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Best Constants in Some Operator Smoothness Estimates

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We provide a short proof of the inequality (cf. Ben-Artzi and Klainerman [Regularity and decay of evolution equations, preprint] and Kato and Yajima [*Rev. Math. Phys.* **1** (1989), 481–496])

$$\int_{-\infty}^{\infty} \|(1+x^2)^{-1/2}(1-\Delta)^{1/4} e^{it\Delta}u\|^2\,dt \leqslant C\,\|u\|^2$$

with explicit (essentially exact) values for C. © 1992 Academic Press, Inc.

Recently Ben-Artzi and Klainerman [1] and Kato and Yajima [5] have focused interest on the estimate

$$\int_{-\infty}^{\infty} \|(1+x^2)^{-1/2}(1-\Delta)^{1/4} e^{it\Delta}u\|^2\,dt \leqslant C\,\|u\|^2, \tag{1}$$

a result that implies and extends a number of results in harmonic analysis, e.g. [6, 8] (see [1]). Our goal here is to provide an elementary proof with explicit constants. Indeed we prove that if $n \geqslant 3$,

$$\int_{-\infty}^{\infty} \|(x^2+1)^{-1/2}\Delta^{1/4}e^{it\Delta}u\|^2\,dt \leqslant \frac{\pi}{2}\,\|u\|^2 \tag{2}$$

$$\int_{-\infty}^{\infty} \|\,|x|^{-1}\,e^{it\Delta}u\|^2\,dt \leqslant \frac{\pi}{(n-2)}\,\|u\|^2, \tag{3}$$

where the constants are best possible.

Inequalities equivalent to (2) and (3) by the Kato theory of smooth perturbations [4], but without best constants, have been known for some time, see e.g. Herbst [2, 3].

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We provide three levels of the proof. First we note that (1) follows immediately from the existence of the trace and Kato's theory of smooth operators [4]. Then we essentially translate what we need from the Kato theory to provide a proof for those who do not know this theory. Finally we provide an elementary proof of the trace theorems we need with optimal constants.

The trace estimate we need (see e.g. Kuroda [7]) is

$$\int_{S^{n-1}} |\hat{\varphi}(k\omega)|^2 k^{n-1} \, d\omega \leqslant C \min(1, k) \, \|(1+x^2)^{1/2} \varphi\|^2 \tag{4}$$

(the stronger result that Kato and Yajima prove).

The result from the Kato theory of smoothness that we need says that

$$\int_{-\infty}^{\infty} \|Ae^{itH}u\|^2 \, dt \leqslant 2\pi\alpha \, \|u\|^2 \tag{5a}$$

with

$$\alpha = \sup_E \|A\delta(H-E)\, A^*\|, \tag{5b}$$

which we also prove below. What we do not prove but use is the fact that α given by (5b) is the optimal constant in (5a).

First proof of (1). By (4)

$$\begin{aligned}(\varphi, \delta(H-E)\varphi) &= \int \delta(k^2-E)\, |\hat{\varphi}(k\omega)|^2 k^{n-1} \, d\omega \, dk \\ &= \left[\frac{1}{2k} \int_{S^{n-1}} |\hat{\varphi}(k\omega)| \, k^{n-1} \, d\omega\right]_{k=E^{1/2}} \\ &\leqslant \frac{1}{2} C \min(E^{-1/2}, 1) \, \|(1+x^2)^{1/2} \varphi\|^2\end{aligned}$$

so

$$(\varphi, (1+E)^{1/2} \delta(H-E)\varphi) \leqslant C_1 \, \|(1+x^2)^{1/2} \varphi\|^2,$$

i.e.,

$$\|(1+x^2)^{-1/2} (1+H)^{1/4} \delta(H-E)(1+H)^{1/4} (1+x^2)^{-1/2}\| \leqslant C_1.$$

By (5) this implies (1). ∎

Second proof of (1). We supplement (1) by proving the part of (5) we need! By the Plancherel theorem, if

$$\hat{f}(E) = \int_{-\infty}^{\infty} (e^{iEt} A e^{-iHt} u)\, dt/(2\pi)^{1/2}$$

then

$$\int_{-\infty}^{\infty} \|Ae^{-itH}u\|^2\, dt = \int_{-\infty}^{\infty} \|\hat{f}(E)\|^2\, dE$$

but

$$\hat{f}(E) = (2\pi)^{1/2}\, A\delta(H-E)u$$

so

$$\begin{aligned} \|\hat{f}(E)\|^2 &\leqslant 2\pi\, \|A\delta(H-E)^{1/2}\|^2\, \|\delta(H-E)^{1/2}\, u\|^2 \\ &\leqslant 2\pi\alpha\, \|\delta(H-E)^{1/2}\, u\|^2. \end{aligned}$$

Since

$$\int \|\delta(H-E)^{1/2}\, u\|^2\, dE = \int \langle u, \delta(H-E)u\rangle\, dE = \|u\|^2,$$

we have proven (5). ∎

These first two proofs have been formal about dealing with $\delta(H-E)$ so we exercise care in our last version of the proof.

Third proof of (1). Define $\delta_\varepsilon(x) = (1/\pi)\, \varepsilon/(\varepsilon^2 + x^2)$. Then

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-2\varepsilon|t|} e^{itE} (Ae^{-itH}u)\, dt = (2\pi)^{1/2}\, A\delta_\varepsilon(H-E)u$$

so as above

$$\int_{-\infty}^{\infty} \|Ae^{itH}u\|^2\, du \leqslant 2\pi\alpha\, \|u\|^2,$$

where

$$\alpha = \sup_{E,\varepsilon} \|A\delta_\varepsilon(H-E)\, A^*\|.$$

We work in k-space, i.e. with Fourier transforms, so, for example, (2) follows from

$$\|(-\Delta+1)^{-1/2}\, x^{1/4}\delta_\varepsilon(x^2-E)\, x^{1/4}(-\Delta+1)^{-1/2}\| \leqslant \tfrac{1}{4}.$$

We turn first to the proof of (2). We suppose $n \geqslant 3$. Write a general u as

$$u(x)=\sum_{l,m} r^{-(n-1)/2}\varphi_{l,m}(r)\, Y_l^m(\theta,\varphi) \tag{6}$$

with

$$\|u\|^2=\sum_{l,m}\int_0^\infty |\varphi_{l,m}(r)|^2\,dr \tag{7a}$$

$$\langle u, -\Delta u\rangle=\sum_{l,m}\int_0^\infty \overline{\varphi_{l,m}(r)}(H_l\varphi_{l,m})(r)\,dr$$

with

$$H_l=-\frac{d^2}{dr^2}+\frac{(n-1)(n-3)}{4r^2}+\frac{l(l+(1/2)(n-1))}{r^2}$$

with $\varphi(0)=0$ boundary conditions.

Consider the one-dimensional δ-potential and $-(d^2/dx^2)$ on $(-\infty,\infty)$. Then $-(d^2/dx^2)-\beta\delta(x)$ has a ground state $\exp(-\frac{1}{2}\beta\,|x|)$ with energy $-(\frac{1}{2}\beta)^2$. Thus (take $\beta=2$)

$$\delta(x)\leqslant\frac{1}{2}\left(-\frac{d^2}{dx^2}+1\right)$$

and we cannot do any better than $\frac{1}{2}$. Since $H_l \geqslant -(d^2/dx^2)$ we see that

$$\begin{aligned}\langle u, \delta(r^2-E)u\rangle &=\frac{1}{2\sqrt{E}}\langle u, \delta(r-\sqrt{E})u\rangle\\ &\leqslant\frac{1}{4\sqrt{E}}\langle u, (-\Delta+1)u\rangle\end{aligned}$$

and by taking $r\to\infty$ we cannot do better than $\frac{1}{4}$. Thus

$$\begin{aligned}\langle u, r^{1/4}\delta_\varepsilon(r^2-E)\,r^{1/4}u\rangle &\leqslant\left[\int_0^\infty \frac{1}{4}\frac{\varepsilon}{(E-r^2)^2+\varepsilon^2}(2r\,dr)\right](u,(-\Delta+1)u)\\ &=\frac{1}{4}(u,(-\Delta+1)u)\end{aligned}$$

as was to be proven.

To prove (3), consider $-\beta\delta(r-r_0)$ perturbations of $H_{l=0}$. For β small this operator is positive. The critical β is when $H\varphi=0$ has a solution which vanishes at 0 and is bounded at infinity. Away from r_0, $H\varphi=0$ means

$$\varphi''=\frac{(n-1)(n-3)}{2}\varphi$$

which has solutions $r^{\alpha_\pm}$ with $\alpha_+=\frac{1}{2}(n-1)$; $\alpha_-=-\frac{1}{2}(n-3)$ so at the critical β, the solution is

$$\varphi(r)=\begin{cases}(r/r_0)^{\alpha_+} & r\leqslant r_0\\ (r/r_0)^{\alpha_-} & r\geqslant r_0\end{cases}$$

and $\beta=r_0^{-1}(\alpha_+-\alpha_-)=(n-2)/r_0$. That is

$$\delta(r-r_0)\leqslant(n-2)^{-1}\,r_0(-\Delta)$$

and $(n-2)^{-1}$ is the best constant. Thus

$$\delta(r^2-r_0^2)=(2r_0)^{-1}\,\delta(r-r_0)\leqslant[2(n-2)]^{-1}\,(-\Delta)$$

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so that

$$\|(-\Delta)^{-1/2}\,\delta(x^2-E)(-\Delta)^{-1/2}\|\leqslant 1/2(n-2)$$

so as above, (3) follows. ∎

Note that since Kato's theory [4] says that the best constant is $2\pi\sup_E\|A\delta(H-E)\,A^*\|$ and our eigenvalue calculations for $H_{l=0}-\beta\delta(r-r_0)$ and $H_{l=0}+1-\beta\delta(r-r_0)$ get optimal β's, we know our constants in (2), (3) are optimal.

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References

1. M. Ben-Artzi and S. Klainerman, Regularity and decay of evolution equations, preprint.
2. I. Herbst, Unitary equivalence of Stark Hamiltonians, *Math. Z.* **155** (1977), 55–70.
3. I. Herbst, Spectral and scattering theory for Schrödinger operators with potentials independent of $|x|$, *Amer. J. Math.*, to appear.

4. T. KATO, Wave operators and similarity for some non-selfadjoint operators, *Man. Ann.* **162** (1977), 258–279.
5. T. KATO AND K. YAJIMA, Some examples of smooth operators and the associated smoothing effect, *Rev. Math. Phys.* **1** (1989), 481–496.
6. C. KENIG, A. RUIZ, AND C. SOGGE, Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators, *Duke Math. J.* **55** (1987), 329–347.
7. S. KURODA, "An Introduction to Scattering Theory," Lecture Note Series, Vol. 51, Aarhus University, Aarhus, Denmark.
8. P. SJÖLIN, Regularity of solutions to the Schrödinger equations, *Duke Math. J.* **53** (1987), 699–715.