$

In order to derive (26) one separately considers the four regions $x \le x' \le x_0$, $x' \le x \le x_0$, $x_0 \le x' \le x$, $x_0 \le x \le x'$ (the cases $x \le x_0 \le x'$, $x' \le x_0 \le x$ being trivial) and uses the mean value theorem to bound

(28)
$$|p_+(\alpha(\lambda), y, x_0) - p_-(\alpha(\lambda), y, x_0)| \le D\alpha(\lambda)|y - x_0|, \quad \lambda \in \overline{\rho_{2n}}, \; \alpha(\lambda) \ge 0 \text{ small enough},$$

with D independent of $n \in \mathbb{N}_0$.

Finally, we introduce Birman-Schwinger type operators and related quantities. We distinguish three cases and again study even (open) gaps ρ_{2n}, $n \in \mathbb{N}_0$ for simplicity.

(a) $W \le 0$. We factorize

(29)
$$w := |W|^{1/2}, \qquad W = -w^2,$$

and define the Birman-Schwinger kernel by

(30)
$$k(\lambda) := -gw(H_0 - \lambda)^{-1}w, \qquad \lambda \in \rho_{2n}, \; n \in \mathbb{N}_0, \; g > 0.$$

Then the selfadjoint Birman-Schwinger kernel satisfies $k(\lambda) \in \mathscr{B}_2(L^2(\mathbb{R}))$ ($\mathscr{B}_2(\cdot)$ the set of Hilbert-Schmidt operators) and due to (24)–(26)

(31)
$$k(\lambda) = -\gamma(\lambda) g P(\lambda) - gM(\lambda), \qquad \lambda \in \rho_{2n},$$
$$\gamma(\lambda) \underset{\lambda \to E_{4n \, (4n-1)}}{=} C_{4n \, (4n-1)} |\alpha(\lambda)|^{-1}, \qquad C_{4n-1} < 0 < C_{4n}, \; n \in \mathbb{N}_0,$$

where $P(\lambda)$, $\lambda \in \overline{\rho_{2n}}$, is a positive rank one projection, $M(\lambda) \in \mathscr{B}_2(L^2(\mathbb{R}))$, $\lambda \in \overline{\rho_{2n}}$, is selfadjoint, and

(32)
$$\|M(\lambda)\| \le CE_{4n-1}^{-1/2}, \qquad \lambda \in \overline{\rho_{2n}}, \; n \in \mathbb{N}_0,$$

with C independent of n. (One can show that $\alpha(\lambda) \underset{\lambda \to E_{4n \, (4n-1)}}{=} d_{4n \, (4n-1)} |\lambda - E_{4n \, (4n-1)}|^{1/2}$ for some constants $d_{4n \, (4n-1)} > 0$.)

(b) $W \ge 0$. Introducing the factorization

(33)
$$w := |W|^{1/2}, \qquad W = w^2,$$

one defines

$$(34) \qquad \hat{k}(\lambda) := gw(H_0 - \lambda)^{-1}w\,, \qquad \lambda \in \rho_{2n}\,,\ n \in \mathbb{N}_0\,,\ g > 0\,.$$

Then $\hat{k}(\lambda) \in \mathscr{B}_2(L^2(\mathbb{R}))$ and (31) and (32) (with $\gamma \to -\gamma$) hold again.

(c) $W = W_+ - W_-$, $W_\pm > 0$ on sets of positive Lebesgue measure. If necessary, we modify $W_\pm$ such that

$$(35) \qquad \begin{aligned} W = W_+ - W_- &= \widetilde{W}_+ - \widetilde{W}_-\,, \\ \widetilde{W}_\pm \geq (1+x^2)^{-1-\varepsilon}\,, \qquad &\varepsilon > 0\,,\ \widetilde{W}_\pm \in L^1(\mathbb{R}, (1+|x|)dx)\,, \\ \tilde{w}_\pm &:= \widetilde{W}_\pm^{1/2}\,. \end{aligned}$$

Following a device of Simon [16] we define the selfadjoint Birman-Schwinger kernel by

$$(36) \qquad \begin{aligned} &\widetilde{K}(\lambda) := g\tilde{w}_+(H_0 - g\widetilde{W}_- - \lambda)^{-1}\tilde{w}_+ \in \mathscr{B}_2(L^2(\mathbb{R}))\,, \\ &\qquad\qquad \lambda \in \rho_{2n} \backslash \sigma_{\mathrm{p}}(H_0 - g\widetilde{W}_-)\,,\ n \in \mathbb{N}_0\,,\ g > 0\,. \end{aligned}$$

The fact that $\widetilde{K}(\lambda)$ is selfadjoint (as opposed to the usual choice

$$|W|^{1/2}\,\mathrm{sgn}(W)(H_0 - \lambda)^{-1}|W|^{1/2})\,,$$

even though W changes sign, will be of crucial importance below. (This trick has also been employed successfully in [7].)

Given all these preliminaries we now turn to the

Proof of Theorem 1. It suffices to treat the even open gaps ρ_{2n}, $n \in \mathbb{N}_0$.

(A) $W \leq 0$. Since

$$(37) \qquad \frac{d}{d\lambda}k(\lambda) = -gw(H_0 - \lambda)^{-2}w \leq 0\,, \qquad \lambda \in \rho_{2n}\,,\ n \in \mathbb{N}_0\,,$$

all eigenvalues of $k(\lambda)$ are monotonically decreasing with respect to $\lambda \in \rho_{2n}$. Moreover, by the Birman-Schwinger principle [12], $H_g = H_0 - g|W|$ has an eigenvalue $E^* \in \rho_n$ iff $k(E^*)$ has an eigenvalue -1 of the same multiplicity. Since E^* is necessarily simple, no eigenvalues of $k(\lambda)$ can cross in ρ_n. Because of (31), $k(\lambda)$ has precisely one eigenvalue decreasing from $+\infty$ at E_{4n-1} to $O(E_{4n-1}^{-1/2})$ near E_{4n} and one eigenvalue branch decreasing from $O(E_{4n-1}^{-1/2})$ near E_{4n-1} to $-\infty$ at E_{4n} (assuming n large enough such that $E_{4n-1} >> 1$). The remaining eigenvalues of $k(\lambda)$ in ρ_{2n} are of order $O(E_{4n-1}^{-1/2})$ for n large enough. Thus choosing n sufficiently large, precisely one eigenvalue of $K(\lambda)$ (the one diverging to $-\infty$) will cross -1. Since $k(\lambda)$ is compact, only finitely many eigenvalues of $k(\lambda)$ cross -1 in each gap ρ_n. This proves (i) and (iii) for $W \leq 0$.

Since $W \geq 0$ can be dealt with analogously, the only difference being that now $\frac{d}{d\lambda}\hat{k}(\lambda) \geq 0$ on ρ_n and hence the eigenvalues of $\hat{k}(\lambda)$ are monotonically increasing (accounting for no eigenvalue crossing of the line -1 on ρ_0 since $\hat{k}(\lambda) \geq 0$ on ρ_0), we immediately turn to the general case.

(B) $\mathrm{sgn}(W) \neq$ constant.

Throughout the rest of the proof we assume that $\lambda \in \overline{\rho_{2_n}}$ with n large enough unless otherwise stated. We start with the elementary identity

$$\begin{aligned}\widetilde{K}_-(\lambda) &:= g\tilde{w}_-(H_0 \dot{-} \widetilde{W}_- - \lambda)^{-1}\tilde{w}_- \\ &= -1 + [1 - g\tilde{w}_-(H_0 - \lambda)^{-1}\tilde{w}_-]^{-1} \\ &:= -1 + [1 + \tilde{k}_-(\lambda)]^{-1}, \qquad \lambda \in \rho_{2n} \backslash \{E_{2n}^*\},\end{aligned} \tag{38}$$

where E_{2n}^* denotes the unique eigenvalue of $H_0 - g\widetilde{W}_-$ in ρ_{2n} determined in Part A. We note that

$$\tilde{k}_-(\lambda) = -\tilde{\gamma}(\lambda) g \widetilde{P}_-(\lambda) - g\widetilde{M}_-(\lambda), \qquad \lambda \in \rho_{2n}, \tag{39}$$

where the selfadjoint rank-one operator $\widetilde{P}_-(\lambda)$, $\lambda \in \overline{\rho_{2n}}$, has the integral kernel

$$\left[\int_{\mathbb{R}} dy \widetilde{W}_-(y) P_{x_0}(\lambda, y, y)\right]^{-1} \tilde{w}_-(x) P_{x_0}(\lambda, x, x') \tilde{w}_-(x'), \qquad \lambda \in \overline{\rho_{2n}}, \tag{40}$$

$$\tilde{\gamma}(\lambda) := \gamma(\lambda) \int_{\mathbb{R}} dx \widetilde{W}_-(y) P_{x_0}(\lambda, x, x,), \qquad \lambda \in \rho_{2n}, \tag{41}$$

and $\widetilde{M}_-(\lambda) \in \mathscr{B}_2(L^2(\mathbb{R}))$, $\lambda \in \overline{\rho_{2n}}$, is selfadjoint with integral kernel

$$\tilde{w}_-(x) G^D_{0, x_0}(\lambda, x, x') \tilde{w}_-(x'), \qquad \lambda \in \overline{\rho_{2n}}. \tag{42}$$

Next we introduce the orthogonal projection

$$\widetilde{Q}_-(\lambda) := 1 - \widetilde{P}_-(\lambda), \qquad \lambda \in \overline{\rho_{2n}}, \tag{43}$$

and insert (39) into (38). Assuming $\varepsilon_n > 0$ sufficiently small, a straightforward computation (inverting 1+rank one + perturbation) then yields for the behavior of $\widetilde{K}_-(\lambda)$ near the band edges E_{4n-1}, E_{4n},

(44)

$$\begin{aligned}\widetilde{K}_-(\lambda) &= -1 - \widetilde{P}_-(\lambda) + [1 - g\widetilde{Q}_-(\lambda)\widetilde{M}_-(\lambda)\widetilde{Q}_-(\lambda)]^{-1} + O(\gamma(\lambda)^{-1}) \\ &= -\widetilde{P}_-(\lambda) + \widetilde{Q}_-(\lambda)\{[1 - g\widetilde{Q}_-(\lambda)\widetilde{M}_-(\lambda)\widetilde{Q}_-(\lambda)]^{-1} - 1\}\widetilde{Q}_-(\lambda) + O(\gamma(\lambda)^{-1}) \\ &= \begin{pmatrix} -1 & O \\ O & [1 - g\widetilde{Q}_-(\lambda)\widetilde{M}_-(\lambda)\widetilde{Q}_-(\lambda)]^{-1} - 1 \end{pmatrix} + O(\gamma(\lambda)^{-1}), \\ &\qquad \lambda \in [E_{4n-1}, E_{4n-1} + \varepsilon_n] \cup [E_{4n} - \varepsilon_n, E_{4n}],\end{aligned}$$

with respect to the decomposition $L^2(\mathbb{R}) = \widetilde{P}_-(\lambda)L^2(\mathbb{R}) \oplus \widetilde{Q}_-(\lambda)L^2(\mathbb{R})$. (Here the symbol $O(\gamma(\lambda)^{-1})$ denotes a compact operator with norm bounded by $C|\gamma(\lambda)|^{-1}$.) In particular,

$$\|\widetilde{K}_-(\lambda)\| = O(1), \qquad \lambda \in [E_{4n-1}, E_{4n-1} + \varepsilon_n] \cup [E_{4n} - \varepsilon_n, E_{4n}] \tag{45}$$

for $\varepsilon_n > 0$ sufficiently small. Noticing that

$$\widetilde{K}(\lambda) = (\tilde{w}_+/\tilde{w}_-)\widetilde{K}_-(\lambda)(\tilde{w}_+/\tilde{w}_-), \qquad \lambda \in \overline{\rho_{2n}} \backslash \{E^*\}, \tag{46}$$

we infer for the behavior of $\widetilde{K}(\lambda)$ near the band edges E_{4n-1}, E_{4n} that

$$\begin{aligned}\widetilde{K}(\lambda) = &- \widetilde{P}(\lambda) + (\tilde{w}_+/\tilde{w}_-)\widetilde{Q}_-(\lambda)\{[1 - g\widetilde{Q}_-(\lambda)\widetilde{M}_-(\lambda)\widetilde{Q}_-(\lambda)]^{-1} - 1\} \\ &\cdot \widetilde{Q}_-(\lambda)(\tilde{w}_+/\tilde{w}_-) + O(\gamma(\lambda)^{-1}) \\ := &- \widetilde{P}(\lambda) + \widetilde{L}(\lambda) + O(\gamma(\lambda)^{-1}), \\ &\qquad \lambda \in [E_{4n-1}, E_{4n-1} + \varepsilon_n] \cup [E_{4n} - \varepsilon_n, E_{4n}].\end{aligned} \tag{47}$$

Here $\widetilde{P}(\lambda)$ has the integral kernel

$$(48)\qquad \left[\int_{\mathbb{R}} dy \widetilde{W}_-(y) P_{x_0}(\lambda, y, y)\right]^{-1} \tilde{w}_+(x) P_{x_0}(\lambda, x, x') \tilde{w}_+(x'), \qquad \lambda \in \overline{\rho_{2n}},$$

and by using a geometric series expansion one checks that $\widetilde{L}(\lambda)$ indeed extends to a $\mathscr{B}_2(L^2(\mathbb{R}))$-operator for $\lambda \in [E_{4n-1}, E_{4n-1} + \varepsilon_n] \cup [E_{4n} - \varepsilon_n, E_{4n}]$ with $\varepsilon_n > 0$ sufficiently small. Moreover,

$$(49)\qquad \|\widetilde{L}(\lambda)\| \underset{n\to\infty}{=} O(E_n^{-1/2}), \qquad \lambda \in [E_{4n-1}, E_{4n-1} + \varepsilon_n] \cup [E_{4n} - \varepsilon_n, E_{4n}].$$

It remains to study $\widetilde{K}(\lambda)$ near E_{2n}^*. By (38) we have

$$(50)\qquad \begin{aligned} \widetilde{K}_-(\lambda) &= -\tilde{k}_-(\lambda)[1 + \tilde{k}_-(\lambda)]^{-1} \\ &= -\mu_1(\lambda) g[1 + \mu_1(\lambda) g]^{-1} P_1(\lambda) - g R_1(\lambda)[1 + g R_1(\lambda)]^{-1}, \\ &\qquad \lambda \in \rho_{2n} \backslash \{E_{2n}^*\}, \end{aligned}$$

where we used the spectral representation for $\tilde{k}_-(\lambda)$,

$$(51)\qquad \begin{aligned} \tilde{k}_-(\lambda) &= \mu_1(\lambda) g P_1(\lambda) + g R_1(\lambda), \\ P_1(\lambda) R_1(\lambda) &= R_1(\lambda) P_1(\lambda) = 0, \qquad \lambda \in \rho_{2n}, \end{aligned}$$

with $\mu_1(\lambda) g$ the unique eigenvalue branch of $\tilde{k}_-(\lambda)$ diverging to $-\infty$ as $\lambda \uparrow E_{4n}$, $P_1(\lambda)$ the associated rank one projection onto the corresponding eigenspace, and

$$(52)\qquad \|R_1(\lambda)\| \le C|E_{4n-1}|^{-1/2}, \qquad \lambda \in \overline{\rho_{2n}},$$

by (32). By (46), (50) yields an analogous formula for $\widetilde{K}(\lambda)$, $\lambda \in \rho_{2n} \backslash \{E_{2n}^*\}$. Given these results one can now finish the proof (similar to Part A). Since

$$(53)\qquad \frac{d}{d\lambda} \widetilde{K}(\lambda) = g \tilde{w}_+ (H_0 - g\widetilde{W}_- - \lambda)^{-2} \tilde{w}_+ \ge 0, \qquad \lambda \in \rho_n,$$

all eigenvalues of $\widetilde{K}(\lambda)$ are monotonically increasing with respect to $\lambda \in \rho_n$. By the Birman-Schwinger principle, $H_g = H_0 \dotplus gW$ has an eigenvalue $E^* \in \rho_n$ iff $\widetilde{K}(E^*)$ has an eigenvalue -1 with multiplicities preserved. Since H_g has only simple eigenvalues, again no eigenvalue crossing of $\widetilde{K}(\lambda)$ occurs in ρ_n. Due to (47), (49), (50), and its analog for $\widetilde{K}(\lambda)$, $\widetilde{K}(\lambda)$ has precisely one eigenvalue branch $\nu_1(\lambda)$ in (E_{2n}^*, E_{4n}) that is monotonically increasing from $-\infty$ at E_{2n}^* to $O(1)$ near E_{4n}, all other eigenvalues of $\widetilde{K}(\lambda)$ in (E_{2n}^*, E_{4n}) being $O(E_{4n-1}^{-1/2})$. Similarly, there is precisely one monotonically increasing eigenvalue branch $\nu_2(\lambda)$ of $\widetilde{K}(\lambda)$ in (E_{4n-1}, E_{2n}^*) that is $O(E_{4n-1}^{-1/2})$ near E_{4n-1} and $+\infty$ at E_{2n}^*, and precisely one eigenvalue branch $\nu_3(\lambda)$ that is $O(1)$ near E_{4n-1} and $O(E_{4n-1}^{-1/2})$ near E_{2n}^*, all other eigenvalues of $\widetilde{K}(\lambda)$ being $O(E_{4n-1}^{-1/2})$ throughout (E_{4n-1}, E_{2n}^*). The $O(1)$ branches near $\underset{(4n-1)}{E_{4n}}$ are of course due to $\widetilde{P}(\lambda)$ in (47) (see also (48)). Given n sufficiently large we thus have the following distinctions:

(a) If $\int_{\mathbb{R}} dx W(x) > O$, then (20), (25), and (48) imply that only $\nu_3(\lambda)$ crosses -1.

(b) If $\int_{\mathbb{R}} dx W(x) < O$, then (20), (25), and (48) imply that only $\nu_1(\lambda)$ crosses -1.

(c) If $\int_{\mathbb{R}} dx W(x) = O$, then $\nu_1(\lambda)$, $\nu_3(\lambda)$ may or may not cross -1 and we have either 0, 1, or 2 eigenvalues in ρ_{2n}.

Since $\widetilde{K}(\lambda)$ is compact, only finitely many eigenvalues can cross -1 in each gap ρ_n. This completes the proof of Theorem 1. □

Since one can replace the phrase "for n large enough" by "$g > 0$ sufficiently small" in every step of the above proof, Theorem 1 can also be viewed as a "weak-coupling" result in the following sense:

Theorem 3. *Assume Hypothesis* (I). *Then*

(i) H_g *has at most two eigenvalues in every open gap* ρ_n, $n \in \mathbb{N}_0$ *for* $g > 0$ *sufficiently small.*

(ii) *Abbreviate*

$$I(\underset{(2n-1)}{E_{2n}}) := \int_{\mathbb{R}} dx W(x) p(\alpha(\underset{(2n-1)}{E_{2n}}), x, x_0)^2, \qquad n \in \mathbb{N}_{(0)}, \tag{54}$$

and assume that $g > 0$ *is small enough. Then* H_g *has no eigenvalues in* $\rho_n = (E_{2n-1}, E_{2n})$, $n \in \mathbb{N}$ *if* $I(E_{2n-1}) < 0$ *and* $I(E_{2n}) > 0$, H_g *has precisely one eigenvalue in* ρ_n *if* $I(E_{2n-1}) < 0$ *and* $I(E_{2n}) < 0$ *or* $I(E_{2n-1}) > 0$ *and* $I(E_{2n}) > 0$, *and* H_g *has two eigenvalues in* ρ_n *if* $I(E_{2n-1}) > 0$ *and* $I(E_{2n}) < 0$. *Moreover,* H_g *has no eigenvalues in* $\rho_0 = (-\infty, E_0)$ *if* $I(E_0) > 0$ *and precisely one eigenvalue in* ρ_0 *if* $I(E_0) \le 0$.

Proof. By the paragraph preceding Theorem 3 we only need to demonstrate the last assertion in the case $I(E_0) = 0$. For that purpose we first prove that $R_0(E_0, x, x')$ (see (22) and (24)) is conditionally positive definite, i. e.,

$$\int_{\mathbb{R}^2} dx\, dx' W(x) p(\alpha(E_0), x, x_0) R_0(E_0, x, x') W(x') p(\alpha(E_0), x', x_0) > 0$$
$$\text{if } I(E_0) = \int_{\mathbb{R}} dx W(x) p(\alpha(E_0), x, x_0)^2 = 0. \tag{55}$$

(We also note that $R_0(E_0, x, x') = G^D_{0,x_0}(E_0, x, x')$.) In order to prove (55) we invoke the eigenfunction expansion associated with H_0. Let

$$\begin{aligned} f(\cdot) &= s - \lim_{R\to\infty} (2\pi)^{-1/2} \int_{|\beta|\le R} d\beta \hat{f}_\pm(\beta) \Psi_\mp(\beta, \cdot), \\ \hat{f}_\pm(\cdot) &= s - \lim_{R\to\infty} (2\pi)^{-1/2} \int_{|y|\le R} dy f(y) \Psi_\pm(\cdot, y), \qquad f \in L^2(\mathbb{R}), \end{aligned} \tag{56}$$

where

$$\begin{aligned} \Psi_\pm(\beta, x) &:= a^{1/2} \left[\int_{x_0}^{x_0+a} dy \psi_-(z(\beta), y, x_0) \psi_+(z(\beta), y, x_0) \right]^{-1/2} \\ &\quad \cdot \psi_\pm(z(\beta) x, x_0), \end{aligned} \tag{57}$$

$$\Psi_\pm(-\beta, x) = \Psi_\mp(\beta, x) = \overline{\Psi_\pm(\beta, x)}, \qquad \beta \in \mathbb{R}, \tag{58}$$

and

$$\cosh[\beta(z)a] = \Delta(z), \qquad \sinh[\beta(z)a] = [\Delta(z)^2 - 1]^{1/2} \tag{59}$$

with $\beta(z)$ an appropriate analytic continuation of arc $\sinh\{[\Delta(z)^2 - 1]^{1/2}\}$ to the Riemann surface $\mathscr{R}$ (see, e.g., [5] for more details). If $f \in L^1(\mathbb{R})$ then the integral for $\hat{f}_\pm$ in (56) becomes an ordinary Lebesgue integral over $\mathbb{R}$ since $\Psi_\pm(\beta, x)$ is uniformly bounded in $x \in \mathbb{R}$. (If $V = 0$ then $\Psi_\pm(\beta, x) = e^{\pm i\beta x}$.) We also note that

$$z(\beta) \underset{\beta \to 0}{=} E_0 + (2\mathscr{H}_0)^{-1}\beta^2 + O(\beta^4) \tag{60}$$

for some $\mathscr{H}_0 > 0$. Next we define

$$\omega(\cdot) := W(\cdot)p(\alpha(E_0), \cdot, x_0) \tag{61}$$

and compute for $\lambda < E_0$,

$$\begin{aligned} \int_{\mathbb{R}^2} dx\, dx' \omega(x) R_0(\lambda, x, x')\omega(x') &= \int_{\mathbb{R}^2} dx\, dx' \omega(x) G_0(\lambda, x, x')\omega(x') \\ &= \int_{\mathbb{R}} d\beta |\hat{\omega}_+(\beta)|^2 [z(\beta) - \lambda]^{-1}, \end{aligned} \tag{62}$$

where we used (22) together with $I(E_0) = 0$ in the first equality and

$$\begin{aligned} ((H_0 - \lambda)^{-1}\Psi_\pm(\beta(z), x) = [z(\beta) - \lambda]^{-1}\Psi_\pm(\beta(z), x), \\ z(\beta) \geq E_0, \quad \beta \in \mathbb{R}, \end{aligned} \tag{63}$$

$$(2\pi)^{-1} \int_{\mathbb{R}} dx \Psi_-(\beta, x)\Psi_+(\beta', x) = \delta(\beta - \beta') \tag{64}$$

(in the distributional sense) and the real-valuedness of ω in the second equality. Since $p(\alpha(E_0), x, x_0)$ is uniformly bounded in $x \in \mathbb{R}$ we have

$$\omega \in L^1(\mathbb{R}; (1 + |x|\, dx)$$

and hence

$$\begin{aligned} \infty &> \int_{\mathbb{R}^2} dx\, dx' \omega(x) R_0(E_0, x, x')\omega(x') \\ &= \int_{\mathbb{R}} d\beta |\hat{\omega}_+(\beta)|^2 [z(\beta) - E_0] > 0 \end{aligned} \tag{65}$$

by (23) and the monotone convergence theorem. This proves (55). It remains to go through the proof of Theorem 1 step-by-step. In fact, let E_0^* be the unique eigenvalue of $H_0 \dot{-} g\widetilde{W}_-$ in $\rho_0 = (-\infty, E_0)$ determined by Part A of the proof of Theorem 1. Since (53) remains valid for $n = 0$, and

$$(H_0 \dot{-} g\widetilde{W}_- - \lambda)^{-1} \geq 0 \quad \text{for } \lambda \in (-\infty, E_0^*), \tag{66}$$

we have

$$\widetilde{K}(\lambda) \geq 0 \quad \text{for } \lambda \in (-\infty, E_0^*). \tag{67}$$

Thus no eigenvalue branch of $\widetilde{K}(\lambda)$ can cross -1 for $\lambda < E_0^*$. In the interval (E_0^*, E_0) there is precisely one eigenvalue branch $\nu_1(\lambda)$ that is monotonically increasing from $-\infty$ at E_0^* to $O(1)$ near E_0, all other eigenvalues of $\widetilde{K}(\lambda)$ being $O(g)$ throughout $[E_0^*, E_0]$. In order to prove that $\nu_1(\lambda)$ actually crosses

-1 for $g > 0$ small enough we next consider $\widetilde{K}(E_0) = n - \lim_{\lambda \uparrow E_0} \widetilde{K}(\lambda)$. In analogy to (44) one proves

$$(68) \qquad \widetilde{K}_-(E_0) = -\widetilde{P}_-(E_0) + g\widetilde{Q}_-(E_0)\widetilde{M}_-(E_0)\widetilde{Q}_-(E_0) + O(g^2),$$

where $O(g^2)$ denotes a compact operator with norm bounded by Cg^2. This yields

$$(69) \quad \widetilde{K}(E_0) = -\widetilde{P}(E_0) + g(\tilde{w}_+/\tilde{w}_-)\widetilde{Q}_-(E_0)\widetilde{M}_-(E_0)\widetilde{Q}_-(E_0)(\tilde{w}_+/\tilde{w}_-) + O(g^2),$$

where $\widetilde{P}(E_0)$ is an orthogonal projection with integral kernel (see (22), (25) and (48))

(70)

$$\left[\int_{\mathbb{R}} dy \widetilde{W}_+(y) p(\alpha(E_0), y, x_0)^2\right]^{-1} \tilde{w}_+(x) p(\alpha(E_0), x, x_0) p(\alpha(E_0), x', x_0) \tilde{w}_+(x')$$

since $I(E_0) = 0$, and $\widetilde{M}_-$, $\widetilde{Q}_-$ have been introduced in (42), (43). A simple computation then yields

$$(71) \quad \begin{aligned} &(\tilde{w}_+ p(\alpha(E_0), \cdot, x_0), \widetilde{K}(E_0)\tilde{w}_+ p(\alpha(E_0), \cdot, x_0))/\|\tilde{w}_+ p(\alpha(E_0), \cdot x_0)\|^2 \\ &\quad = -1 + g \iint_{\mathbb{R}^2} dx\, dx' \omega(x) R_0(E_0, x, x')\omega(x') + O(g^2). \end{aligned}$$

By (55) this indeed proves that $\nu_1(\lambda)$ crosses -1 for $g > 0$ sufficiently small. □

Remark 4. To the best of our knowledge the fact that $R_0(E_0, x, x')$ is conditionally positive definite (in the sense of (55)) and that for $g > 0$ small enough H_g has precisely one eigenvalue in $\rho_0 = (-\infty, E_0)$ if $I(E_0) = 0$ appears to be new. It generalizes a corresponding result of [15] (extended in [9]) in the special case where $V \equiv 0$.

Evidently, our strategy of using a selfadjoint Birman-Schwinger kernel, even if $\mathrm{sgn}(W) \neq$ constant, extends to perturbed one-dimensional periodic Dirac operators and weakly perturbed second-order finite difference operators.

Finally, we remark that Theorem 1, in particular, implies that N-soliton solutions of the Korteweg-de Vries equation relative to a periodic background solution (i.e., relative reflectionless solutions) will in general not decay as $x \to +\infty$ and $x \to -\infty$ since by definition they are associated with the insertion of N eigenvalues in the spectral gaps of the period background Hamiltonian.

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