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L^p Norms of Non-critical Schrödinger Semigroups

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We consider Schrödinger semigroups e^{-tH}, $H=-\Delta+V$ on $\mathbb{R}^n$ with $V\sim -c|x|^{-2}$ as $|x|\to\infty$, $0<c<[(1/2)(n-2)]^2$ with $H\geqslant 0$. We determine the exact power law divergence of $\|e^{-tH}\|_{p,p}$ and of some $\|e^{-tH}\|_{q,p}$ as maps from L^p to L^q. The results are expressed most naturally in terms of the power α for which there exists a positive resonance η such that $H\eta=0$, $\eta(x)\sim|x|^{-\alpha}$.

1. INTRODUCTION

We study the asymptotic behavior as $t\to\infty$ of the L^p norms of e^{-tH} where $H=-\Delta+V$ is a non-negative Schrödinger operator on $L^2(\mathbb{R}^N)$. If H has a zero energy resonance η such that $\eta(x)\sim|x|^{-\alpha}$ as $|x|\to\infty$ we find that the L^p norm remains bounded as $t\to\infty$ if $2\leqslant p<N/\alpha$. When $N/\alpha<p<\infty$, we find the precise power law which governs the divergence of the norm as $t\to\infty$. See Theorems 11 and 15 for the precise statements of these laws. We also obtain pointwise bounds on the heat kernel which indicate the increasing influence of the resonance as $t\to+\infty$. See Theorems 16 and 18. Our results apply under a variety of somewhat different technical conditions on V and η, but are relevant when $V(x)\sim -c|x|^{-2}$ as $|x|\to\infty$ for some $c>0$. Such potentials just fail to lie in the class $L^{N/2-\varepsilon}\cap L^{N/2+\varepsilon}$ to which most earlier results concerning resonance phenomena have been restricted.

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We suppose that $H = -\Delta + V$ where V_+ is K_N^{loc} and V_- lies in the Kato class K_N. We consider various subcriticality and resonance conditions and assume throughout that $H \geqslant 0$ as an operator on $L^2(\mathbb{R}^N)$. We only consider the problem in dimension $N \geqslant 3$, since resonance behavior is different for $N = 1, 2$. It turns out that a few results are different or simpler to treat when $N \geqslant 5$, but we discuss this when it becomes relevant.

One says that V is short range if $V \in L^{N/2-\varepsilon} \cap L^{N/2+\varepsilon}$ for some $\varepsilon > 0$. Since the study of resonances and criticality is fairly extensive under this condition (see Pinchover [10], Zhao [15], and references therein), we consider potentials outside this class. One also says that V is subcritical if for all $W \in C_c^\infty$ one has $H - \varepsilon W \geqslant 0$ for small enough $\varepsilon > 0$. Of more importance in this paper is a modified notion. We say that H is strongly subcritical if $H - \varepsilon V_- \geqslant 0$ for small enough $\varepsilon > 0$. The status of this definition is clarified by the following lemma.

LEMMA 1. *If we put*

$$A = V_-^{1/2}(-\Delta + V_+)^{-1}\, V_-^{1/2}$$

then $H \geqslant 0$ *if and only if* $\|A\| \leqslant 1$. *The following are equivalent*:

(i) *V is strongly subcritical.*

(ii) $\|A\| < 1$.

If, moreover, $V_- \in L_w^{N/2}$, *then* (i), (ii) *are equivalent to*

(iii) *For all* $0 \leqslant W \in L_w^{N/2}$ *one has* $H - \varepsilon W \geqslant 0$ *for all small enough* $\varepsilon > 0$.

Proof. Assuming (i) we have

$$-\Delta + V_+ \geqslant (1 + \varepsilon)\, V_- \geqslant 0$$

and hence

$$1 \geqslant (-\Delta + V_+)^{1/2}\,(1+\varepsilon)\, V_-(-\Delta + V_+)^{-1/2} = (1+\varepsilon)\, BB^*,$$

where

$$B = (-\Delta + V_+)^{-1/2}\, V_-^{1/2}.$$

Thus

$$\|A\| = \|B^*B\| = \|BB^*\| \leqslant (1+\varepsilon)^{-1}.$$

This proves (ii). The proof of (ii) $\Rightarrow$ (i) is similar. (ii) $\Rightarrow$ (iii). If $0 \leqslant W \in L_w^{N/2}$ then there exists $c < \infty$ such that

$$W \leqslant c(-\Delta) \leqslant c(-\Delta + V_+).$$

Therefore

$$0 \leqslant (-\Delta + V_+)^{-1/2} W(-\Delta + V_+)^{-1/2} \leqslant c$$

so

$$\|(-\Delta + V_+)^{-1/2} (V_- + \varepsilon W)(-\Delta + V_+)^{-1/2}\| \leqslant \|A\| + \varepsilon c \leqslant 1$$

for small enough $\varepsilon > 0$. This implies that

$$0 \leqslant (-\Delta + V_+)^{-1/2} (V_- + \varepsilon W)(-\Delta + V_+)^{-1/2} \leqslant 1$$

so

$$V_- + \varepsilon W \leqslant -\Delta + V_+$$

and

$$H - \varepsilon W \geqslant 0.$$

(iii) $\Rightarrow$ (i) is trivial if $V_- \in L_w^{N/2}$. ∎

We comment that if V_- is short range then A is a compact operator on $L^2(\mathbb{R}^N)$. Thus strong subcriticality is equivalent to assuming that the largest eigenvalue of A is less than one, and this is also equivalent to subcriticality. We show in Example 5 that subcriticality and strong subcriticality are not equivalent in general.

Our main interest is in finding upper and lower bounds on $\|e^{-Ht}\|_{p,p}$ where $\|X\|_{q,p}$ denotes the norm of an operator X from L^p to L^q. It turns out that if $V(x) \sim -c|x|^{-2}$ as $|x| \to \infty$, where $c > 0$, then a much wider variety of phenomena can occur than are found in the short range case.

From the point of view of the potentials V, it appears that we are analyzing very special cases, albeit borderline and thus interesting ones. For if $c > 0$ and $V \sim -cx^{-\alpha}$ at infinity, then $\alpha > 2$ means short range and $\alpha < 2$ means not subcritical. However, if one thinks of Dirichlet forms and writes $\tilde{H} = UHU^{-1}$ as

$$(\varphi, \tilde{H}\varphi) = \int |\nabla\varphi|^2 \eta^2 \, dx; \qquad (\varphi, \varphi) = \int \varphi^2 \eta^2 \, dx,$$

where $H\eta = 0$, then all short range cases correspond to $\eta \sim c$ at infinity while we are looking at $\eta \sim |x|^{-\alpha}$. From this point of view it is the short range case that looks special.

Finally we close by noting why $\|\cdot\|_{q,p}$ are of considerable interest:

(1) $\|e^{-tH}\|_{\infty,1}$ bounds provide pointwise bounds on the heat kernel.

(2) $\|e^{-tH}\|_{\infty,2}$ bounds provide pointwise bounds on eigenfunctions.

Since bounds on $\|e^{itH}\|_{q,p}$ imply bounds on $\|e^{-tH}\|_{q,p}$ by interpolation, our results show that those of Journé *et al.* [5] do not extend beyond short range potentials.

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2. Results Not Depending on Resonances

We start with a result of Simon [11, 12].

PROPOSITION 2. *If $H \geqslant 0$ then there exists $c < \infty$ such that*

$$\|e^{-Ht}\|_{\infty,\infty} \leqslant c(1+t)^{N/2}$$

for all $t \geqslant 0$. Moreover, $\|e^{-Ht}\|_{q,p} < \infty$ for all $p \leqslant q$ and $t > 0$.

Under strengthened hypotheses on the negative part of V we may say more.

THEOREM 3. *If $H \geqslant 0$ and $V_- \in L^p_w$ for some $p > 2$ then*

$$\|e^{-Ht}\|_{\infty,\infty} \leqslant c(1+t)^{p/2}. \tag{2.1}$$

If also V is strongly subcritical then

$$\|e^{-Ht}\|_{\infty,\infty} \leqslant c(1+t)^{p/2-1/2}. \tag{2.2}$$

Proof. If we put $K = -\Delta + V_+$ then

$$e^{-Ht} = e^{-Kt} + \int_{s=0}^{t} e^{-Hs} V_- e^{-K(t-s)}\, ds.$$

Therefore

$$0 \leqslant e^{-Ht}1 = e^{-Kt}1 + \int_{s=0}^{t} e^{-Hs} V_- e^{-K(t-s)} 1\, ds$$

$$\leqslant 1 + \int_{s=0}^{t} e^{-Hs} V_-(1)\, ds$$

and

$$0 \leqslant e^{-H(t+1)}1 \leqslant e^{-H}1 + \int_{s=0}^{t} e^{-H(s+1)}(V_-)\, ds.$$

We deduce (by interpolation) that

$$\|e^{-H(t+1)}\|_{\infty,\infty} = \|e^{-H(s+1)}1\|_{\infty}$$

$$\leqslant c_1 + \int_{s=0}^{t} \|e^{-H(s+1)}(V_-)\|_{\infty}\, ds \tag{2.3}$$

$$\leqslant c_1 + c_2 \int_{s=0}^{t} \|e^{-H(s+1)}\|_{\infty,2}^{2/p}\, \|e^{-H(s+1)}\|_{\infty,\infty}^{1-2/p}\, \|V_-\|_{p,w}\, ds. \tag{2.4}$$

If we put

$$n(t) = \sup\{\|e^{-H(s+1)}\|_{\infty,\infty} : 0 \leqslant s \leqslant t\}$$

and use the estimate

$$\|e^{-H(s+1)}\|_{\infty,2} \leqslant \|e^{-H}\|_{\infty,2}\, \|e^{-Hs}\|_{2,2} \leqslant c_3$$

then $0 < t \leqslant T$ implies

$$\|e^{-H(t+1)}\|_{\infty,\infty} \leqslant c_1 + c_4 n(T)^{1-2/p}\, T.$$

This easily yields

$$n(T) \leqslant c_5 T n(T)^{1-2/p}$$

for all $T \geqslant 1$, and hence

$$\|e^{-Ht}\|_{\infty,\infty} \leqslant n(T) \leqslant c_6 T^{p/2}$$

as required to prove the first statement of the theorem.

If H is strongly subcritical, then $H \geqslant \alpha^2 H_0$ for some $\alpha > 0$, where $H_0 = -\Delta$. Therefore

$$\|H^{-1/2}f\|_2 \leqslant \alpha^{-1}\|{}_0^{-1/2}f\|_2 \leqslant c_1\|f\|_{2N/(N+2)}$$

for all $f \in L^2 \cap L^{2N/(N+2)}$ by a standard Sobolev inequality. Therefore

$$\begin{aligned}\|e^{-Ht}f\|_2 &= t^{-1/2}\|e^{-Ht}(Ht)^{1/2}\, H^{-1/2}f\|_2 \\ &\leqslant c_2 t^{-1/2}\|H^{-1/2}f\|_2 \\ &\leqslant c_3 t^{-1/2}\|f\|_{2N/(N+2)}.\end{aligned}$$

It follows that

$$\|e^{-Ht}\|_{2N/(N-2),2} = \|e^{-Ht}\|_{2,2N/(N+2)} \leqslant c_3 t^{-1/2} \tag{2.5}$$

and this implies that

$$\|e^{-H(t+1)}\|_{\infty,2} \leqslant c_4(1+t)^{-1/2}.$$

We now substitute this into (2.4) to get

$$\begin{aligned} n(T) &\leqslant c_1 + c_5 n(T)^{1-2/p} T^{1-1/p} \\ &\leqslant c_t n(T)^{1-2/p}\, T^{1-1/p} \cdot \end{aligned}$$

for all $T \geqslant 1$. This yields (2.2), as before. ∎

Sometimes by further interpolation, one can do better. For example, if $N=8$ and $p=2$, then we can interpolate between $\|e^{-tH}\|_{\infty,\infty} \leqslant Ct$ and $\|e^{-tH}\|_{8/3,2} < Ct^{-1/2}$ to see that $\|e^{-tH}\|_{32/9,\,8/3} \leqslant Ct^{-1/8}$ and so $\|e^{-tH}\|_{32/9,\,2} \leqslant Ct^{-5/8}$ and thus $\|e^{-tH}\|_{\infty,\infty} \leqslant Ct^{1-5/8}$ which we could iterate and improve.

THEOREM 4. *If* $H \geqslant 0$, $V_- \in L_w^{N/2}$, $N \geqslant 3$, *and* V *is strongly subcritical then*

$$\|e^{-Ht}\|_{\infty,\infty} \leqslant c(1+t)^{N/4-1/2}$$

for all $t \geqslant 0$.

Proof. If $N \geqslant 5$, this is an immediate corollary of Theorem 3, so suppose $N=3$ or 4. If $t \geqslant 0$ then

$$\begin{aligned} &\|e^{-H(t+1)}\|_{\infty,2N/(N+2)} \\ &\quad \leqslant \|e^{-H/3}\|_{\infty,2N/(N-2)} \|e^{-H(t/2+1/3)}\|_{2N/(N-2),2} \|e^{-H(t/2+1/3)}\|_{2,2N/(N+2)} \\ &\quad \leqslant c_3(1+t)^{-1}. \end{aligned}$$

We combine this with the bound

$$\begin{aligned} \|e^{-H(t+1)}\|_{\infty,\infty} \leqslant c_1 + c_3 \int_0^t & \|e^{-H(s+1)}\|_{\infty,\infty}^{1-\lambda} \\ & \times \|e^{-H(s+1)}\|_{\infty,2N/(N+2)}^{\lambda} \|V_-\|_{N/2,w}\, dt \end{aligned}$$

where $(1-\lambda)/\infty + \lambda((N+2)/2N) = 2/N$, to obtain

$$n(T) \leqslant c_1 + c_4 n(T)^{1-\lambda}\, T^{1-\lambda}$$

for all $T \geqslant 1$. This implies

$$n(T) \leqslant c_5 T^{1/\lambda - 1} = c_5 T^{(N+2)/4-1} = c_5 T^{N/4-1/2}. \quad ∎$$

3. Resonance Eigenfunctions and Upper Bounds

If $H=-\Delta+V\geqslant 0$ and $V\in K_N^{loc}$ then there always exists at least one positive continuous function η on $\mathbb{R}^N$ such that $H\eta=0$ in the sense of distributions. This function need not be unique or bounded, but in the applications we have in mind, it is both. Since this problem has already been studied in detail by Murata [6–8], we simplify our treatment by *defining* a resonance to be a positive continuous bounded function η on $\mathbb{R}^N$ such that $e^{-Ht}\eta=\eta$ for all $t\geqslant 0$, where e^{-Ht} is the semigroup on L^∞ consistent with the usual self-adjoint semigroup on L^2. We do not exclude the possibility that $\eta\in L^2$ and so is a proper bound state. We start with an example discussed by Murata [6–8].

Example 5. Let $N\geqslant 3$ and put $H=-\Delta+V$ where

$$V(x)=\begin{cases}0 & \text{if } |x|\leqslant 1\\ -c/|x|^2 & \text{if } |x|>1\end{cases}$$

so that $H\geqslant 0$ if and only if $c\leqslant((N-2)/2)^2$. The same condition ensures that V is subcritical, but for strong subcriticality we need $c<((N-2)/2)^2$. Assuming $0<c<((N-2)/2)^2$, there are two radial solutions of $H\eta=0$ on $\{x\colon |x|>1\}$ namely $|x|^{-\alpha}$ and $|x|^{-\alpha'}$ where

$$0<\alpha=\frac{N-2}{2}-\sqrt{\left(\frac{N-2}{2}\right)^2-c}<\frac{N-2}{2}$$

$$\frac{N-2}{2}<\alpha'=\frac{N-2}{2}+\sqrt{\left(\frac{N-2}{2}\right)^2-c}<N-2.$$

There is one positive radial solution η of $H\eta=0$ on the whole of $\mathbb{R}^N$ and this satisfies $\eta(x)\sim|x|^{-\alpha}$ as $|x|\to\infty$.

We say that a resonance $\eta>0$ is slowly varying with index $\alpha\geqslant 0$ if

$$\frac{\eta(x)}{\eta(y)}\leqslant c_1(1+|x-y|)^\alpha \tag{3.1}$$

for all $x, y\in\mathbb{R}^N$. We say that η is regularly varying if

$$c_2^{-1}\leqslant\frac{\eta(x)}{\eta(y)}\leqslant c_2$$

whenever

$$|x-y|<\tfrac{1}{2}(1+|y|).$$

Both of these conditions are satisfied if

$$c_3^{-1}(1+|x|)^{-\alpha} \leqslant \eta(x) \leqslant c_3(1+|x|)^{-\alpha} \tag{3.2}$$

for all $x \in \mathbb{R}^N$.

LEMMA 6. *If $H = -\Delta + V \geqslant 0$ has a slowly varying resonance η with index $\alpha < (N-2)/2$ then V is subcritical. If (3.2) holds for some $\alpha < (N-2)/2$ then V is strongly subcritical.*

Proof. We define the unitary operator U from $L^2(\mathbb{R}^N, \eta^2\,dx)$ to $L^2(\mathbb{R}^N, dx)$ by $Uf = \eta f$ and put $\tilde{H} = U^{-1}HU$ so that $\tilde{H}$ has the quadratic form

$$\tilde{Q}(f) = \int |\nabla f|^2\, \eta^2\, dx.$$

We next observe from (3.1) that

$$\eta(x) \geqslant c_2(1+|x|)^{-\alpha}$$

for all $x \in \mathbb{R}^N$, so $\eta > c_3\eta_1$, where η_1 is the zero energy resonance of the operator H_1 of Example 5, with $c = \alpha(N-2-\alpha)$. Since the potential V_1 of Example 5 is subcritical, given $W \in C_c^\infty$ there exists $\varepsilon > 0$ such that

$$\begin{aligned}\tilde{Q}(f) &\geqslant c_3^2 \int |\nabla f|^2\, \eta_1^2\, dx \\ &\geqslant c_3^2 \int |f|^2\, \varepsilon W \eta_1^2\, dx \\ &= c_3^2 \int |f|^2\, \varepsilon \frac{W\eta_1^2}{\eta_1^2}\, \eta^2\, dx.\end{aligned}$$

Therefore

$$H \geqslant \varepsilon \frac{W\eta_1^2}{\eta^2}$$

and H is subcritical.

If $0 < \alpha < ((N-2)/2)^2$ then (3.2) implies

$$c_3^{-1}\eta_1 \leqslant \eta \leqslant c_3\eta_1.$$

If $W \in L_w^{N/2}$ then by the strong subcriticality of H_1 there exists $\varepsilon > 0$ such that

$$\int |\nabla f|^2\, \eta_1^2\, dx \geqslant \varepsilon \int |\nabla f|^2\, W\eta_1^2\, dx.$$

Therefore

$$\int |\nabla f|^2 \eta^2 \, dx \geqslant \varepsilon c_3^{-4} \int |\nabla f|^2 W \eta^2 \, dx$$

and

$$H \geqslant \varepsilon c_3^{-4} W. \quad \blacksquare$$

We now define $K(t, x, y)$ to be the heat kernel of e^{-Ht}. In our next proposition one can replace 5 by $(4+\varepsilon)$ for any $\varepsilon > 0$.

PROPOSITION 7. *For any H, one has*

$$0 \leqslant K(t, x, y) \leqslant ct^{-N/2} \exp\left[-\frac{|x-y|^2}{5t} \right]$$

for all $0 < t < 1$. If $H \geqslant 0$ has a slowly varying resonance one also has

$$0 \leqslant K(t, x, y) \leqslant c \exp\left[-\frac{|x-y|^2}{5t} \right]$$

for all $t \geqslant 1$.

The first statement is taken from [13] and the second fom [3].

THEOREM 8. *If $H \geqslant 0$ has a slowly varying resonance η with index α, then*

$$\|e^{-Ht}\|_{\infty,\infty} \leqslant c_\varepsilon (1+t)^{\alpha/2+\varepsilon}$$

for all $t \geqslant 0$.

Note. Apart from the possible elimination of $\varepsilon > 0$, it follows from Theorem 14 below that this is the strongest result possible under the stated hypothesis.

Proof. We put

$$(e^{-Ht}1)(x) = \int K(t, x, y) \, dy = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \int_{|x-y| \leqslant R} K(t, x, y) \, dy \\ &\leqslant \int_{|x-y| \leqslant R} K(t, x, y) \frac{\eta(y)}{\eta(x)} c(1 + |x-y|)^\alpha \, dy \\ &\leqslant c(1+R)^\alpha \eta(x)^{-1} \int_{|x-y| \leqslant R} K(t, x, y) \eta(y) \, dy \\ &\leqslant c(1+R)^\alpha. \end{aligned}$$

Assuming $t \geqslant 1$ we use Proposition 7 to obtain

$$I_2 = \int_{|x-y|>R} K(t, x, y)\, dy \leqslant c \int_R^\infty e^{-r^2/5t} r^{N-1}\, dr$$

$$= ct^{N/2} \int_{Rt^{-1/2}}^\infty e^{-s^2/5} s^{N-1}\, ds.$$

We now put ron $R = t^{1/2+\varepsilon/\alpha}$ to obtain

$$0 \leqslant (e^{-Ht}1)(x) \leqslant c(1 + t^{1/2+\varepsilon/\alpha})^\alpha$$

$$+ ct^{N/2} \int_{t^{\varepsilon/\alpha}}^\infty e^{-s^2/5} s^{N-1}\, ds$$

$$= c(1 + t^{1/2+\varepsilon/\alpha})^\alpha + o(1) \leqslant c'(1+t)^{\alpha/2+\varepsilon}.$$

The proof is completed by using the identity

$$\|e^{-Ht}\|_{\infty,\infty} = \|e^{-Ht}1\|_\infty. \quad \blacksquare$$

If the resonance η does not lie in L^2 then the "projection"

$$Pf = \langle f, \eta \rangle \eta$$

is not a bounded operator on any L^p space. Nevertheless the presence of η increasingly dominates the heat kernel as t increases. The following upper bound on the heat kernel gives an impression of its effect. See also Theorem 18 for a lower bound.

THEOREM 9. *If $H \geqslant 0$ has a regularly varying resonance then*

$$0 \leqslant K(t, x, y) \leqslant c_\delta a(t, x)\, a(t, y) \exp\left[-\frac{|x-y|^2}{(4+\delta)t} \right]$$

for all $\delta > 0$ and $t > 0$, where

$$a(t, x) = \max\{(1+|x|)^{-N/2}, t^{-N/4}\}.$$

Proof. We first note that if we transfer the problem to the weighted space $L^2(\mathbb{R}^N, \eta^2\, dx)$ in the usual way, the new heat kernel $\tilde{K}$ is related to K by

$$\tilde{K}(t, x, y) = \frac{K(t, x, y)}{\eta(x)\, \eta(y)}. \tag{3.3}$$

We now apply the methods of Theorem 3 of [3] taking the bounded geometry radius $r(x)$ at $x \in \mathbb{R}^N$ to be

$$r(x) = \tfrac{1}{2}(1 + |x|).$$

We obtain

$$0 \leqslant \tilde{K}(t, x, y) \leqslant c|B(x, s_1^{1/2})|^{-1/2} |B(y, s_2^{1/2})|^{-1/2} \cdot \exp\left[-\frac{(|x-y| - s_1^{1/2} - s_2^{1/2})_+^2}{4(t+s_1+s_2)}\right] \tag{3.4}$$

provided $0 < s_1 < t$, $2s_1^{1/2} < r(x)$, $2s_2^{1/2} < r(y)$ where, according to [3]

$$|B(x, r)| = \int_{|y-x|<r} \eta(y)^2 \, dy.$$

By the assumption that η is regularly varying, we see that

$$c^{-1}\eta(x)^2 r^N \leqslant |B(x, r)| \leqslant c\eta(x)^2 r^N \tag{3.5}$$

for all $r < r(x)$. Combining (3.3), (3.4), and (3.5) we obtain

$$0 \leqslant K(t, x, y) \leqslant c_1 s_1^{-N/4} s_2^{-N/4} \exp\left[-\frac{(|x-y| - s_1^{1/2} - s_2^{1/2})_+^2}{4(t+s_1+s_2)}\right] \tag{3.6}$$

under the stated conditions on s_1, s_2. We now put

$$s_1 = \min\left\{\frac{r(x)^2}{4}, \varepsilon^2 t\right\}$$

$$s_2 = \min\left\{\frac{r(y)^2}{4}, \varepsilon^2 t\right\}$$

and estimate the exponential factor in (3.6). If $|x-y| \geqslant t^{1/2}$ then

$$\frac{|x-y|^2 (1-2\varepsilon)^2}{4t(1+2\varepsilon^2)} \leqslant \frac{(|x-y| - s_1^{1/2} - s_2^{1/2})_+^2}{4(t+s_1+s_2)} \leqslant \frac{|x-y|^2}{4t}$$

and if $|x-y| < t^{1/2}$ then

$$0 \leqslant \frac{(|x-y| - s_1^{1/2} - s_2^{1/2})_+^2}{4(t+s_1+s_2)} \leqslant \frac{|x-y|}{4t} \leqslant 1.$$

In both cases the theorem follows upon putting

$$1 + \delta = \frac{1+2\varepsilon^2}{(1-2\varepsilon)^2}. \quad \blacksquare$$

An immediate corollary of this theorem is that

$$\lim_{t \to \infty} K(t, x, x) \leqslant c(1 + |x|)^{-N}. \tag{3.7}$$

However, it follows directly from the spectral theorem that

$$\lim_{t \to \infty} K(t, x, x) = 0$$

unless $\eta \in L^2(\mathbb{R}^N)$, in which case

$$\lim_{t \to \infty} K(t, x, x) = \eta(x)^2$$

assuming η is normalized. Note that the RHS of (3.7) just fails to lie in $L^1(\mathbb{R}^N)$.

4. Central Potentials

In this section we obtain some essential improvements on the upper bounds of $\|e^{-Ht}\|_{p,p}$ as $t \to \infty$ under the assumption that the potential V is central, or approximately so. We start with a result of Murata [6–8].

Proposition 10. *Let $H = -\Delta + V$ where $V \leqslant 0$ is a subcritical potential which is radial and increasing with $V(r) \sim -c/r^2$ as $r \to \infty$, where $0 < c < (N-2)/2)^2$. Then there is a unique positive radial resonance η of H, and this satisfies*

$$\eta(r) \sim r^{-\alpha}$$

as $r \to \infty$ where

$$\alpha = \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 - c}.$$

In our terms the resonance η is both slowly and regularly varying, since it satisfies (3.2). Moreover V is strongly subcritical by Lemma 6.

We say that an operator A is bounded on L^p_w if

$$\|Af\|_{p,w} \leqslant c\|f\|_{p,w}$$

for all $f \in L^p_w$ where

$$\|f\|_{p,w} = \sup\{|\{x: f(x) \geqslant \lambda\lambda\}|^{1/p}\,\lambda: 0 < \lambda < \infty\}.$$

Note that this is not a norm (but it is equivalent to a norm [14]).

THEOREM 11. *Under the conditions of Proposition* 10, e^{-Ht} *are uniformly bounded on* $L_w^{N/\alpha}$ *for* $0 \leqslant t \leqslant \infty$. *Hence*

$$\|e^{-Ht}\|_{p,p} \leqslant c_p < \infty$$

for all $t > 0$ *and* $2 \leqslant p < N/\alpha$. *Also*

$$\|e^{-Ht}\|_{p,p} \leqslant c_{p,\varepsilon}(1+t)^{\alpha/2 - N/2p + \varepsilon}$$

for all $t > 0$, $\varepsilon > 0$, *and* $N/\alpha \leqslant p \leqslant \infty$.

Proof. Let $*$ denote the (non-linear) operator of symmetric decreasing rearrangement [1]. If $0 \leqslant f \in L_w^{N/\alpha}$ then

$$\|f^*\|_{N/\alpha, w} = \|f\|_{N/\alpha, w}$$

and

$$0 \leqslant f^*(x) \leqslant c\|f\|_{N/\alpha, w}|x|^{-\alpha}.$$

The Brascamp–Lieb–Luttinger theorem [1] implies that

$$\begin{aligned} 0 &\leqslant (e^{-Ht}f)^* \leqslant e^{-Ht}(f^*) \\ &\leqslant c\|f\|_{N/\alpha, w} e^{-Ht}(|x|^{-\alpha}) \leqslant c\|f\|_{N/\alpha, w} e^{-H(t-1)} g, \end{aligned}$$

where

$$0 \leqslant g = e^{-H}(|x|^{-\alpha}) \in L_w^{N/\alpha} \cap L^\infty$$

is a bounded symmetric decreasing function. Hence

$$0 \leqslant g(x) \leqslant c_1 \eta(x)$$

for some $c_1 < \infty$ and all $x \in \mathbb{R}^N$. We deduce that

$$0 \leqslant (e^{-Ht}f)^*(x) \leqslant c\|f\|_{N/\alpha, w} c_1 e^{-H(t-1)}\eta = c\|f\|_{N/\alpha, w} c_1 \eta$$

and this implies that

$$\begin{aligned} \|e^{-Ht}f\|_{N/\alpha, w} &= \|(e^{-Ht}f)^*\|_{N/\alpha, w} \\ &\leqslant cc_1\|f\|_{N/\alpha, w}\|\eta\|_{N/\alpha, w} \leqslant c_2\|f\|_{N/\alpha, w}. \end{aligned}$$

The second statement of the theorem follows by interpolation between the bound of Theorem 8 and

$$\|e^{-Ht}\|_{N/\alpha - \delta, N/\alpha - \delta} \leqslant c_{N/\alpha - \delta}. \quad \blacksquare$$

Remark 12. One can extend this result to some non-central potentials if the spherical rearrangement of the potential yields a subcritical one.

5. Lower Bounds on Heat Kernel Norms

In this section we show that the upper bounds on $\|e^{-Ht}\|_{p,p}$ given by Theorem 11 are essentially sharp. We obtain corresponding lower bounds on the norms by adapting some estimates of Nash [9, 4, 2]. In this section it is not necessary for V to be a short range perturbation of a central potential. [From Theorem 18 we can also obtain lower bounds similar to those in Theorem 14; see Section 6].

Throughout this section we assume that $H \geqslant 0$, that V is strongly subcritical, and that H has a resonance $\eta \geqslant 0$ in $L_w^{N/\alpha}$ which is slowly varying with index α where $0 < \alpha < (N-2)/2$.

Lemma 13. *Let $2 \leqslant p \leqslant \infty$ and let*

$$n_p(t) = \sup\{\|e^{-Hs}\|_{p,p} : 0 \leqslant s \leqslant t\}.$$

Then

$$\|e^{-Ht}\|_{p,2} \leqslant ct^{-\gamma} n_p(t)$$

for all $t > 0$, where $\gamma = N(p-2)/4p$.

Proof. We first note that if $1/p + 1/p' = 1$ then

$$\|f\|_2 \leqslant \|f\|_{2N/(N-2)}^{2} \|f\|_{p'}^{1-\lambda},$$

where

$$\frac{1}{2} = \lambda \frac{N-2}{2N} + \frac{1+\lambda}{p'}$$

or equivalently

$$\lambda^{-1} - 1 = \frac{2p}{N(p-2)} = \frac{1}{2\gamma}.$$

Putting this into the Sobolev inequality

$$\|f\|_{2N/(N-2)}^2 \leqslant cQ(f)$$

we get the Nash-type inequality

$$\|f\|_2^{2/\lambda} \leqslant cQ(f) \, \|f\|_{p'}^{2(1-\lambda)/\lambda}.$$

If $u_t = \|f_t\|_2^2$ where $f_t = e^{-Ht}f$, we deduce that

$$\frac{d}{dt}(u_t^{1-1/\lambda}) = \left(1 - \frac{1}{\lambda}\right) u^{-1/\lambda} \frac{du}{dt} \geqslant c_1 \|f_t\|_{p'}^{-2(1-\lambda)/\lambda}.$$

Integrating this we obtain

$$\|f_t\|_2^{2-2/\lambda} \geqslant \int_0^t \frac{d}{ds}(u_s^{1-1/\lambda})\, ds$$
$$\geqslant c_1 t n_p(t)^{-2(1-\lambda)/\lambda} \|f\|_{p'}^{-2(1-\lambda)/\lambda}$$

and hence

$$\|f_t\|_2 \leqslant c_2 t^{-\lambda/2(1-\lambda)} \|f\|_{p'} n_p(t)$$
$$= c_2 t^{-\gamma} \|f\|_{p'} n_p(t).$$

The inequality

$$\|e^{-Ht}\|_{2,p'} \leqslant c t^{-\gamma}$$

which follows yields the lemma by taking adjoints. ∎

THEOREM 14. *For any $\varepsilon > 0$ and $0 < t < \infty$ we have*

$$c_{1,\varepsilon}(1+t)^{\alpha/2-\varepsilon} \leqslant \|e^{-Ht}\|_{\infty,\infty} \leqslant c_{2,\varepsilon}(1+t)^{\alpha/2+\varepsilon} \tag{5.1}$$

and

$$c_{3,\varepsilon} t^{-N/4}(1+t)^{\alpha/2-\varepsilon} \leqslant \|e^{-Ht}\|_{\infty,2} \leqslant c_{4,\varepsilon} t^{-N/4}(1+t)^{\alpha/2+\varepsilon}. \tag{5.2}$$

Proof. The upper bound of (5.1) was proved in Theorem 8. Lemma 13 with $p = \infty$ now yields the upper bound of (5.2). Assuming $t \geqslant 1$ we obtain by interpolation

$$\|\eta\|_\infty = \|e^{-Ht}\eta\|_\infty \leqslant \|e^{-Ht}\|_{\infty,\infty}^{1-\lambda} \|e^{-Ht}\|_{\infty,2}^{\lambda} \|\eta\|_{N/\alpha,w},$$

where

$$\frac{\alpha}{N} = \frac{\lambda}{2} + \frac{1-\lambda}{\infty}.$$

Therefore

$$\|\eta\|_\infty \leqslant \|e^{-Ht}\|_{\infty,\infty}^{1-\lambda} c_1 t^{(\alpha/2-N/4+\varepsilon)\lambda}$$

and

$$\|e^{-Ht}\|_{\infty,\infty} \geqslant c_2 t^{-(\alpha/2-N/4+\varepsilon)(\lambda/(1-\lambda))} = c_2 t^{-\alpha/2-\varepsilon'}.$$

The lower bound on $\|e^{-Ht}\|_{\infty,2}$ for $t \geqslant 1$ follows in a similar manner.

Finally the lower bounds for $0<t\leqslant 1$ follow on general grounds without any of the conditions on V of this section [13, 2]. ▌

We finally give a converse to Theorem 11, admittedly under somewhat different conditions.

THEOREM 15. *If* $t\geqslant 0$ *and* $2\leqslant p\leqslant N/\alpha$, *then*

$$\|e^{-Ht}\|_{p,p}\geqslant 1.$$

If $t\geqslant 0$, $\varepsilon>0$, *and* $N/\alpha<p<\infty$ *then*

$$\|e^{-Ht}\|_{p,p}\geqslant c_{p,\varepsilon}(1+t)^{\alpha/2-N/2p-\varepsilon}.$$

Proof. The first inequality is a simple interpolation:

$$1=\|e^{-Ht}\|_{2,2}\leqslant\|e^{-Ht}\|_{p,p}^{\lambda}\|e^{-Ht}\|_{p',p'}^{1-\lambda}=\|e^{-Ht}\|_{p,p},$$

where

$$\frac{1}{2}=\frac{\lambda}{p}+\frac{1-\lambda}{p'}.$$

In order to prove the second inequality we combine Theorem 11 and Lemma 13 to get

$$\|e^{-Ht}\|_{p,2}\leqslant c_1 t^{-N(p-2)/4p}(1+t)^{(p\alpha-N)/2p+\varepsilon}\leqslant c_2 t^{(2\alpha-N)/4+\varepsilon}$$

if $t\geqslant 1$. Also

$$\|\eta\|_p\leqslant\|e^{-Ht}\|_{p,2}^{\lambda}\|e^{-Ht}\|_{p,p}^{1-\lambda}\|\eta\|_{N/\alpha,w},$$

where

$$\frac{\alpha}{N}=\frac{\lambda}{2}+\frac{1-\lambda}{p}$$

or equivalently

$$\frac{1}{\lambda}-1=\frac{p(N-2\alpha)}{2(p\alpha-N)}.$$

This implies

$$\|e^{-Ht}\|_{p,p}\geqslant c_3\|e^{-Ht}\|_{p,2}^{-(\lambda^{-1}-1)}\geqslant c_4 t^{\mu},$$

where

$$\begin{aligned}\mu &= -\left(\frac{2\alpha-N}{4}+\varepsilon\right)\frac{2(p\alpha-N)}{p(N-2\alpha)}\\ &=\frac{p\alpha-N}{2p}-\varepsilon'\\ &=\frac{\alpha}{2}-\frac{N}{2p}-\varepsilon'.\end{aligned}$$

If $0<t<1$, then the second inequality of the theorem follows from the first, which is actually valid for all p.

6. Further Pointwise Heat Kernel Bounds

In this section we obtain pointwise upper and lower bounds on heat kernels which go beyond Proposition 7 and Theorem 9.

THEOREM 16. *Suppose that V is strongly subcritical and that $H\geqslant 0$ has a resonance $\eta>0$ which is slowly varying with index $\alpha<(N-2)/2$. Then for any $\varepsilon>0$ and $t\geqslant 1$ we have*

$$0\leqslant K(t,x,y)\leqslant c_\varepsilon t^{\alpha-N/2+\varepsilon}\exp[-|x-y|^2/a_\varepsilon t]$$

for some positive constants a_ε, c_ε.

Proof. By combining Theorem 8 with Lemma 13 for $p=\infty$, we obtain for $t\geqslant 1$

$$\|e^{-Ht}\|_{\infty,2}\leqslant c_{1,\alpha}t^{\alpha-N/4+\varepsilon/2}.$$

This implies that

$$\|e^{-Ht}\|_{\infty,1}\leqslant \|e^{-Ht/2}\|^2_{\infty,2}\leqslant c_{2,\varepsilon}t^{\alpha-N/2+\varepsilon}.$$

We then note that

$$\begin{aligned}\|e^{-Ht}\|_{\infty,1}&=\sup\{K(t,x,y): x,y\in\mathbb{R}^N\}\\ &=\sup\{K(t,x,x): x\in\mathbb{R}^N\}\end{aligned}\tag{6.1}$$

to obtain

$$0\leqslant K(t,x,y)\leqslant c_{2,\varepsilon}t^{-\alpha-N/2+\varepsilon}.$$

We also have the upper bound

$$0 \leqslant K(t, x, y) \leqslant c \exp\left[-\frac{|x-y|^2}{5t}\right]$$

by Proposition 7. Therefore

$$0 \leqslant K(t, x, y) \leqslant \{c_{2,\varepsilon/2} t^{\alpha - N/2 + \varepsilon/2}\}^{\lambda} \left\{c \exp\left[-\frac{|x-y|}{5t}\right]\right\}^{1-\lambda}$$

for any $0 < \lambda < 1$. The theorem follows by taking λ close enough to 1, specifically

$$\lambda = \frac{N/2 - \alpha - \varepsilon}{N/2 - \alpha - \varepsilon/2}.$$

COROLLARY 17. *Under the same conditions the Green function G of H^{-1} satisfies*

$$0 \leqslant G(x, y) \leqslant c_1 |x-y|^{2-N} + c_2 |x-y|^{2+2\alpha-N+\varepsilon}.$$

Proof. We have

$$\begin{aligned} G(x, y) &= \int_0^\infty K(t, x, y)\, dt \\ &= \int_0^1 K(t, x, y)\, dt + \int_1^\infty K(t, x, y)\, dt. \end{aligned}$$

The two integrals are estimated using Proposition 7 and Theorem 16, respectively.

Our final theorem is of a rather limited character but indicates that the power of t in Theorem 16 is the correct one.

THEOREM 18. *Let V be a subcritical central potential whose resonance $\eta > 0$ satisfies*

$$\lim_{r \to \infty} r^\alpha \eta(r) = a$$

$$\lim_{r \to \infty} \frac{r\eta'(r)}{\eta(r)} = -\alpha,$$

where $a > 0$, $0 < \alpha < (N-2)/2$. Then for large enough $t > 0$ and all x with $|x| < t^{1/2}$ we have

$$K(t, x, x) \geqslant c_1 \eta(x)^2\, t^{\alpha - N/2}.$$

In particular

$$\|e^{-Ht}\|_{\infty,1} \geqslant c_2 t^{\alpha - N/2}.$$

Remark. Note that the proof of Theorem 18 is independent of the proof of Theorem 14, so it provides an alternate proof of the lower bound half of (5.2) since

$$\|e^{-2Ht}\|_{1,\infty} \leqslant \|e^{-Ht}\|_{1,2} \|e^{-Ht}\|_{2,\infty} = \|e^{-Ht}\|_{2,\infty}^2.$$

Proof. Let $\varphi > 0$ be the ground state eigenfunction of $-\Delta$ on $\{x: |x| < 1\}$ normalized by $\|\varphi\|_2 = 1$ and subject to Dirichlet boundary conditions. Let $E > 0$ be the corresponding eigenvalue. We define a spherical symmetric function f on $\{x: |x| < R\theta_R\}$ by

$$f(r) = \begin{cases} \eta(r) & \text{if } \ 0 \leqslant r \leqslant R \\ \gamma_R \varphi(r\theta_R/R) & \text{if } \ R < r < R/\theta_R, \end{cases}$$

where γ_R and θ_R are determined by the conditions

$$\begin{gathered} \eta(R) = \gamma_R \varphi(\theta_R) \\ \eta'(R) = \gamma_R \theta_R R^{-1} \varphi'(\theta_R) \\ 0 < \theta_R < 1. \end{gathered}$$

Eliminating γ_R, one of the conditions is

$$\frac{\theta_R \varphi'(\theta_R)}{\varphi(\theta_R)} = \frac{R\eta'(R)}{\eta(R)}.$$

As θ_R increases from 0 to 1, the LHS decreases from 0 to $-\infty$, so far large enough R this equation has a unique solution. Moreover

$$\lim_{R \to \infty} \theta_R = \theta,$$

where $0 < \theta < 1$ and

$$\theta \varphi'(\theta)/\varphi(\theta) = -\alpha.$$

The first consistency condition now yields

$$\gamma_R = \eta(R)/\varphi(\theta_R)$$

so

$$\lim_{r \to \infty} R^\alpha \gamma_R = a/\varphi(\theta).$$

The function f is the ground state of the Schrödinger operator

$$H_1 = -\Delta + V_1$$

on $\{x: |x| < R/\varphi_R\}$ subject to Dirichlet boundary conditions where

$$V_1(x) = \begin{cases} V(x) & \text{if } |x| \leqslant R \\ -E\theta_R^2/R^2 & \text{if } R < |x| < R/\theta_R. \end{cases}$$

Since

$$V_1 + E\theta_R^2/R^2 \geqslant V$$

within $\{x: |x| < R/\theta_R\}$ one has

$$\begin{aligned} K(t, x, x\} &\geqslant K_1(t, x, x) \exp[-E\theta_R^2 t/R^2] \\ &\geqslant \frac{f(x)^2}{\|f\|_2^2} \exp[-E\theta_R t/R^2] \end{aligned}$$

whenever $|x| < R/\theta_R$. Now

$$\|f\|_2^2 = c_1 \int_0^R \eta(r)^2 r^{N-1}\, dr + c_1 \int_R^{R/\theta_R} f(r)^2 r^{N-1}\, dr \sim R^{N-2\alpha}$$

as $R \to \infty$. If we put $R = t^{1/2}$ then for large enough t and all $|x| < t^{1/2}$ we obtain the first statement of the theorem. The second follows from the identity (6.1).

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